On the Inverse Conductivity Problem

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Preface

This thesis presents the results of my Ph.D. studies at the Department of Mathematical Sciences, Aalborg University, Denmark, in the period August 1 1999 – August 1 2002.

The aim of the thesis is two-fold. First, the thesis is a presentation of the scientific results obtained. These new results concern three different aspects of the inverse conductivity problem and include the results obtained jointly in the papers [**KT01, CK02, CKS02**]. Second, the thesis is intended as a self-contained and convenient introduction to the inverse conductivity problem. To accomplish this the following exposition contains both a number of well-known results, which are either explained briefly or proved rigorously, and the new results, which are then given in a suitable context. Furthermore, at the end of each chapter a section with notes can be found. Here further results, references, and comments are given.

The mathematical notation used in the thesis is standard; a review is given in Appendix A. References are displayed as [**Knu02**] and refers to the bibliography pp. 97.

Part of the work was done in the fall 2001 while I was visiting the Mathematical Sciences Research Institute in Berkeley, California, as a Ph.D. student associate in the Inverse Problem program. I thank Professor Gunther Uhlmann for making this visit possible and hereby giving me a wonderful experience.

I am deeply indebted to my collaborators Horia Cornean, Samuli Siltanen, and Alexandru Tamasan. Thanks to Horia for always being open for discussions on all kinds of subjects, thanks to Samu for having a great spirit in our joint work and for the generosity in sharing with me the matlab code, and thanks to Alex for many good discussions in Berkeley. I am grateful also to my fellow students Lasse and Steen for both our mathematical discussions and a good climate in the office.

I would also like to thank my advisor, Professor Arne Jensen, for suggesting this interesting problem to me in the first place, for encouragement through the studies, and for giving me constructive comments on the manu-script.

I am grateful to Professor Russell Brown, Professor Gunther Uhlmann, and Associate Professor Svend Berntsen for being in the assessment committee for this thesis. Thanks also to Svend for sharing with us the ideas that lead to the joint project with Horia.

Finally, I wish to thank my wife, Janne, for coming with me to Berkeley, for understanding (at least some of the mathematics), encouraging support, and love!

Aalborg, Denmark, September 2002 Kim Knudsen

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Summary

This thesis concerns the inverse conductivity problem. The physical motivation for the study of this mathematical problem is a new technology for medical imaging called electrical impedance tomography. In this method one looks to find the interior conductivity in body from static electric measurements on the boundary of the body. The underlying mathematical problem is then to determine and reconstruct a coefficient in an elliptic partial differential equation on a smooth and bounded domain from full or partial knowledge of the associated Dirichlet-to-Neumann on the boundary, i.e. the map that takes a voltage potential on the boundary (Dirichlet data) into the resulting current flux through the boundary (Neumann data).

In the literature many of the important questions concerning uniqueness, reconstruction, and stability have been answered affirmatively for sufficiently regular conductivities, but for less regular conductivities including merely bounded conductivities the questions still remain open. This thesis surveys the available methods and results. The emphasis is put on the following three aspects, where also new results are obtained:

Reconstruction of conductivities in the plane (Chapter 3): focusing on the interplay between the two-dimensional inverse conductivity problem and complex analysis, a review of the uniqueness proof and reconstruction algorithm due to Nachman for conductivities having essentially two derivatives is given. Moreover, the uniqueness proof due to Brown and Uhlmann for conductivities having essentially one derivative is expounded. Then by making each step constructive in the last uniqueness proof, a new reconstruction algorithm for less regular conductivities is proposed, and this algorithm is compared to the method of Nachman's. Furthermore, the algorithm is implemented numerically and tested on synthetic, noiseless data.

The inverse conductivity problem in higher dimensions (Chapter 4): the uniqueness proof for the higher dimensional problem is reviewed as well as the existing reconstruction method. Then a result concerning absence of exceptional points near zero for merely bounded conductivities is proved. As a byproduct of this result, a new algorithm for the reconstruction of a sufficiently regular conductivity close to constant is proposed.

Inverse boundary value problem with a spherical potential (Chapter 5): here an inverse problem for the related Schrödinger equation in three dimensions is considered. The problem is from a finite number of boundary measurements to reconstruct the potential. We prove that a square integrable and sufficiently small potential, which is assumed to be independent of the radial coordinate, can be reconstructed in a stable way by just two boundary measurements.

Dansk resumé (summary in Danish)

Denne afhandling omhandler det inverse ledningsproblem. Den fysiske problemstilling, der motiverer studierne af dette matematiske problem, er en nyere teknologi for billeddannelse til eksempelvis medicinsk diagnostik kaldet Elektrisk Impedans Tomografi. Idéen bag denne metode er udfra statiske elektriske målinger på overfladen af et objekt at bestemme den indre ledningsevne i objektet. Det bagvedliggende matematiske problem er da at bestemme og rekonstruere en koefficient i en elliptisk partiel differentialligning på et glat og begrænset område udfra helt eller delvist kendskab til den tilhørende Dirichlet-til-Neumann afbildning. Denne afbildning er defineret som afbildningen, der tager et spændingsfelt på overfladen af området (Dirichlet data) til den resulterende strøm gennem overfladen (Neumann data).

I litteraturen er mange af de vigtige spørgsmål omkring entydighed, rekonstruktion og stabilitet besvaret for tilstrækkeligt glatte ledningsevner, men for mindre regulære ledningsevner er flere af spørgsmålene stadig åbne. Denne afhandling giver et overblik over metoder og resultater. Hovedvægten er især lagt på tre aspekter af problemet, indenfor hvilke der også gives en række nye resultater:

Rekonstruktion af ledningsevner i planen (Kapitel 3): Fokus er her på samspillet mellem det todimensionale inverse problem og kompleks analyse. Der redegøres for Nachmans metode til rekonstruktion af ledningsevner, der er to gange differentiable, og for et nyere entydighedsbevis af Brown og Uhlmann for ledningsevner, der kun er en gang differentiable.

Ved at lave hvert skridt i denne senere metode konstruktivt, gives en ny rekonstruktionsalgoritme for en klasse af mindre regulære ledningsevner. Endvidere gives en foreløbig numerisk implementation af denne algoritme.

Det inverse ledningsproblem i flere dimensioner (Kapitel 4): Der redegøres for løsning af spørgsmålene omkring entydighed og rekonstruktion for det inverse problem i flere dimensioner. Endvidere gives et nyt resultat, som udelukker eksistensen af exceptionelle punkter nær origo. Som en konsekvens af dette resultat foreslås en ny rekonstruktionsalgoritme for tilpas glatte ledningsevner, som er næsten konstante.

Et inverst randværdiproblem med et sfærisk potentiale (Kapitel 5): Her behandles et inverst problem for den beslægtede Schrödinger ligning med et sfærisk potentiale. Problemet er udfra et begrænset sæt af målinger på randen at rekonstruere potentialet. Det vises, at et kvadratisk integrabelt potentiale, som ikke afhænger af den radiale variabel, og som desuden har tilpas lille norm, kan rekonstrueres udfra kun to målinger på randen.

Contents

Chapter 1

Introduction

The inverse conductivity problem is the mathematical problem behind a new method for medical imaging called Electrical Impedance Tomography (EIT). In EIT one has a conductive body with unknown conductivity, and from static electric measurements on the boundary, i.e. by applying a voltage potential on the boundary and measuring the current flux through the boundary, one would like to monitor the interior conductivity. Since muscle tissue, fat tissue, bones, inner organs, lungs etc. have different conductive properties (see [**BB90**]), an image of the conductivity distribution inside a body may be used for medical diagnostics.

The expected advantages of this technology are many. First of all EIT is inexpensive compared to other imaging technologies such as Computerized Tomography and Magnetic Resonance Imaging. A second perspective is that images from EIT may complement the images obtained by the other technologies, i.e. a combination of images obtained from different technologies may give a better basis for medical diagnostics. In other situations where other technologies do not distinguish the features one wants to observe, EIT may be the only alternative for medical imaging. A third perspective is that EIT is considered to be less harmful and more comfortable to the patient. Finally, since the data collection in EIT can be done very fast, it may be possible with EIT to monitor certain dynamic features in real-time. The limitation of the technology is, however, that the expected resolution of images obtained by EIT is very low when compared to images obtained by the other imaging methods. This is in part due to the ill-posedness of the underlying mathematical problem.

In the wide-spread literature on EIT, many concrete applications have been proposed, for example in non-invasive monitoring of blood-flow, of heart and lung function, in the detection of pulmonary emboli and breast cancer, and in imaging of the brain function. Besides medical imaging EIT may also be useful in other imaging problems. We mention only geoelectrical exploration of ground water flow [**CS98**], an application in the diagnoses of trees, where EIT is proposed as a method for estimating the extent of interior damage to trees, and hence the danger for the surroundings due to falling, and a recent application in the determination of the fat marbling and hence the quality in beef. Even though there are many possible applications, the technology is still considered to be at an early stage, and there are many practical and theoretical problems to be solved before successful commercial applications of EIT can be seen. These problems include questions concerning the design of the measurement device, the design of the experiments, construction of the inversion algorithm. We refer to [**CIN99**] for a thorough review of the state of the art of EIT.

In this thesis the mathematical issues of EIT are considered. In the next section we will in greater detail introduce these mathematical problems and give an overview of the contents of the thesis.

1.1. The inverse conductivity problem

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset with smooth boundary $\partial\Omega$. Define the open subset

$$L^\infty_+(\Omega) = \{\gamma \in L^\infty(\Omega) \ : \ \gamma > 0 \text{ and } \gamma^{-1} \in L^\infty(\Omega)\}$$

of $L^\infty(\Omega)$ and assume henceforth that the conductivity $\gamma \in L^\infty_+(\Omega)$. Then γ satisfies a uniform ellipticity condition, i.e. there is a $\theta > 0$ such that $\gamma(x) \geq \theta > 0,\ a.e.\ x \in \Omega$. Assuming that there are no sources or sinks of current in Ω, the application of a voltage potential f on $\partial\Omega$ induces a voltage potential u inside Ω defined as the unique solution to the boundary value problem

$$\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega, \qquad u = f \text{ on } \partial\Omega. \tag{1.1}$$

The solvability of this elliptic boundary value problems is well-known. When $f \in H^{1/2}(\partial\Omega)$ there is a unique solution $u \in H^1(\Omega)$ which can be estimated by

$$\|u\|_{H^1(\Omega)} \leq C(\|\gamma\|_{L^\infty(\Omega)}\theta^{-1} + 1)\|f\|_{H^{1/2}(\partial\Omega)}, \tag{1.2}$$

where the constant C depends only on Ω. A proof of this statement can be found in any of the monographs [**Eva98, GT83, Gri85**]. The estimate (1.2) is obtained by keeping track of constants in the specialization of the general theory to the conductivity problem. The solution to (1.1) is slightly more regular in that the normal derivative on the boundary can be defined in a coherent way as an element of $H^{-1/2}(\partial\Omega)$ by

$$\langle \gamma(\partial_\nu u)|_{\partial\Omega}, g\rangle = \int_\Omega \gamma(x)\nabla u(x)\cdot\nabla v(x)dx, \tag{1.3}$$

where $v \in H^1(\Omega)$ satisfies $v|_{\partial\Omega} = g \in H^{1/2}(\partial\Omega)$. Here $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $H^s(\partial\Omega)$ and its dual space $H^{-s}(\partial\Omega)$. Using that u is a weak solution to (1.1), it is straightforward to see that the definition (1.3) is independent of the extension v of g. Note that when $u, v \in C^2(\overline{\Omega})$ and $\gamma \in C^1(\overline{\Omega})$ integration by parts in (1.3) gives

$$\langle \gamma(\partial_\nu u)|_{\partial\Omega}, g\rangle = \int_{\partial\Omega} \gamma(x)\partial_\nu u(x)v(x)d\sigma(x),$$

where $d\sigma$ is the usual Euclidean surface measure on $\partial\Omega$. Hence the generalized definition (1.3) of a normal derivative at the boundary coincides with the classical definition, when this makes sense.

The distribution $\gamma(\partial_\nu u)|_{\partial\Omega}$ describes the current flux through the boundary, and it is the natural Neumann data for the equation (1.1). Thus we can define the Dirichlet-to-Neumann map $\Lambda_\gamma\colon H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Lambda_\gamma f = \gamma(\partial_\nu u)|_{\partial\Omega}. \tag{1.4}$$

This map encodes all possible boundary measurements which can in principle be considered for EIT.

Consider the map

$$\Lambda : \gamma \mapsto \Lambda_\gamma$$

defined on $L^\infty_+(\Omega)$ with values in $\mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$. This map is continuous with respect to the norm topologies on the Banach spaces $L^\infty(\Omega)$ and $\mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$ (see section 2.3 below), and hence the problem of computing Λ_γ from γ is well-posed in the sense of Hadamard **[Had53]**, i.e. for any $\gamma \in L^\infty_+(\Omega)$ there is a well-defined Dirichlet-to-Neumann map which depends continuously on γ. This problem is our forward problem. Note that Λ is a non-linear map since the integrand in the right hand side of (1.3) depends on γ in a non-linear way.

The inverse conductivity problem concerns the inverse of the map Λ. There are several important questions:

- The injectivity of Λ (uniqueness).
- The continuity of Λ^{-1} (stability).
- Reconstruction of γ from Λ_γ.
- Numerical realization of a reconstruction algorithm.
- Determination of the range of Λ (existence).

- Finite number of measurements, i.e. the action of Λ_γ on a finite dimensional subspace of $H^{1/2}(\partial\Omega)$ is known.

The uniqueness question is extremely relevant, since an affirmative answer tells that two different conductivities can in principle be distinguished from boundary measurements, and hence EIT is in principle possible. Stability is important for the analysis of the propagation of errors in the data and also for numerical algorithms. The reconstruction question is relevant both from a theoretical and applied point of view, and a numerical realization of an algorithm may on one hand be a practical tool in applications and on the other hand give new insight into the theoretical problem. Concerning the existence question little seems to be known, even though it is a very interesting mathematical question. Moreover, in applications it may be advantageous to project noisy data onto the range of Λ before an inversion is carried out. Finally the question of having only a finite number of measurements is in some sense more realistic and a greater understanding of this problem and the limitations of finite data is definitely useful. Note that the issues listed are by no means independent. Indeed, as we shall see, a uniqueness proof can provide a stable reconstruction algorithm, which can be realized numerically.

An interesting fact about the inverse conductivity problem is that the amount of data given in Λ_γ is depending on the dimension of the underlying space. The data Λ_γ is a map from a space with $n-1$ variables into a space with the same number of variables, so the Schwartz kernel of that operator is a function of $2*(n-1)$ variables. On the other hand the function γ, we are looking to recover, is depending on n variables. So, in dimension $n=2$ the problem is formally determined, whereas in higher dimensions it is over determined. This motivates somewhat that to solve the two-dimensional problem we have to invoke a different method than the one used for the higher dimensional problem.

In the literature many of the inverse problems have been treated, and for sufficiently regular conductivities affirmative answers have been given. However, when the conductivity is merely bounded, only little seems to be know; this is the starting point of this project. In this thesis we will both review available results and give a number of new results. The emphasis will be put on the uniqueness and the reconstruction questions for the inverse conductivity problem in both two and higher dimensions, and on the reconstruction problem for a related Schrödinger equation from two boundary measurements.

The thesis is divided into 4 chapters. In Chapter 2 we describe two basic reductions of the inverse conductivity problem, which simplifies the problem. Then we will introduce the fundamental concept of exponentially growing solutions and the notion of exceptional points, and we will see that these concepts are equivalent, and moreover characterized by a boundary integral equation.

In Chapter 3 we draw the attention to the two-dimensional problem. Focusing on the interplay between the two-dimensional inverse conductivity problem and complex analysis we will review the uniqueness proof and reconstruction algorithm due to Nachman for conductivities having essentially two derivatives. Moreover, we will review the uniqueness proof due to Brown and Uhlmann for conductivities having essentially one derivative, and by making each step constructive in the last uniqueness proof we propose a reconstruction algorithm for a class of less regular conductivities. This reconstruction algorithm is then compared to the method of Nachman's. Furthermore, the algorithm is implemented numerically and tested on synthetic, noiseless data. The theoretical results in this chapter are in part joint with Alexandru Tamasan, University of Washington, and the implementation is joint with Samuli Siltanen, Instrumentarium Corp., Finland.

In Chapter 4 we focus on the three-dimensional problem and consider also here the uniqueness and reconstruction issues. The uniqueness proof for the higher dimensional problem is reviewed as well as the reconstruction method due to Nachman. Moreover, a new result concerning absence of exceptional points near zero for merely bounded conductivities is proved. As a byproduct of this result, a new reconstruction algorithm for sufficiently small and regular conductivities is proposed. These results are obtained jointly with Horia Cornean, Aalborg University, Denmark, and Samuli Siltanen, Instrumentarium Corp., Finland.

Finally in Chapter 5 we study a related inverse problem for a Schrödinger equation in three dimensions with a spherical potential. We prove that if the the potential is square integrable, sufficiently small, and independent of the radial coordinate, then it can be reconstructed in a stable way by just two boundary measurements. Moreover, we see how this result relates to the inverse conductivity problem. This result is joint work with Horia Cornean, Aalborg University, Denmark.

1.2. Notes

The mathematical formulation of the inverse conductivity problem as described here was given by Calderón in **[Cal80]**, a paper that put the direction for an intensive research in a new area in applied PDE. Calderón was motivated by a problem in geophysics he worked on as an engineer in the 1940's. He considered instead of the Dirichlet-to-Neumann map the unbounded quadratic form Q_γ in $L^2(\partial\Omega)$ given on the domain $H^{1/2}(\partial\Omega)$

by

$$Q_\gamma(f) = \int_\Omega \gamma(x)|\nabla u(x)|^2 dx, \tag{1.5}$$

where u solves (1.1). This form can be interpreted as the power needed to maintain the voltage potential f on $\partial\Omega$. We note that since Q_γ is positive, symmetric and closed, it defines a unique selfadjoint operator in $L^2(\partial\Omega)$, which is exactly Λ_γ. We will return to the results of Calderón in section 2.5. Although Calderón was the first author to state the problem as we do here, the problem has roots far back. In fact Langer considered a related problem in 1933 [**Lan33**]. For a thorough review on the history and developments of the inverse conductivity problem we refer to the review papers [**SU90, IN99, Uhl99**].

We note that the same inverse problems can be stated in case the conductivity is anisotropic, i.e. when the coefficient $\gamma = (\gamma_{ij})_{ij}$ entering (1.1) is a strictly positive definite $n \times n$ matrix. In fact in medical imaging the conductive properties of certain tissue does depend on directions; for instance muscle tissue has different conductivity in the longitudinal and transversal directions. For the anisotropic inverse conductivity problem uniqueness is known not to hold. Let $\Phi : \Omega \mapsto \Omega$ be a diffeomorphism with $\Phi|_{\partial\Omega} = I$. Then for

$$\Phi_*\gamma(x) = \frac{(D\Phi)^T \gamma(x)(D\Phi)}{|D\Phi|} \circ \Phi^{-1}(x),$$

where $D\Phi$ is the Jacobian of Φ, it can be seen [**KV84b**] that $\Lambda_\gamma = \Lambda_{\Phi_*\gamma}$. Hence this kind of diffeomorphic transformations violates uniqueness. In two dimensions these transformations are known to be the only obstruction to uniqueness [**Syl90, Nac96**], but in higher dimensions this question is open. We refer to [**Uhl00**] and the references given there for anisotropic inverse problems.

The assumptions that $\partial\Omega$ is smooth is taken mainly for simplicity. Most results to be presented are valid for more general domains, for instance when $\partial\Omega$ is assumed only to be Lipschitz. However, this assumption is, as we shall see in Chapter 2, in general not a restriction.

Chapter 2

Preliminary reductions and results

In this chapter we describe two reductions of the inverse conductivity problem to problems, which are in some sense simpler. Furthermore, we define the important concepts of exponentially growing solutions and exceptional points, and give a number of results concerning these objects.

The outline of the chapter is the following: in section 2.1 the general inverse problem is reduced to a special problem, where the conductivity is constant near the boundary. In section 2.2 the conductivity equation is then reduced to a Schrödinger equation, and for that equation a related inverse problem is posed. In section 2.3 we derive a simple integral identity, which links a difference of two Dirichlet-to-Neumann maps to the difference of the conductivities. This identity motivates the interest in finding solutions to the equations with exponential growth at infinity, the so-called exponentially growing solutions. The construction and properties of such solutions are the contents of section 2.4.

2.1. Reduction to the case $\gamma = 1$ near $\partial\Omega$

In this section the inverse conductivity problem is reduced to a problem, where the conductivity is constant near the boundary of the domain. The main idea behind this reduction is based on three steps. First the boundary value of the unknown conductivity and its derivatives are calculated from the Dirichlet-to-Neumann map. Then from the boundary values the conductivity is extended outside the domain of interest in a way such that the regularity is preserved. Finally, the Dirichlet-to-Neumann map on the boundary of some larger domain is computed from the known extension and the original Dirichlet-to-Neumann map. We emphasize that this reduction is constructive.

The first step concerns reconstruction of the conductivity at the boundary. Results of this kind rely both on the regularity of the conductivity and the regularity of the boundary of the domain. A formulation suitable for our purpose is the following.

Lemma 2.1.1. *Assume that $\gamma \in W^{1,r}(\Omega)$, $r > n$. Then the boundary value $\gamma|_{\partial\Omega}$ can be reconstructed from Λ_γ. If further $\gamma \in W^{2,p}(\Omega) \cup C^{1+\epsilon}(\overline{\Omega})$, $p > n/2, \epsilon > 0$, then $(\partial_\nu \gamma)|_{\partial\Omega}$ can be computed from Λ_γ.*

Proof. A proof of the result concerning reconstruction of conductivities in Sobolev spaces can be found in [**Nac96**], and a proof of the result concerning Hölder continuous conductivities can be found in [**KY01**]. □

Note that by the Sobolev embedding theorem the conductivities in Lemma 2.1.1 are continuous functions on $\overline{\Omega}$.

The next results concerns the extension to $\mathbb{R}^n$ of a function given on the boundary of a domain. We require in this context that the procedure is constructive, and that the extension relies only on the boundary values of the function and its derivatives.

Lemma 2.1.2. *Let Ω be a smooth bounded domain and let $u \in C^{l+\epsilon}(\Omega)$, $\epsilon > 0$. Then from the trace $(u|_{\partial\Omega}, (\partial_\nu u)|_{\partial\Omega}, \dots, ((\partial_\nu)^l u)|_{\partial\Omega}) \in \prod_{j=0}^{l} C^{l-j+\epsilon}(\partial\Omega)$, we can construct function $u' \in C^{l+\epsilon}(\mathbb{R}^2)$ such that $u' = u$ on Ω.*

Furthermore, let $u \in W^{s,p}(\Omega)$ and let $l \in \mathbb{Z}_+$ satisfy $s - l - 1/p > 0$. Then from $(u|_{\partial\Omega}, (\partial_\nu u)|_{\partial\Omega}), \dots, ((\partial_\nu)^l u)|_{\partial\Omega}) \in \prod_{j=0}^{l} W^{s-j-1/p,p}(\partial\Omega)$ we can construct a function $u' \in W^{s,p}(\mathbb{R}^n)$ such that $u' = u$ on Ω.

Proof. The first claim follows from the construction of the extension operator of Whitney type [**Ste70**, pp. 176], which takes into account only the boundary values of the function and its derivative.

To prove the second claim we use the fact that the trace operator

$$u \mapsto (\gamma|_{\partial\Omega}, (\partial_\nu u)|_{\partial\Omega}, \dots, ((\partial_\nu)^l u)|_{\partial\Omega})$$

is bounded from $W^{s,p}(\mathbb{R}^n)$ into $\prod_{j=0}^{l} W^{s-j-1/p,p}(\partial\Omega)$ and has a continuous right inverse $\mathcal{E}$ (see for instance [**Gri85**, Theorem 1.5.1.2]). Then we define the extension u' by

$$u' = \begin{cases} u & \text{on } \Omega, \\ \mathcal{E}(\gamma|_{\partial\Omega}, (\partial_\nu u)|_{\partial\Omega}, \dots, ((\partial_\nu)^l u)|_{\partial\Omega}) & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

This function is easily seen to be in $W^{s,p}(\mathbb{R}^n)$. □

When an extension has been constructed, it is straightforward to modify this function by multiplying and adding smooth functions supported away from $\overline{\Omega}$ and thereby construct another extension u, with the property that $u - 1$ has compact support.

We will finally prove that if the Dirichlet-to-Neumann map is known at the boundary of some domain and the conductivity is known in the exterior, then the Dirichlet-to-Neumann map at the boundary of a larger encapsulating domain can be computed.

Lemma 2.1.3. *Let $\Omega_e, \Omega \subset \mathbb{R}^n$ be smooth and bounded domains such that $\Omega \subset \Omega_e$. Let $\gamma \in L^\infty_+(\Omega_e)$, let Λ_γ be the Dirichlet-to-Neumann map on $\partial\Omega$ corresponding to $\gamma|_\Omega$ and let Λ_{γ_e} be the Dirichlet-to-Neumann map on $\partial\Omega_e$ corresponding to γ. Then Λ_{γ_e} can be reconstructed from Λ_γ and $\gamma|_{\Omega_e \setminus \overline{\Omega}}$.*

Proof. From the definition of the Dirichlet-to-Neumann map we have for any $f, g \in H^{1/2}(\partial\Omega_e)$, that

$$\begin{aligned}\langle \Lambda_{\gamma_e} f, g\rangle &= \int_{\Omega_e} \gamma \nabla u \cdot \nabla v \\ &= \int_{\Omega_e \setminus \overline{\Omega}} \gamma \nabla u \cdot \nabla v + \langle \Lambda_\gamma(u|_{\partial\Omega}), v|_{\partial\Omega}\rangle,\end{aligned}$$

where $u \in H^1(\Omega_e)$ denotes the unique solution to

$$\begin{aligned}\nabla \cdot \gamma \nabla u &= 0, \text{ in } \Omega_e, \\ u &= f, \text{ on } \partial\Omega_e,\end{aligned} \tag{2.1}$$

and $v \in H^1(\Omega_e)$ is any function with $v|_{\partial\Omega_e} = g$. Hence we see that Λ_{γ_e} can be found from $\gamma|_{\Omega_e \setminus \overline{\Omega}}$ and Λ_γ without explicit knowledge of γ in Ω provided that the solution u to (2.1) can be found in $\Omega_e \setminus \overline{\Omega}$.

We claim that u in $\Omega_e \setminus \overline{\Omega}$ can be found as the unique solution to boundary value problem

$$\begin{aligned}\nabla \cdot \gamma \nabla u &= 0 \text{ in } \Omega_e \setminus \overline{\Omega}, \\ u &= f \text{ on } \partial\Omega_e, \\ \gamma(\partial_\nu u)|_{\partial\Omega} &= \Lambda_\gamma(u|_{\partial\Omega}) \text{ on } \partial\Omega.\end{aligned} \tag{2.2}$$

Here $\gamma(\partial_\nu u)|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ is the normal derivative on $\partial\Omega$ defined in the weak sense for $g \in H^{1/2}(\partial\Omega)$ as

$$\langle \gamma(\partial_\nu u)|_{\partial\Omega}, g\rangle = \int_{\Omega \setminus \overline{\Omega}} \gamma \nabla u \cdot \nabla v, \tag{2.3}$$

with $v \in H^1(\Omega \setminus \overline{\Omega}), v|_{\partial\Omega} = g, v|_{\partial\Omega_e} = 0$.

That $u|_{\Omega_e\setminus\overline{\Omega}}$ solves (2.2) is trivial. For the uniqueness we assume that $u_0 \in H^1(\Omega_e \setminus \overline{\Omega})$ solves (2.2) with $f = 0$. Extend u_0 into Ω as the solution to

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \qquad u = u_0 \text{ on } \partial\Omega.$$

Since the extended u_0 is in $H^1(\Omega_e)$ and solves (2.1) with $f = 0$, we conclude that $u_0 = 0$ in Ω_e. □

Note that the solution to (2.2) can be constructed explicitly using for instance pseudodifferential calculus, see **[Gru96]**.

The conclusion of this section is now that if the domain Ω is smooth and the conductivity is known a priori to be in either $W^{1,p}(\Omega),\ p > n,\ W^{2,p}(\Omega)$, $p > n/2$, or $C^{1+\epsilon}(\overline{\Omega}), \epsilon > 0$, then the inverse problem can be transformed into an inverse problem on a larger domain (a ball for instance) for an extended conductivity γ, which has exactly the same regularity as before, and the property that $\gamma = 1$ near the boundary of the larger domain. Hence if we can solve the reduced problem on the larger domain, we will implicitly solve the original problem.

2.2. Reduction to a Schrödinger equation

In this section we review the well-known transformation of the conductivity problem (1.1) to a Dirichlet problem for a Schrödinger operator, and for this operator we pose an inverse problem. This reduction is central in the solution of the inverse conductivity problem, and moreover it motivates the studies in Chapter 5.

Since the reduction requires a certain amount of smoothness, we will assume that $\gamma \in W^{2,\infty}(\Omega)$. Let u be a solution to $\nabla \cdot \gamma \nabla u = 0$ and introduce $v = \gamma^{1/2} u$. Then

$$\begin{aligned} 0 &= \nabla \cdot \gamma (v \nabla \gamma^{-1/2} + \gamma^{-1/2} \nabla v) \\ &= v \nabla \cdot \gamma \nabla \gamma^{-1/2} + (\gamma \nabla \gamma^{-1/2} + \nabla \gamma^{1/2}) \cdot \nabla v + \gamma^{1/2} \Delta v \\ &= -v \Delta \gamma^{1/2} + \gamma^{1/2} \Delta v, \end{aligned}$$

which implies that

$$(-\Delta + q) v = 0 \tag{2.4}$$

for

$$q = \gamma^{-1/2} \Delta \gamma^{1/2}. \tag{2.5}$$

Conversely we could take any solution v to (2.4) and show that $u = \gamma^{-1/2} v$ solves the conductivity equation. Hence there is a one to one correspondence between solutions to the conductivity equation and solutions to the

Schrödinger equation. This shows that the problem

$$(-\Delta + q)v = 0 \text{ in } \Omega, \qquad v = f \text{ on } \partial\Omega \tag{2.6}$$

has a unique solution $v \in H^1(\Omega)$ for any $f \in H^{1/2}(\partial\Omega)$, i.e. zero is not a Dirichlet eigenvalue for $(-\Delta + q)$.

Consider now (2.6) with a general potential $q \in L^\infty(\Omega)$, and assume that zero is not a Dirichlet eigenvalue for $(-\Delta + q)$. Define the Dirichlet-to-Neumann map $\Lambda_q : H^{1/2}(\Omega) \to H^{-1/2}(\Omega)$ associated with q by

$$\Lambda_q f = (\partial_\nu v)|_{\partial\Omega}, \tag{2.7}$$

where $v \in H^1(\Omega)$ is the unique solution to (2.6) with trace $f \in H^{1/2}(\partial\Omega)$. Here the normal derivative on the boundary is defined in the weak sense by

$$\langle (\partial_\nu v)|_{\partial\Omega}, g\rangle = \int_\Omega \nabla v \cdot \nabla w + qvw, \tag{2.8}$$

where $w \in H^1(\Omega)$ has trace $g \in H^{1/2}(\partial\Omega)$ (cf. (1.3)).

A useful fact is that when q is defined from (2.5), then Λ_q can be recovered from Λ_γ. Let $u \in H^1(\Omega)$ denote the unique solutions to $\nabla \cdot \gamma\nabla u = 0$ with boundary value $\gamma^{-1/2} f \in H^{1/2}(\partial\Omega)$. Then $\gamma^{1/2}u$ solves (2.6), and for $w \in H^1(\Omega)$ with $w|_{\partial\Omega} = g \in H^{1/2}(\partial\Omega)$ we have that

$$\begin{aligned}
\langle \Lambda_q f, g\rangle &= \int_\Omega \nabla(\gamma^{1/2}u)\cdot\nabla w + \int_\Omega \Delta\gamma^{1/2}uw \\
&= \int_\Omega (\gamma\nabla u \cdot \nabla(\gamma^{-1/2}w) + \nabla\gamma^{1/2}\cdot\nabla(uw)) \\
&\quad + \int_{\partial\Omega} (\partial_\nu\gamma^{1/2})uw - \int_\Omega \nabla\gamma^{1/2}\cdot\nabla(uw) \\
&= \langle \Lambda_\gamma(\gamma^{-1/2}f), \gamma^{-1/2}g\rangle + \langle (\partial_\nu\gamma^{1/2})(\gamma^{-1/2}f), g\rangle.
\end{aligned}$$

Therefore the Dirichlet-to-Neumann map Λ_q associated to the Schrödinger equation can be obtained from the Dirichlet-to-Neumann map Λ_γ associated to the conductivity equation by first calculating the boundary values of γ and $\partial_\nu\gamma$ by Lemma 2.1.1 and then Λ_q by

$$\Lambda_q = \gamma^{-1/2}\left(\frac{1}{2}\partial_\nu\gamma + \Lambda_\gamma\right)(\gamma^{-1/2}\cdot). \tag{2.9}$$

Note that when $\gamma = 1$ near $\partial\Omega$ then $(\partial_\nu\gamma^{1/2})|_{\partial\Omega} = 0$ and hence $\Lambda_q = \Lambda_\gamma$.

We can now pose the inverse problems concerning injectivity, stability, invertibility etc. of the map

$$\begin{aligned}
\tilde{\Lambda} : L^\infty &\to \mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)) \\
q &\mapsto \Lambda_q.
\end{aligned}$$

As we shall see below in section 4.1, an answer to this inverse problem will give at least a partial answer to the inverse conductivity problem.

We note that for a general potential $q \in L^\infty(\Omega)$, zero may be a Dirichlet eigenvalue of $(-\Delta + q)$, and then the Dirichlet-to-Neumann map is not well-defined. The relevant data for the inverse problem is then the set of Cauchy data given by

$$\mathcal{C}_q = \{(u|_{\partial\Omega}, (\partial_\nu u)|_{\partial\Omega}) \mid u \in H^1(\Omega),\ (-\Delta + q)u = 0 \text{ in } \Omega\}. \tag{2.10}$$

For a potential with a well-defined Dirichlet-to-Neumann the set of Cauchy data is just the graph of this map, i.e.

$$\mathcal{C}_q = \{(f, \Lambda_q f) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)\}.$$

We will return to this in section 4.1.

2.3. An integral identity

The inverse conductivity problem is a problem where one from cleverly chosen Dirichlet data and the observation of the resulting Neumann data looks for a function inside a domain. The next result is an integral identity, which turns the focus from the boundary values of solutions to the solutions inside the domain.

Lemma 2.3.1. *Let $\gamma_1, \gamma_2 \in L^\infty(\Omega)$. Then for any solutions $u_1, u_2 \in H^1(\Omega)$ to* (1.1) *with boundary values $f_1, f_2 \in H^{1/2}(\partial\Omega)$ we have*

$$\langle(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})f_1, f_2\rangle = \int_\Omega (\gamma_1 - \gamma_2)\nabla u_1 \cdot \nabla u_2. \tag{2.11}$$

Proof. This is a direct consequence of the symmetry of the Dirichlet-to-Neumann map, i.e.

$$\langle(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})f_1, f_2\rangle = \langle\Lambda_{\gamma_1} f_1, f_2\rangle - \langle\Lambda_{\gamma_2} f_2, f_1\rangle = \int_\Omega (\gamma_1 - \gamma_2)\nabla u_1 \cdot \nabla u_2.$$

□

Note that (2.11) implies that the forward map $\Lambda : \gamma \mapsto \Lambda_\gamma$ is continuous. In fact the estimate (1.2) gives that

$$\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{B(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))} \leq C\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)}, \tag{2.12}$$

where the constant $C = C_1(\|\gamma_1\|_{L^\infty}/\theta_1 + 1)(\|\gamma_2\|_{L^\infty}/\theta_2 + 1)$, C_1 depends only on Ω, and θ_i is the ellipticity constant for γ_i, $i = 1, 2$.

For the Dirichlet-to-Neumann maps associated the Schrödinger equation we find a similar equation. In this case the relation is

$$\langle(\Lambda_{q_1} - \Lambda_{q_2})f_1, f_2\rangle = \int_\Omega (q_1 - q_2)u_1 u_2, \tag{2.13}$$

where $u_i \in H^1(\Omega)$ solve $(-\Delta + q_i)u_i = 0$ with $f_i = u_i|_{\partial\Omega}$, $i = 1, 2$.

The identity (2.11) is especially useful in relation to the uniqueness question. If two Dirichlet-to-Neumann maps agree then the difference of the underlying coefficients multiplied by a product of gradients of solutions to the relevant equations will integrate to zero. Hence uniqueness will follow, if products of gradients of solutions to the conductivity problem are dense in $L^1(\Omega)$.

2.4. Exponentially growing solutions

In the following we will consider the Schrödinger equation and the conductivity equation in the entire space $\mathbb{R}^n$. Tacitly we assume that the given potential and conductivity is extended by zero and one respectively outside the domain Ω of interest.

Let $\mathcal{V} = \{\mathbb{C}^n \setminus \{0\} \mid \zeta \cdot \zeta = 0\}$, where $\zeta \cdot \zeta = \sum_{j=1}^n \zeta_j^2$ is the real inner product. With this definition $e^{ix\cdot\zeta}$ is harmonic if and only if $\zeta \in \mathcal{V}$. The idea due to Sylvester and Uhlmann [**SU87**] is then to look for a family of special solutions $\psi(x,\zeta), \zeta \in \mathcal{V}$, to the Schrödinger equation, which are asymptotically exponential, i.e.

$$\begin{aligned} (-\Delta + q)\psi(x,\zeta) &= 0 \text{ in } \mathbb{R}^n, \\ \psi(x,\zeta) &\sim e^{ix\cdot\zeta} \text{ when } |x| \to \infty. \end{aligned} \tag{2.14}$$

The asymptotic property means that the function ω defined by

$$\omega(x,\zeta) = \mu(x,\zeta) - 1, \tag{2.15}$$

with

$$\mu(x,\zeta) = e^{-ix\cdot\zeta}\psi(x,\zeta), \tag{2.16}$$

decays to zero when x tends to infinity. Note that by plugging (2.15) into (2.14) we get the equation

$$(-\Delta - 2i\zeta \cdot \nabla)\omega(x,\zeta) + q(x)\omega(x,\zeta) = -q(x) \text{ in } \mathbb{R}^n. \tag{2.17}$$

The function ψ is called an exponentially growing solution or a complex geometrical optics solution.

In this section we outline the construction and analyze the properties of exponentially growing solutions. First we will define Faddeev's Green's functions and review the properties of these. This leads then to the construction of exponentially growing solutions and the definition of the so-called exceptional points. We then characterize the exceptional points in terms of the solvability of a certain boundary integral equation, an equation that enables a coherent definition of exponentially growing solutions and exceptional points in relation to the conductivity equation with a non-smooth conductivity.

2.4.1. Faddeev's Green's functions. Consider the equation

$$(-\Delta - 2i\zeta \cdot \nabla)u = f$$

and note that on $\mathcal{S}'(\mathbb{R}^n)$

$$(-\Delta - 2i\zeta \cdot \nabla)u = \mathcal{F}^{-1}((|\xi|^2 + 2\xi \cdot \zeta)\hat{u}(\xi)).$$

Hence formally

$$(-\Delta - 2i\zeta \cdot \nabla)^{-1} f = \mathcal{F}^{-1}((|\xi|^2 + 2\xi \cdot \zeta)^{-1}\hat{u}(\xi)) = g_\zeta * f,$$

where

$$g_\zeta(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\xi \cdot \zeta} d\xi, \qquad G_\zeta(x) = e^{ix\cdot\zeta} g_\zeta(x) \tag{2.18}$$

are the Faddeev's Green's functions for $(-\Delta - 2i\zeta \cdot \nabla)$ and $(-\Delta)$ respectively. Note that for fixed $\zeta \in \mathcal{V}$, the zeroes of $|\xi|^2 + 2\xi \cdot \zeta$ are located in the hyper plane $\xi \cdot \operatorname{Im} \zeta = 0$ on the circle $|\xi + \operatorname{Re} \zeta| = |\operatorname{Re} \zeta|$. Moreover, $(|\xi|^2 + 2\xi \cdot \zeta)^{-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and hence the integral in the definition of g_ζ makes sense when interpreted as the inverse Fourier transform defined on the space of tempered distributions.

We now collect a few useful facts about g_ζ. First note that $\zeta = \operatorname{Re}(\zeta) + i \operatorname{Im}(\zeta) \in \mathcal{V}$ satisfies $\zeta \cdot \zeta = \operatorname{Re}(\zeta)^2 - \operatorname{Im}(\zeta)^2 + 2i \operatorname{Re}(\zeta) \cdot \operatorname{Im}(\zeta) = 0$, which implies that

$$|\operatorname{Re}(\zeta)| = |\operatorname{Im}(\zeta)|, \quad \operatorname{Re}(\zeta) \cdot \operatorname{Im}(\zeta) = 0.$$

Hence ζ has the form

$$\zeta = \kappa(k_\perp + ik), \tag{2.19}$$

where $k_\perp, k \in \mathbb{R}^n$, $|k_\perp| = |k| = 1/\sqrt{2}$, $k \cdot k_\perp = 0$, and $|\zeta| = \kappa$. We have now the following result, which is useful in the analysis of g_ζ for growing κ.

Proposition 2.4.1. *Let $\zeta \in \mathcal{V}$ be decomposed as* (2.19). *Then*

$$g_\zeta(x) = \kappa^{n-2} g_{k_\perp + ik}(\kappa x). \tag{2.20}$$

Let further $\mathbf{R}$ *be a real orthogonal $n \times n$ matrix with* $\det(\mathbf{R}) = 1$. *Then*

$$g_\zeta(x) = g_{\mathbf{R}\zeta}(\mathbf{R}x). \tag{2.21}$$

Proof. We prove the results by manipulating the formal integral (2.18) (a rigorous but tedious proof by distribution theory can also be given). The substitution $\tilde{\xi} = \xi/\kappa$ implies

$$\begin{aligned} g_\zeta(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\zeta\cdot\xi} d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i\kappa x\cdot\tilde{\xi}}}{\kappa^2(|\tilde{\xi}|^2 + 2\frac{\zeta}{\kappa}\cdot\tilde{\xi})}\kappa^n d\tilde{\xi} \\ &= \kappa^{n-2} g_{k_\perp + ik}(\kappa x), \end{aligned}$$

which shows (2.20).

To prove (2.21), write $\xi' = \mathbf{R}^T\xi$ and $d\xi' = d\xi$ to get

$$\begin{aligned} g_\zeta(\mathbf{R}x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i(\mathbf{R}x)\cdot\xi}}{|\xi|^2 + 2\zeta\cdot\xi} d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix\cdot(\mathbf{R}^T\xi)}}{|\xi|^2 + 2\zeta\cdot\xi} d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi'}}{|(\mathbf{R}^T)^{-1}\xi'|^2 + 2\zeta\cdot((\mathbf{R}^T)^{-1}\xi')} d\xi' \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi'}}{|\xi'|^2 + 2(\mathbf{R}^{-1}\zeta)\cdot\xi'} d\xi' \\ &= g_{\mathbf{R}^{-1}\zeta}(x). \end{aligned}$$

□

The next result concerns the mapping properties of the convolution operator with kernel g_ζ. Consult the appendix for the definition of the weighted Sobolev spaces $H^s_\delta(\mathbb{R}^n)$.

Proposition 2.4.2. *Let $n \geq 2$, $\zeta \in \mathcal{V}$ with $|\zeta| > \epsilon > 0$ and take $-1 < \delta < 0$. Then there is a constant $C > 0$ depending on δ, ϵ, n such that*

$$\|g_\zeta * f\|_{H^s_\delta(\mathbb{R}^n)} \leq \frac{C}{|\zeta|}\|f\|_{H^s_{\delta+1}(\mathbb{R}^n)}, \tag{2.22}$$

$$\|g_\zeta * f\|_{H^s_\delta(\mathbb{R}^n)} \leq C\|f\|_{H^{s-1}_{\delta+1}(\mathbb{R}^n)}, \tag{2.23}$$

for any $s \geq 0$.

Proof. The key estimate (2.22) is proved in [**SU87**]. By a careful analysis in the Fourier domain it is proved that locally $g_\zeta *$ can be seen as the Fourier multiplier given by the function $(\xi_1 + i\xi_2)^{-1}$. An estimate for this two-dimensional operator due to Nirenberg and Walker [**NW73**] gives the result.

The estimate (2.23) is proved using (2.22), see [**Bro96**]. □

The estimates (2.22) and (2.23) are sufficient for our purpose. Consult the notes for further results along this line.

2.4.2. Existence of exponentially growing solutions and definition of exceptional points. The question considered in this section concerns the existence and uniqueness of exponentially growing solutions. To proceed we convolve with g_ζ in (2.17) and obtain the so-called Lippmann-Schwinger-Faddeev integral equation

$$(I + g_\zeta * (q\cdot))\omega(\cdot, \zeta) = g_\zeta * q, \tag{2.24}$$

which is essentially equivalent to the equation for ψ

$$\psi(x, \zeta) + \int_{\mathbb{R}^n} G_\zeta(x - y)q(y)\psi(y, \zeta)dy = e^{ix\cdot\zeta}. \tag{2.25}$$

We have then the following fundamental result concerning the existence of exponentially growing solutions for large $|\zeta|$:

Theorem 2.4.3. *Let $q \in L^\infty(\Omega)$ and let $-1 < \delta < 0$. Then there is a constant $C > 0$ depending only on Ω and δ such that for $|\zeta| > C\|q\|_{L^\infty(\Omega)}$, there is a unique solution $\omega(\cdot, \zeta) \in H^1_\delta(\mathbb{R}^n)$ to (2.17). Furthermore, we have the estimate*

$$\|\omega(\cdot, \zeta)\|_{L^2_\delta(\mathbb{R}^n)} \leq \frac{C}{|\zeta|}\|q\|_{L^\infty(\Omega)}. \tag{2.26}$$

Proof. We intend to solve (2.24) in $L^2_\delta(\mathbb{R}^n)$ Let $|\zeta| > \epsilon > 0$. Since multiplication by the compactly supported $q \in L^\infty(\mathbb{R}^n)$ maps $L^2_\delta(\mathbb{R}^n)$ into $L^2_{\delta+1}(\mathbb{R}^n)$, the operator $g_\zeta * (q\cdot)$ is bounded on $L^2_\delta(\mathbb{R}^n)$ with norm

$$\|g_\zeta * (q\cdot)\|_{B(L^2_\delta(\mathbb{R}^n))} \leq \frac{C}{|\zeta|}\|q\|_{L^\infty(\mathbb{R}^n)}.$$

Hence the operator $(I + g_\zeta * (q\cdot))$ can be inverted in $L^2_\delta(\mathbb{R}^n)$ by a Neumann series whenever $|\zeta| > C\|q\|_{L^\infty(\Omega)}$. Note that $q \in L^\infty(\Omega)$ extended by zero into $\mathbb{R}^n \setminus \Omega$ is in $L^2_{\delta+1}(\mathbb{R}^n)$. This implies by (2.22) that $g_\zeta * q \in H^1_\delta(\mathbb{R}^n)$ and we can then define

$$\omega(\cdot, \zeta) = (I + g_\zeta * (q\cdot))^{-1}(g_\zeta * q),$$

and derive the norm estimate of $\omega(\cdot, \zeta)$ from

$$\|g_\zeta * q\|_{L^2_\delta(\mathbb{R}^n)} \leq \frac{C}{|\zeta|}\|q\|_{L^\infty(\Omega)}.$$

Since $g_\zeta * q \in H^1_\delta(\mathbb{R}^n)$ and $g_\zeta * (q\omega) \in H^1_\delta(\mathbb{R}^n)$ it follows that $\omega \in H^1_\delta(\mathbb{R}^n)$. □

We note that a simple bootstrapping argument shows that in fact $\psi \in H^2_{\text{loc}}(\mathbb{R}^n)$ and smooth wherever $q = 0$.

Having established the existence and uniqueness of exponentially growing solutions for sufficiently large $|\zeta|$, we now consider the same questions for general $\zeta \in \mathcal{V}$. It turns out that this question is highly related to the existence of a non-vanishing solution to the homogeneous equation

$$(-\Delta - 2i\zeta \cdot \nabla)\omega_0(x,\zeta) + q(x)\omega_0(x,\zeta) = 0 \text{ in } \mathbb{R}^n. \tag{2.27}$$

Lemma 2.4.4. *Let $\zeta \in \mathcal{V}$. Then there is a unique exponentially growing solution to* (2.14) *with $e^{-ix\cdot\zeta}\psi(x,\zeta) - 1 \in H^1_\delta(\mathbb{R}^n)$ if and only if* (2.27) *have in $H^1_\delta(\mathbb{R}^n)$ only the trivial solution.*

Proof. Decompose $g_\zeta * (q\cdot)$ on $H^1_\delta(\mathbb{R}^n)$ as

$$H^1_\delta(\mathbb{R}^n) \xrightarrow{r} H^1(\Omega) \overset{j}{\hookrightarrow} L^2(\Omega) \xrightarrow{q\cdot} L^2(\Omega) \xrightarrow{e} L^2_{\delta+1}(\mathbb{R}^n) \xrightarrow{g_\zeta*\cdot} H^1_\delta(\mathbb{R}^n),$$

where r is the restriction operator from $\mathbb{R}^n$ to Ω, j is the identity operator, e is the extension outside Ω by zero. Since $j : H^1(\Omega) \mapsto L^2(\Omega)$ is compact by Rellich's theorem (see for instance **[Eva98]**), the compound operator is compact on $H^1_\delta(\mathbb{R}^n)$. Moreover, the right hand side $g_\zeta * q \in H^1_\delta(\mathbb{R}^n)$ by (2.23). Hence (2.24) is a Fredholm equation of the second kind in $H^1_\delta(\mathbb{R}^n)$, so if the homogeneous equation (2.27) has only the trivial solution then $I + g_\zeta * (q\cdot)$ is invertible by the Fredholm alternative. This proves the claim. □

The idea of converting the question of existence of exponentially growing solutions into a uniqueness question for a Fredholm equation of the second kind is in particular useful for the two-dimensional inverse problem for potentials q coming from a conductivity, since ideas from complex analysis (see Chapter 3) provide us with a uniqueness proof. In dimension $n \geq 3$ the uniqueness question seems harder to settle and hence this approach has not yet been useful.

Those points $\zeta \in \mathcal{V}$, for which there is no unique exponentially growing solution, are called exceptional. By Lemma 2.4.4 these are the points where (2.27) has a non trivial solution $\omega_0 \in H^1_\delta(\mathbb{R}^n)$. Since $\psi_0 = e^{ix\cdot\zeta}\omega_0$ solves the Schrödinger equation we define:

Definition 2.4.5. *A point $\zeta \in \mathcal{V}$ is said to be exceptional for q, if there is a non-vanishing solution $\psi_0(\cdot,\zeta)$ to* (2.14) *with $e^{-ix\cdot\zeta}\psi_0(x,\zeta) \in H^1_\delta(\mathbb{R}^n)$.*

The content of Theorem 2.4.3 can be formulated in terms of absence of exceptional points:

Corollary 2.4.6. *Let $q \in L^\infty(\Omega)$. Then there is a constant C depending on q such that there are no exceptional points ζ for q with $|\zeta| > C$.*

We will later (Theorem 4.2.4) the possibility of having exceptional points near $\zeta = 0$.

2.4.3. A boundary integral equation. Let $\mathbf{S}_\zeta$ be the single layer potential corresponding to Faddeev's Green's function defined initially on $C(\partial\Omega)$ by

$$\mathbf{S}_\zeta\phi(x) = \int_{\partial\Omega} G_\zeta(x-y)\phi(y)d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \partial\Omega, \tag{2.28}$$

and let

$$S_\zeta\phi(x) = \int_{\partial\Omega} G_\zeta(x-y)\phi(y)d\sigma(y), \qquad x \in \partial\Omega$$

be the boundary single layer potential. Denote by G_0 the standard Green's function for $(-\Delta)$ defined by

$$G_0(x) = -\frac{1}{2\pi}\log|x|, \quad n = 2, \tag{2.29}$$

$$G_0(x) = \frac{1}{n(n-2)\alpha(n)|x|^{n-2}}, \quad n \geq 3, \tag{2.30}$$

where $\alpha(n), n \geq 3$, is the volume of the unit ball in $\mathbb{R}^n$. Since $G_\zeta - G_0$ is harmonic and hence smooth in $\mathbb{R}^n$, the operators $\mathbf{S}_\zeta, S_\zeta$ inherit many useful properties from the standard single layer potentials

$$\mathbf{S}_0\phi(x) = \int_{\partial\Omega} G_0(x-y)\phi(y)d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \partial\Omega, \tag{2.31}$$

$$S_0\phi(x) = \int_{\partial\Omega} G_0(x-y)\phi(y)d\sigma(y), \qquad x \in \partial\Omega.$$

We collect a few facts in the following proposition. To fix notation we define the boundary operator K'_ζ by

$$K'_\zeta\phi(x) = p.v.\int_{\partial\Omega} \frac{\partial G_\zeta(x-y)}{\partial\nu(x)}\phi(y)d\sigma(y). \tag{2.32}$$

Moreover, for $u \in C^1(\mathbb{R}^n \setminus \partial\Omega)$ we write $\partial_{\nu^\pm} u = \lim_{\epsilon\to 0} \nu \cdot \nabla u(x \pm \epsilon\nu)|_{\partial\Omega}$ for the interior and exterior normal derivatives, and use the weak extension for less regular functions (cf. (2.3)). Then we have

Proposition 2.4.7. *Let $-1 \leq s \leq 0$. Then $S_\zeta \in \mathcal{B}(H^s(\partial\Omega), H^{s+1}(\partial\Omega))$ and $\mathbf{S}_\zeta \in \mathcal{B}(H^s(\partial\Omega), H^{s+3/2}(\Omega))$. For $\phi \in H^s(\partial\Omega)$ and $u = \mathbf{S}_\zeta\phi$ we have*

(1) $\Delta u = 0$ *in* $\mathbb{R}^n \setminus \partial\Omega$.

(2) *The trace of u on $\partial\Omega$ is given by*

$$u|_{\partial\Omega} = S_\zeta\phi. \tag{2.33}$$

(3) *The operator* $K'_\zeta \in \mathcal{B}(H^s(\partial\Omega))$, *and the normal derivative has across* $\partial\Omega$ *the jump*

$$\partial_{\nu^\pm} u(x) = (\mp\frac{1}{2} + K'_\zeta)\phi(x). \tag{2.34}$$

In particular

$$(\partial_{\nu^-} - \partial_{\nu^+})u(x) = \phi(x), \qquad x \in \partial\Omega. \tag{2.35}$$

Proof. Since $G_\zeta - G_0$ is harmonic it is sufficient to validate the properties for $\mathbf{S_0}$. It is a classical result that $S_0 \in \mathcal{B}(L^2(\partial\Omega), H^1(\partial\Omega))$, and that $\mathbf{S}_0\phi$ is harmonic in $\mathbb{R}^n \setminus \partial\Omega$ and has trace $\mathbf{S}_0\phi|_{\partial\Omega} = S_0\phi$ for $\phi \in L^2(\partial\Omega)$ (see **[CK98]** for the case of a smooth boundary and **[Ver84]** for the case of a Lipschitz boundary). This shows that $\mathbf{S}_0 \in \mathcal{B}(L^2(\partial\Omega), H^{3/2}(\Omega))$.

Since the adjoint $S'_0 \in \mathcal{B}(L^2(\partial\Omega), H^1(\partial\Omega))$, a duality argument shows that $S_0 \in \mathcal{B}(H^{-1}(\partial\Omega), L^2(\partial\Omega))$. The claim $S_0 \in \mathcal{B}(H^s(\partial\Omega), H^{s+1}(\partial\Omega))$, $-1 < s < 0$, then follows by interpolation (see for instance [**BL76**]). Now (1) follows by an approximation argument and (2) follows by elliptic regularity.

It is also classical that $K'_0 \in \mathcal{B}(L^2(\partial\Omega))$ (defined by (2.32) with G_ζ replaced by G_0) and that the $L^2(\partial\Omega)$ adjoint is bounded on $H^1(\partial\Omega)$. By duality we then see that $K'_0 \in \mathcal{B}(H^{-1}(\partial\Omega))$, and by interpolation that $K'_0 \in \mathcal{B}(H^s(\partial\Omega))$ for $-1 \le s \le 0$. Hence (2.34) and (2.35) follows by a density argument. □

As a consequence of (2.33) we will abuse notation and let S_ζ denote both the single layer potential and the boundary single layer potential corresponding to G_ζ.

An exponentially growing solution is inside Ω a solution to a boundary value problem for the Schrödinger equation and outside Ω a solution to Laplace's equation with a certain exponential growth condition near infinity. We can think of the boundary of Ω as the manifold, where these properties meet. In fact the boundary value of the exponentially growing solution is easily seen to satisfy a single equation, which takes into account both behaviors. Assume zero is not a Dirichlet eigenvalue of $(-\Delta + q)$ and let Λ_q be the associated Dirichlet-to-Neumann map. Assume further that ζ is not exceptional and let ψ be the unique exponentially growing solution. Then for $x \in \mathbb{R}^n \setminus \overline{\Omega}$ we find from (2.13) and (2.25) that

$$\begin{aligned}(S_\zeta(\Lambda_q - \Lambda_0)\psi)(x) &= \langle G_\zeta(x - \cdot), (\Lambda_q - \Lambda_0)\psi(\cdot, \zeta)\rangle \\ &= \int_{\mathbb{R}^n} G_\zeta(x-y)q(y)\psi(y,\zeta)dy \\ &= e^{ix\cdot\zeta} - \psi(x,\zeta),\end{aligned}$$

and by the continuity of the trace operator (2.33) we see that $f_\zeta = \psi(\cdot,\zeta)|_{\partial\Omega}$ solves

$$f_\zeta = e^{ix\cdot\zeta} - S_\zeta(\Lambda_q - \Lambda_0)f_\zeta. \tag{2.36}$$

The following theorem relates the set of exceptional points and hence the existence of exponentially growing solutions to the solvability of this equation.

Theorem 2.4.8. *A point $\zeta \in \mathcal{V}$ is not exceptional if and only if the equation* (2.36) *has a unique solution $f_\zeta \in H^{1/2}(\partial\Omega)$.*

Proof. We will first see that any solution $f_\zeta \in H^{1/2}(\partial\Omega)$ to (2.36) can be extended to an exponentially growing solution. Define $\psi(\cdot,\zeta)$ on Ω to be the solution to $(-\Delta + q)\psi = 0$ with boundary value $\psi|_{\partial\Omega} = f_\zeta$, and define ψ on $\mathbb{R}^n \setminus \overline{\Omega}$ by

$$\begin{aligned}\psi(x,\zeta) &= e^{ix\cdot\zeta} - S_\zeta(\Lambda_q - \Lambda_0)f_\zeta \\ &= e^{ix\cdot\zeta} - \int_\Omega G_\zeta(x-y)q(y)\psi(y,\zeta)dy, \qquad x \text{ in } \mathbb{R}^n \setminus \overline{\Omega}.\end{aligned} \tag{2.37}$$

By assumption on f_ζ

$$\psi|_{\partial\Omega} = f_\zeta,$$

where the trace is taken from either side of $\partial\Omega$. Furthermore, since $e^{ix\cdot\zeta} - S_\zeta(\Lambda_q - \Lambda_0)f_\zeta$ is harmonic in Ω and equals f_ζ on $\partial\Omega$, we see that

$$\partial_{\nu^-}(e^{ix\cdot\zeta} - S_\zeta(\Lambda_q - \Lambda_0)f_\zeta) = \Lambda_0 f_\zeta.$$

The jump relation (2.35) now shows that

$$(\partial_{\nu^+} - \partial_{\nu^-})(e^{ix\cdot\zeta} - S_\zeta(\Lambda_q - \Lambda_0)f_\zeta) = (\Lambda_q - \Lambda_0)f_\zeta,$$

and hence we conclude that

$$\partial_{\nu^+}\psi(\cdot,\zeta) = \Lambda_q f_\zeta.$$

This shows that $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ is in fact a weak solution to the Schrödinger equation across $\partial\Omega$. Moreover, Proposition 2.4.2 applied to (2.37) shows that ψ is exponentially growing.

To prove the theorem assume ζ is not exceptional and let ψ be the unique exponentially growing solution. Then $\psi(\cdot,\zeta)|_{\partial\Omega}$ solves (2.36). If f_ζ is another solution, then it has by the above argument an extension to an exponentially growing solution, which differs from Ψ; this violates the uniqueness of the exponentially growing solution and hence the assumption on ζ.

Conversely, assume (2.36) has a unique solution f_ζ, which extends to an exponentially growing solution ψ. Let $\tilde{\psi}$ be a different exponentially growing solution. Then $\tilde{\psi}(\cdot,\zeta)|_{\partial\Omega}$ solves (2.36) and thus $\tilde{\psi}(\cdot,\zeta)|_{\partial\Omega} = \psi(\cdot,\zeta)|_{\partial\Omega} = f_\zeta$. By uniqueness for the Dirichlet problem we see that $\psi(\cdot,\zeta) = \tilde{\psi}(\cdot,\zeta)$ in Ω and then by (2.25) that $\psi(\cdot,\zeta) = \tilde{\psi}(\cdot,\zeta)$ everywhere. This shows the uniqueness of exponentially growing solutions, and by Lemma 2.4.4 we see that ζ is not exceptional. □

2.4.4. Exceptional points for the conductivity equation. When the conductivity $\gamma \in W^{2,\infty}(\Omega)$ and $\gamma = 1$ near $\partial\Omega$ the Dirichlet-to-Neumann maps Λ_γ and Λ_q coincide. We can then write the boundary integral equation (2.36) as

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - S_\zeta(\Lambda_\gamma - \Lambda_1)\psi(\cdot,\zeta). \tag{2.38}$$

By Theorem 2.4.8 the solvability of this equation is equivalent to ζ not being exceptional, but since (2.38) makes sense for $\gamma \in L^\infty_+(\Omega)$ we can generalize the definition of exceptional points in a coherent way.

Definition 2.4.9. *Let $\gamma \in L^\infty_+(\Omega)$ and assume that $\gamma = 1$ near $\partial\Omega$. Then a point $\zeta \in \mathcal{V}$ is said not to be exceptional for γ if the boundary integral equation* (2.38) *has a unique solution in $H^{1/2}(\partial\Omega)$.*

The definition of exceptional points for a potential (Definition 2.4.5) was motivated by the existence of exponentially growing solutions to the Schrödinger equation. The following theorem shows that a exceptional point in the sense of Definition 2.4.9 has an analogue property.

Theorem 2.4.10. *Let $\gamma \in L^\infty_+(\Omega)$ and assume $\gamma = 1$ near $\partial\Omega$ and in $\mathbb{R}^n \setminus \Omega$. Then for ζ not exceptional, there is a solution $\phi(\cdot,\zeta)$ to*

$$\nabla \cdot \gamma \nabla \phi = 0 \in \mathbb{R}^n \tag{2.39}$$

with

$$(\phi(\cdot,\zeta)e^{-ix\cdot\zeta} - 1) \in L^2_\delta(\mathbb{R}^n) \tag{2.40}$$

for $-1 < \delta < 0$.

Proof. Let $f_\zeta \in H^{1/2}(\partial\Omega)$ be the unique solution to (2.38), and define $\phi(\cdot,\zeta)$ in Ω as the solution to the conductivity equation with $\phi|_{\partial\Omega} = f_\zeta$ and in $\mathbb{R}^n \setminus \Omega$ by

$$\phi(x,\zeta) = e^{ix\cdot\zeta} - S_\zeta(\Lambda_\gamma - \Lambda_1)f_\zeta.$$

This defines a solution $\phi \in H^1_{\text{loc}}(\mathbb{R}^n)$ to (2.39) (cf. the proof of Theorem 2.4.8).

To prove (2.40) we see by (2.11) that

$$\begin{aligned} e^{ix\cdot\zeta} - \phi(x,\zeta) &= S_\zeta(\Lambda_\gamma - \Lambda_1)\phi(\cdot,\zeta) \\ &= \langle G_\zeta(x-\cdot), (\Lambda_\gamma - \Lambda_1)\phi(\cdot,\zeta)\rangle \\ &= \int_\Omega (\gamma(y)-1)\nabla_y G_\zeta(x-y)\cdot\nabla\phi(y)dy \end{aligned}$$

for $x \in \mathbb{R}^n \setminus \overline{\Omega}$. Multiplying this equation by $e^{-ix\cdot\zeta}$ and replacing G_ζ by g_ζ gives

$$\begin{aligned} 1 - \phi(x,\zeta)e^{-ix\cdot\zeta} &= \int_\Omega (\gamma(y)-1)\nabla_y(e^{-iy\cdot\zeta}g_\zeta(x-y))\cdot\nabla\phi(y)dy \\ &= \int_\Omega (\gamma(y)-1)(\nabla_y - i\zeta)g_\zeta(x-y)\cdot e^{-iy\cdot\zeta}\nabla\phi(y)dy \\ &= -(\nabla_x + i\zeta)\cdot\int_\Omega g_\zeta(x-y)(\gamma(y)-1)e^{-iy\cdot\zeta}\nabla\phi(y)dy. \end{aligned}$$

Proposition 2.4.2 now implies for $-1 < \delta < 0$ that

$$\|\phi(x,\zeta)e^{-ix\cdot\zeta} - 1\|_{L^2_\delta(\mathbb{R}^n)} \le C\|\gamma - 1\|_{L^\infty(\Omega)}\|e^{-iy\cdot\zeta}\nabla\phi\|_{L^2_{\delta+1}(\Omega)}, \tag{2.41}$$

which shows the claim. □

We note that it seems impossible to obtain a decay estimate like (2.26) for $(\phi(x,\zeta)e^{-ix\cdot\zeta} - 1)$ when ζ grows. In the estimate (2.41) above, the constant C is independent of ζ, but $\|e^{-iy\cdot\zeta}\nabla\phi\|_{L^2_{\delta+1}(\mathbb{R}^n)}$ seems to be growing. This estimate may, however, not be optimal. We note further that the integral equation for $\tilde{\omega}(x,\zeta) = \phi(x,\zeta)e^{-ix\cdot\zeta} - 1$ is

$$(I + (\nabla_x + i\zeta)\cdot g_\zeta * ((\gamma-1)(\nabla + i\zeta)\cdot))\tilde{\omega} = -i\zeta\cdot\nabla g_\zeta * (\gamma - 1),$$

but the estimates in Proposition 2.4.2 seem insufficient for solving this equation.

To actually solve the boundary integral equation the following result is useful. Here it is shown that $S_\zeta(\Lambda_\gamma - \Lambda_1)$ is compact, and thus (2.38) is a Fredholm equation of the second kind. When γ is sufficiently regular this fact can be proved by transforming the equation into the Fredholm equation (2.24), but when γ is only bounded we have to argue differently. Most probably the result is well-known, but in the lack of a proper reference, we give the proof.

Lemma 2.4.11. *Assume $\gamma \in L^\infty_+(\Omega)$ satisfies $\gamma = 1$ near $\partial\Omega$. Then $(\Lambda_\gamma - \Lambda_1) \in \mathcal{B}(H^{1/2}(\partial\Omega), H^s(\partial\Omega))$ for any $s \in \mathbb{R}$, and we have the estimate*

$$\|\Lambda_\gamma - \Lambda_1\|_{\mathcal{B}(H^{1/2}(\partial\Omega),H^s(\partial\Omega))} \le C\|\gamma - 1\|_{L^\infty(\Omega)}, \tag{2.42}$$

where C depends on s, Ω, the ellipticity constant for γ, and the support of $(\gamma - 1)$.

Moreover, the composite operator $S_\zeta(\Lambda_\gamma - \Lambda_1)$ is compact on $H^{1/2}(\partial\Omega)$.

Proof. We will prove that $\Lambda_\gamma - \Lambda_1 \in \mathcal{B}(H^{1/2}(\partial\Omega), H^{m+1/2}(\partial\Omega))$ for any $m \in \mathbb{Z}_+$. Take an open domain $\Omega' \subset \Omega$ such that $\gamma = 1$ on $\Omega_0 = \Omega \setminus \overline{\Omega'}$. Take further a set of smooth cut-off functions $\{\phi_k\}_{k=0}^m$ supported near $\partial\Omega$ such that

(1) $0 \le \phi_k \le 1, \quad k = 0, \dots, m$
(2) $\phi_m = 1$ near $\partial\Omega$
(3) $\phi_0 = 0$ near $\partial\Omega'$
(4) $\phi_k = 1$ on $\Omega_{k+1} := \operatorname{supp}(\phi_{k+1})$ for $k = 0, \dots, m-1$

Then since $\phi_k(u-v) \in H_0^1(\Omega_k)$ solves

$$\Delta(\phi_k(u-v)) = \Delta\phi_k(u-v) + \nabla\phi_k \cdot \nabla(u-v) \text{ in } \Omega_k,$$

we can estimate

$$\begin{aligned}\|\phi_k(u-v)\|_{H^{k+2}(\Omega)} &\le C\|\Delta\phi_k(u-v) + \nabla\phi_k \cdot \nabla(u-v)\|_{H^k(\Omega_k)} \\ &\le C\|u-v\|_{H^{k+1}(\Omega_k)} \\ &\le C\|\phi_{k-1}(u-v)\|_{H^{k+1}(\Omega)}, \quad k = 1, 2, \dots, m, \end{aligned} \tag{2.43}$$

where C depends on the cut-off function ϕ_k. A similar argument shows that

$$\|\phi_0(u-v)\|_{H^2(\Omega)} \le C\|u-v\|_{H^1(\Omega)}. \tag{2.44}$$

Now since $u - v \in H_0^1(\Omega)$ satisfies

$$\nabla \cdot \gamma\nabla(u-v) = -\nabla \cdot (\gamma - 1)\nabla v \text{ in } \Omega,$$

standard elliptic estimates show that

$$\begin{aligned}\|u-v\|_{H^1(\Omega)} &\le C\|\nabla \cdot (\gamma-1)\nabla v\|_{H^{-1}(\Omega)} \\ &\le C\|\gamma - 1\|_{L^\infty(\Omega)}\|v\|_{H^1(\Omega)} \\ &\le C\|\gamma - 1\|_{L^\infty(\Omega)}\|f\|_{H^{1/2}(\partial\Omega)}, \end{aligned} \tag{2.45}$$

where C now depends on the ellipticity constant for γ and Ω. Combining (2.43), (2.44) and (2.45) gives

$$\begin{aligned}\|(\Lambda_\gamma - \Lambda_1)f\|_{H^{m+1/2}(\partial\Omega)} &\le \|\partial_\nu(\phi_m(u-v))\|_{H^{m+1/2}(\partial\Omega)} \\ &\le C\|\phi_m(u-v)\|_{H^{m+2}(\Omega)} \\ &\le C\|\phi_0(u-v)\|_{H^2(\Omega)} \\ &\le C\|u-v\|_{H^1(\Omega)} \\ &\le C\|\gamma-1\|_{L^\infty(\Omega)}\|f\|_{H^{1/2}(\partial\Omega)}, \end{aligned}$$

which proves the claim.

The compactness of $S_\zeta(\Lambda_\gamma - \Lambda_1)$ in $H^{1/2}(\partial\Omega)$ follows from the first part of the lemma, since $S_\zeta \in \mathcal{B}(L^2(\partial\Omega), H^1(\partial\Omega))$ (see Proposition 2.4.7) and the inclusion $H^s(\partial\Omega) \subset H^{s-\epsilon}(\partial\Omega)$ is compact for $s \in \mathbb{R}$, $\epsilon > 0$. $\square$

Since $S_\zeta(\Lambda_\gamma - \Lambda_1)$ is compact on $H^{1/2}(\partial\Omega)$, $(I + S_\zeta(\Lambda_\gamma - \Lambda_1))$ is invertible when ζ is not exceptional. This gives a formula for the computation of $\psi(\cdot,\zeta)|_{\partial\Omega}$, i.e.

$$\psi(\cdot,\zeta)|_{\partial\Omega} = (I + S_\zeta(\Lambda_\gamma - \Lambda_1))^{-1}(e^{ix\cdot\zeta}),$$

which will later be useful in the reconstruction procedures.

2.5. Notes

Concerning the determination and reconstruction of the conductivity γ and its derivatives on the boundary from Λ_γ, there are many results available. The first result was due to Kohn and Vogelius [**KV84a**], who showed that if γ is smooth on a smooth domain Ω, then Λ_γ uniquely determines $\partial_\nu^m \gamma|_{\partial\Omega}$ for any $m \geq 0$. Hereby they also solved the uniqueness question for the class of real-analytic conductivities, a result that was later generalized to piecewise real-analytic conductivities [**KV85**]. The result on the boundary was improved by Sylvester and Uhlmann [**SU88**], who proved by pseudodifferential techniques that if $\partial\Omega$ is smooth and $\gamma \in C^k(\overline{\Omega})$ then Λ_γ determines $\partial_\nu^m \gamma|_{\partial\Omega}$ for $m \leq k$. Moreover, a stability estimate for the boundary values was given. For Lipschitz domains Alessandrini [**Ale88, Ale90**] proved uniqueness at the boundary of the conductivity $\gamma \in W^{1,\infty}(\Omega)$ and for the derivatives $\partial_\nu^m \gamma|_{\partial\Omega}$ $m \leq k$, provided that γ is C^k near the boundary. More recent results include [**Bro01**],[**NT01**], and [**KY01**], where the last two references treat the determination of anisotropic conductivities as well.

The results in Lemma 2.1.1 concerning $\gamma \in W^{1,p}(\Omega), p > n$ and $\gamma \in W^{2,p}(\Omega)$, $p > n/2$ are valid also when $\partial\Omega$ is only Lipschitz. Also Lemma 2.1.2 is valid for less regular domains. The result for functions in the Hölder spaces holds for any Lipschitz domain, while the result for functions in Sobolev spaces seems to require the boundary to be $C^{k,1}, k+1 > s$. Hence the method outlined in section 2.1 may be used to reduce an inverse conductivity problem on a non-smooth domain into a problem on a smooth domain.

The reduction to the case $\gamma = 1$ near $\partial\Omega$ was first given by Nachman [**Nac96**]. He considered the case $\gamma \in W^{2,p}(\Omega), p > n$, and gave an explicit formula for the Dirichlet-to-Neumann map on the outer boundary: let $\Omega_1 := \Omega$ and let $\Omega_2 := \Omega_e$. Define the operators $\Lambda^{ij} : H^{1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_j), i,j \in \{1,2\}$, by

$$\langle \Lambda_{ij} f, g \rangle = (-1)^i \int_{\Omega_1 \setminus \overline{\Omega}_2} \gamma \nabla u_i \cdot \nabla v_j,$$

where $\nabla \cdot \gamma \nabla u_i = 0, u_i|_{\partial\Omega_i} = f, u_1|_{\partial\Omega_2} = 0 = u_1|_{\partial\Omega_1}$ and $v_j|_{\partial\Omega_j} = g, v_1|_{\partial\Omega_2} = 0 = v_2|_{\partial\Omega_1}$. This defines Λ_{11} and Λ_{22} as reduced Dirichlet-to-Neumann maps on $\partial\Omega_1$ and $\partial\Omega_2$ respectively, and $\Lambda_{12}, \Lambda_{21}$ as transition Dirichlet-to-Neumann maps from one boundary to the other. Then it is proved that

(1) $(\Lambda_\gamma - \Lambda_{11})$ is invertible,

(2) $\Lambda_{\gamma_e} = \Lambda_{22} + \Lambda_{21}(\Lambda_\gamma - \Lambda_{11})^{-1}\Lambda_{12}$,

which gives the formula for Λ_{γ_e}. The proof of these statements (see the preprint [**Nac93**]) relies on the assumption that $\gamma \in W^{2,p}(\Omega), p > 1$. However, in a recent paper by Ikehata [**Ike02**], (1) was proved in a different context using only the assumption that $\gamma \in L^\infty_+(\Omega)$. Moreover, by using Ikehata's result, (2) is easily seen to be valid in the more general case. This constitutes a different proof of Lemma 2.1.3; the proof given in section 2.1 is based on [**KT01**].

An interesting question related to Lemma 2.1.3 is, whether it is possible to propagate the Dirichlet-to-Neumann inwards through a known medium, i.e. whether Λ_γ is determined by Λ_{γ_e} and $\gamma|_{\Omega_e \setminus \overline{\Omega}}$. Ikehata [**Ike02**] gave an affirmative answer assuming $\gamma \in C^{0,1}(\overline{\Omega}_e)$ when $n \geq 3$ and $\gamma \in L^\infty(\Omega_e)$ when $n = 2$. The method of proof relies on the Runge approximation theorem and is thus in essence non-constructive.

The study of the exponentially growing solutions is motivated by Calderón's fundamental paper [**Cal80**]. Consider the bilinear form associated the quadratic form (1.5)

$$Q_\gamma(f,g) = \int_\Omega \gamma \nabla u \cdot \nabla v,$$

where u, v solves (1.1) with f, g on the boundary. Calderón showed in his paper that the Fréchet derivative of Q_γ in direction $h \in L^\infty(\Omega)$ is

$$(dQ_\gamma(h))(f,g) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(Q_{\gamma+\epsilon h}(f,g) - Q_\gamma(f,g)) = \int_\Omega h \nabla u \cdot \nabla v, \quad (2.46)$$

where again u, v solves (1.1) with f, g on the boundary. Hence the derivative is injective if products of gradients to the conductivity equation are dense in $L^1(\Omega)$. For $\gamma = 1$ and $\zeta \in \mathcal{V}$ the harmonic functions $e^{ix\cdot\zeta}$, $e^{-ix\cdot\overline{\zeta}}$ are solutions to the conductivity equation, and by using these solutions in (2.46) Calderón proved that $dQ_1(h) = 0$ if and only if $h = 0$. However, since, as it was pointed out by Calderón, $\mathrm{Ran}(dQ_1)$ is not closed (see [**IN99**] for a proof), the inverse mapping theorem cannot be applied to give local uniqueness around constant conductivities. Calderón [**Cal80**] gave also using this differential an approximate reconstruction method for γ close to a constant.

In [**Isa91**] Isakov proved by a different method the existence of exponentially growing solutions by taking into account the fact that the potential is compactly supported. The idea is to show the existence of a Green's operator $E_\zeta : L^2(\Omega) \to H^2(\Omega)$ with

$$(\Delta + 2i\zeta \cdot \nabla)(E_\zeta f) = f,$$

such that

$$\|E_\zeta f\|_{L^2(\Omega)} \leq \frac{C}{|\zeta|}\|f\|_{L^2(\Omega)},$$
$$\|E_\zeta f\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

where the generic C is independent of ζ. The ideas in the proof goes back to the work of Ehrenpreis and Malgrange in the 1950's and is based on a general method for constructing Green's functions. This method works in fact for more general types of equations, and hence exponentially growing solutions for these equations can be constructed. In [**Häh96**] Hähner gave an elegant and elementary proof of a similar result using Fourier series techniques.

Another result concerning the convolution operator $g_\zeta *$ in dimension $n \geq 3$ is the following due to Jerison and Kenig (as found in [**Cha90**]): let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Assume that $q \in L^r(\Omega)$ for some r with $n/2 \leq r \leq (n+1)/2$. Then for $\zeta \in \mathcal{V}$ and $p = 2r/(r-1)$

$$\|g_\zeta * (qf)\|_{L^p(\mathbb{R}^n)} \leq \frac{C}{|\zeta|^{2-n/r}}\|q\|_{L^r(\Omega)}\|f\|_{L^p(\mathbb{R}^n)}.$$

This result follows by applying to $g_{\zeta/|\zeta|}$ the uniform estimates in [**KRS87**] for second order differential operators with constant coefficients. As a consequence we obtain for

$$C|\zeta|^{-(2-n/r)}\|q\|_{L^r(\Omega)} < 1,$$

the existence of a unique exponentially growing solution with

$$\|\omega(\cdot,\zeta)\|_{L^p(\mathbb{R}^n)} \leq \frac{C}{|\zeta|^{2-n/r}}\|q\|_{L^r(\Omega)}. \tag{2.47}$$

We refer to Chapter 3 for an improvement of this L^p-estimate in two dimensions.

We mention also the following extension of Proposition 2.4.2 due to Brown [**Bro96**, Theorem 0.3]. By interpolating (2.22) and (2.23) we get

$$\|g_\zeta * f\|_{H^s_\delta(\mathbb{R}^n)} \leq \frac{C}{|\zeta|^{1-s}}\|f\|_{L^2_{\delta+1}(\mathbb{R}^n)}, \qquad 0 \leq s \leq 1,\ -1 < \delta < 0,$$

which by duality implies that

$$\|g_\zeta * f\|_{L^2_\delta(\mathbb{R}^n)} \leq \frac{C}{|\zeta|^{1-s}}\|f\|_{H^{-s}_{\delta+1}(\mathbb{R}^n)}, \qquad 0 \leq s \leq 1,\ -1 < \delta < 0.$$

Interpolating these two estimates with interpolation parameter $\theta = 1/2$ gives

$$\|g_\zeta * f\|_{H^s_\delta(\mathbb{R}^n)} \leq \frac{C}{|\zeta|^{1-2s}}\|f\|_{H^{-s}_{\delta+1}(\mathbb{R}^n)}, \qquad 0 \leq s \leq \frac{1}{2},\ -1 < \delta < 0.$$

The study of the special Green's function goes back to Faddeev [**Fad65**], who considered the inverse scattering problem for the Schrödinger equation. Faddeev suggested a generalization of the method for the one-dimensional inverse scattering problem due to Gelfand-Levitan-Marchenko, but a major obstacle was the presence of real exceptional points. For a comprehensive survey on the inverse scattering problem in dimension $n = 3$ we refer to the monograph [**New89**]. In [**NA84, BC85, SU87**] Faddeev's Green's functions were rediscovered independently.

The boundary integral equation was derived in Nachman [**Nac88**] and Novikov [**NK87**] independently. Nachman derived the slightly different boundary integral equation

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - (S_\zeta\Lambda_q - K_\zeta - \frac{1}{2}I)\psi(\cdot,\zeta) \tag{2.48}$$

based on a combined single and double layer potential representation of the exponentially growing solution. Here K_ζ is the boundary double layer potential based the Faddeev Green's function. Since an easy argument shows that $(1/2I + K_\zeta) = S_\zeta\Lambda_0$ we see that (2.48) is equivalent to (2.36).

Chapter 3

Uniqueness and reconstruction in two dimensions

In this chapter we will turn our attention to the two-dimensional inverse conductivity problem. As stated in Chapter 1 this problem seems at first glance harder than the higher dimensional problem, since it is only formally determined. One great advantage in two dimensions is, however, the applicability of the well developed tools from complex analysis.

The problem of uniqueness and reconstruction was solved by Nachman in [**Nac96**] for conductivities having essentially two derivatives. Shortly after Brown and Uhlmann relaxed the assumption and gave an affirmative answer to the uniqueness question for conductivities having only one derivative [**BU97**]. The technique of proof in the two papers are very similar in that both methods exploit results from complex analysis and are based on the so-called $\bar{\partial}$-method of inverse scattering. Based on Brown and Uhlmann's approach in a joint work with Alexandru Tamasan we gave a reconstruction algorithm for the less regular class of conductivities [**KT01**].

In this chapter we will first review a few basic and useful results from complex analysis. Next we review the uniqueness proof and reconstruction method for conductivities having two derivatives. We will then focus on the uniqueness proof for less regular conductivities and then show how this method can be turned into a reconstruction algorithm. In section 3.5 the two reconstruction methods are compared, and we will see that in some sense the new method is a direct generalization of the previous one. In the last section we describe a numerical implementation of the algorithm and comment on the results.

As seen in section 2.1 the general inverse conductivity problem can be reduced to the case, where the conductivity $\gamma = 1$ near the boundary of the domain Ω. We will throughout this chapter take this assumption as well as the ellipticity assumption $\gamma \in L^\infty_+(\Omega)$.

3.1. A few results from complex analysis

Let $z = x_1 + ix_2 \in \mathbb{C}$ and define the complex differential operators $\partial = \partial/\partial z$ and $\overline{\partial} = \partial/\partial \overline{z}$ by

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \qquad \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

To define the inverse of these operators we introduce

$$\mathfrak{g}(z) = \mathcal{F}^{-1}\left(\frac{2}{i(\xi_1 + i\xi_2)}\right) = \frac{1}{\pi z}, \tag{3.1}$$

which is a Green's function for $\overline{\partial}$. In the definition of $\mathfrak{g}$ and henceforth we identify the complex number $z = x_1 + ix_2 \in \mathbb{C}$ and $(x_1, x_2) \in \mathbb{R}^2$. Then we define on $\mathcal{S}(\mathbb{R}^2)$ the weakly singular integral operators $\partial^{-1}, \overline{\partial}^{-1}$ by

$$\partial^{-1} f(z) = \overline{\mathfrak{g}} * f(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z')}{\overline{z} - \overline{z}'} d\mu(z'),$$
$$\overline{\partial}^{-1} f(z) = \mathfrak{g} * f(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z')}{z - z'} d\mu(z'),$$

where $d\mu(z) = dz_1 dz_2$ is the usual Lebesgue measure in the plane. The mapping properties of these operators in conventional spaces are well analyzed in the literature. We denote below by $\tilde{p},\ p'$ the parameters defined by

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2} \quad \text{for } 1 \leq p \leq 2$$
$$\frac{1}{p'} = 1 - \frac{1}{p} \quad \text{for } 1 \leq p \leq \infty.$$

Then we can state the following useful facts.

Proposition 3.1.1. *Let $f \in \mathcal{S}(\mathbb{R}^2)$ and let T be either ∂^{-1} or $\overline{\partial}^{-1}$. Define $u = Tf$ in $\mathbb{R}^2$. Then for $1 < p < 2$*

$$\|u\|_{L^{\tilde{p}}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}, \tag{3.2}$$

and hence $T \in \mathcal{B}(L^p(\mathbb{R}^2), L^{\tilde{p}}(\mathbb{R}^2))$.

Moreover, if $f \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)$ *for some* $1 < p_1 < 2 < p_2 < \infty$ *then*

$$u \in C^\alpha(\mathbb{R}^2), \qquad \text{for } \alpha = 1 - \frac{2}{p_2}. \tag{3.3}$$

Finally, if $f \in C^\alpha(\mathbb{R}^2)$ *is compactly supported then* $u \in C^{1+\alpha}_{loc}(\mathbb{R}^2)$.

Proof. The first result is a consequence of the Hardy-Littlewood-Sobolev theorem of fractional integration for the Riesz transform, see [**Ste70**, Theorem 1, p. 119]. The results concerning Hölder regularity are from the classical monograph of Vekua, see [**Vek62**, Theorem 1.21 and 1.32]. □

Note that since T commutes with differential operators, more regularity in f implies more regularity in Tf.

We will later need the following extension of Proposition 3.1.1. Define the unimodular function

$$e(z,k) = \exp(i(zk + \overline{zk})) = \exp(2i\,\mathrm{Re}(zk)), \qquad z,k \in \mathbb{C}.$$

Then

$$(\partial + ik)u = e(z,-k)\partial(e(z,k)u)$$

motivates the definition

$$(\partial + ik)^{-1} f = e(z,-k)\partial^{-1}(e(z,k)f).$$

For this operator we have the following result.

Proposition 3.1.2. *Let* $1 < p < 2$ *and assume that* $v \in L^{\tilde{p}}(\mathbb{R}^2)$ *has* $\overline{\partial} v \in L^p(\mathbb{R}^2)$. *Then for* $k \in C \setminus \{0\}$, *the function* $u = (\partial + ik)^{-1} v \in W^{1,\tilde{p}}(\mathbb{R}^2)$ *and we have the estimates*

$$\|(\partial + ik)^{-1} v\|_{L^{\tilde{p}}(\mathbb{R}^2)} \leq \frac{C}{|k|}(\|v\|_{L^{\tilde{p}}(\mathbb{R}^2)} + \|\overline{\partial} v\|_{L^p(\mathbb{R}^2)}), \tag{3.4}$$

$$\|(\partial + ik)^{-1} v\|_{W^{1,\tilde{p}}(\mathbb{R}^2)} \leq C\left(1 + \frac{1}{|k|}\right)(\|v\|_{L^{\tilde{p}}(\mathbb{R}^2)} + \|\overline{\partial} v\|_{L^p(\mathbb{R}^2)}), \tag{3.5}$$

where C *is independent of* k.

Proof. We refer to [**Nac96**]. □

A fundamental result in the theory of complex functions is Liouville's theorem, which states that any bounded function u satisfying $\overline{\partial} u = 0$ in $\mathbb{C}$ must be constant. A consequence is that the only entire function in $L^p(\mathbb{R}^2), 1 \leq p < \infty$, is the zero function. This fundamental result can be generalized to a class of functions satisfying a more complicated equation.

In the following we will say that a function u is pseudoanalytic in $\mathbb{C}$ with coefficients a, b if and only if it satisfies the equation

$$\bar{\partial} u = a\overline{u} + bu \text{ in } \mathbb{C}. \tag{3.6}$$

The generalized Liouville theorem for pseudoanalytic functions is then

Theorem 3.1.3. *If $u \in L^p(\mathbb{R}^2), 1 \leq p < \infty$ is pseudoanalytic in $\mathbb{C}$ with the coefficients $a, b \in L^2(\mathbb{R}^2)$, then $u = 0$.*

Proof. We will give the proof when $a, b \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)$ for $1 \leq p_1 < 2 < p_2 < \infty$. Let

$$\tilde{v} = a\frac{\overline{u}}{u} + b \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2).$$

Note that $\bar{\partial}^{-1}\tilde{v} \in L^\infty(\mathbb{R}^2)$ by (3.3). Let $v = u \exp(-\bar{\partial}^{-1}\tilde{v})$ and note that $v \in L^p(\mathbb{R}^2)$ by assumption on u. Since $\bar{\partial} v = 0$ everywhere, v is an entire function in $L^p(\mathbb{R}^2)$, and by the Liouville theorem we conclude that $v = 0$. This implies $u = 0$.

When $p_1 = 2 = p_2$ the proof is more complicated, since in general $\bar{\partial}^{-1} f$ is only in $\mathrm{BMO}(\mathbb{R}^2)$ and not bounded for $f \in L^2(\mathbb{R}^2)$. For a proof of the theorem in this case we refer to **[BU97]**. □

In the following we will several times have to solve a certain pseudoanalytic equation with a decay condition at infinity. The next result gives an exact condition for this to be possible.

Corollary 3.1.4. *Let $1 < p < 2$ and assume that $a \in L^p(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Then the equation*

$$\bar{\partial} m = a\overline{m} \tag{3.7}$$

has a unique solution m with $m - 1 \in L^{\tilde{p}}(\mathbb{R}^2)$.

Proof. The equation for $m - 1$ is

$$\bar{\partial}(m-1) = a\overline{(m-1)} + a$$

or equivalently

$$(I - \bar{\partial}^{-1}(a\, \bar{\cdot}))(m-1) = \bar{\partial}^{-1} a. \tag{3.8}$$

Since $a \in L^p(\mathbb{R}^2)$ implies $\bar{\partial}^{-1} a \in L^{\tilde{p}}(\mathbb{R}^2)$ by (3.2), it makes sense to consider this integral equation in $L^{\tilde{p}}(\mathbb{R}^2)$.

From **[Nac93]** it is known that when $a \in L^2(\mathbb{R}^2)$, the operator $\bar{\partial}^{-1}(a\, \bar{\cdot})$ is real-linear and compact in the $L^r(\mathbb{R}^2)$, $r > 2$. Hence invertibility of $(I - \bar{\partial}^{-1}(a\, \bar{\cdot}))$ in $L^{\tilde{p}}(\mathbb{R}^2)$ is by the Fredholm alternative a consequence of uniqueness of a solution to the homogeneous equation. However, a solution $h \in L^{\tilde{p}}(\mathbb{R}^2)$ to the homogeneous equation would satisfy $\overline{\partial} h = a\overline{h}$, and hence $h = 0$ by the generalized Liouville theorem, Theorem 3.1.3. This proves the result. □

We note that when a is only in $L^2(\mathbb{R}^2)$, then the generalized Liouville theorem does give uniqueness of a solution to (3.7). But for $a \in L^2(\mathbb{R}^2)$, $\overline{\partial}^{-1} a$ is generally only of bounded mean oscillation, and hence it seems not possible to actually solve (3.8) in a conventional Lebesgue space.

In applications of the above result we will need a result concerning the dependence of the solution on the parameter a. We have the following continuity result.

Lemma 3.1.5. *Let $1 < p_1 < 2 < p_2 < \infty$ and assume that $a_i \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)$, $i = 1, 2$. Let m_i, $m_i - 1 \in L^{\tilde{p}_1}(\mathbb{R}^2)$ be the unique solution to* (3.7) *with coefficient a_i. Then*

$$\|m_1 - m_2\|_{L^{\tilde{p}_1}(\mathbb{R}^2)} \le C_2(\|a_1 - a_2\|_{L^{p_1}(\mathbb{R}^2)} + C_1\|a_1 - a_2\|_{L^2(\mathbb{R}^2)}\|a_1\|_{L^{p_1}(\mathbb{R}^2)}),$$

where

$$C_i = C\exp(C(\|a_i\|_{L^{p_1}(\mathbb{R}^2)} + \|a_i\|_{L^{p_2}(\mathbb{R}^2)})).$$

Proof. From [**BBR01**, Lemma 2.6] we have for the solution to $\overline{\partial} u = a\overline{u} + b$ the stability estimate

$$\|u\|_{L^{\tilde{p}}(\mathbb{R}^2)} \le C\|b\|_{L^p(\mathbb{R}^2)} \exp(C(\|a\|_{L^{p_1}(\mathbb{R}^2)} + \|a\|_{L^{p_1}(\mathbb{R}^2)})) \tag{3.9}$$

for $1 < p < 2$ and $1 < p_1 < 2 < p_2 < \infty$. This estimate applied to (3.7) gives

$$\begin{aligned}\|m_i - 1\|_{L^{\tilde{p}_1}(\mathbb{R}^2)} &\le C\|a_i\|_{L^{p_1}(\mathbb{R}^2)} \exp(C(\|a_i\|_{L^{p_1}(\mathbb{R}^2)} + \|a_i\|_{L^{p_2}(\mathbb{R}^2)})) \\ &= C_i\|a_i\|_{L^{p_1}(\mathbb{R}^2)}. \end{aligned} \tag{3.10}$$

Since

$$\overline{\partial}(m_1 - m_2) = a_1\overline{(m_1 - m_2)} + (a_1 - a_2)\overline{m_2},$$

the estimate (3.9) applied to $m_1 - m_2$ then gives

$$\begin{aligned}\|m_1 - m_2\|_{L^{\tilde{p}_1}(\mathbb{R}^2)} &\le C_1\|(a_1 - a_2)\overline{m_2}\|_{L^{p_1}(\mathbb{R}^2)} \\ &\le C_1(\|a_1 - a_2\|_{L^{p_1}(\mathbb{R}^2)} + \|a_1 - a_2\|_{L^2(\mathbb{R}^2)}\|\overline{m}_2 - 1\|_{L^{\tilde{p}_1}(\mathbb{R}^2)})\end{aligned}$$

by Hölders inequality. The claim now follows by combining this with (3.10). □

The last result we need from complex analysis is a result concerning the single layer operator associated with the Green's functions for ∂ and $\overline{\partial}$. Let

$D \subset \mathbb{C}$ be an open, bounded and smooth domain and let $f \in C^{\alpha}(\partial D), 0 < \alpha < 1$. Define the Cauchy integral operator

$$\mathcal{S}f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z')}{z'-z}dz', \qquad z \in \mathbb{R}^2 \setminus \partial D,$$

where the integral is understood as a path integral along a positively oriented path describing the boundary $\partial \mathcal{D}$. This integral defines a function which is holomorphic away from ∂D. When $z \in \partial D$ the integral kernel is not integrable, so to define an operator on the boundary we have to work with principal values of the integral. Thus we define the singular integral operator

$$\mathcal{K}f(z) = \frac{1}{2\pi i} p.v. \int_{\partial D} \frac{f(z')}{z'-z}dz', \qquad z \in \partial D.$$

A few facts about $\mathcal{K}$ and the relation to $\mathcal{S}$ is collected in the next result.

Proposition 3.1.6. *The singular integral operator $\mathcal{K} \in \mathcal{B}(C^{m+\alpha}(\partial D))$ for $m \in \mathbb{Z}_+$ and $0 < \alpha < 1$.*

Let ν be the outpointing normal to ∂D. Then on the boundary we have the jump condition

$$\lim_{\epsilon \to 0} \mathcal{S}f(z \pm \nu\epsilon) = \mp \frac{1}{2} f(z) + \mathcal{K}f(z), \qquad z \in \partial D. \tag{3.11}$$

Proof. For a proof we refer to the classical monograph of Muskhelishvili on singular integral equations [**Mus53**, p. 42]. ☐

3.2. Uniqueness and reconstruction for $\gamma \in W^{2,p}(\Omega)$

In this section we briefly describe Nachman's [**Nac96**] reconstruction method for the two-dimensional inverse conductivity problem. The main result is

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^2$ be open, bounded and smooth, and let $\gamma \in W^{2,p}(\Omega)$, $p > 1$. Then γ can be reconstructed uniquely from Λ_γ.*

The method of proof is based on the reduction of the conductivity equation to the Schrödinger equation as described in section 2.2, and the construction of exponentially growing solutions to this equation. In two dimensions the variety $\mathcal{V} = \{\zeta \in \mathbb{C}^2 \setminus \{0\} \mid \zeta \cdot \zeta = 0\} = \{\zeta \in \mathbb{C}^2 \setminus \{0\} \mid \zeta = (k, \pm ik), k \in \mathbb{C}\}$, and since an exponentially growing solution ψ to the Schrödinger equation has the symmetry $\psi(x,(k,ik)) = \overline{\psi(x,(-\overline{k}, i\overline{k}))}$ when $q = \Delta\gamma^{1/2}/\gamma^{1/2}$ is real, it suffices to consider only $\zeta = (k, ik), k \in \mathbb{C} \setminus \{0\}$. In this case we note that $x \cdot \zeta = x_1 k + i x_2 k = zk$ for $z = x_1 + ix_2$. Hence we will in the following abuse notation and write $\psi(z,k) := \psi(x,(k,ik))$ indicating that this is an exponentially growing solution with growth e^{izk}.

From Theorem 2.4.6 it is known, that there are no exceptional points outside a sufficiently large ball. In two dimensions it can be shown (see Lemma 3.2.3 below) that when the potential comes from a conductivity, then there are no exceptional points at all, i.e. the exponentially growing solutions $\psi(\cdot, k)$ are defined for any $k \in \mathbb{C} \setminus \{0\}$. This allows the introduction of the function

$$\mathbf{t}(k) = \int_\Omega e^{i\overline{kz}} q(z)\psi(z,k)d\mu(z), \qquad k \in \mathbb{C} \setminus \{0\}, \tag{3.12}$$

the so-called non-physical scattering transform of the potential q. We note that $\mathbf{t}$ is a nonlinear transform of q, since in the integrand both q itself and ψ depend on the potential. Note further that since $\psi(z,k) \sim e^{izk}$, the integrand has the asymptotics $e^{i(\overline{kz}+zk)}q(x) \sim e^{2i(x_1k_1 - x_2k_2)}q(x)$. This implies $\mathbf{t}(k) \sim 2\pi\hat{q}(-2k_1, 2k_2)$ for large k. Not only is $\mathbf{t}$ asymptotically close to the Fourier transform of the potential, but it has also many similar properties. It can for instance be shown that smoothness in q corresponds to decay in $\mathbf{t}$, and symmetries in q implies symmetries in $\mathbf{t}$, see **[Sil99]**.

The non-physical scattering transform is the key intermediate object in the reconstruction algorithm consisting of the two steps:

(1) Reconstruct $\mathbf{t}$ from Λ_γ.
(2) Reconstruct γ from $\mathbf{t}$.

In the following subsections we will first review the result concerning absence of exceptional points. Then we will be more explicit about the details in the two steps in the reconstruction.

3.2.1. Absence of exceptional points. Before giving the result we state the following extension of Proposition 2.4.2. Let g_ζ be Faddeev's Green's function defined in (2.18) and denote

$$g_k(z) := g_{(k,ik)}(x), \quad k \in \mathbb{C} \setminus \{0\}.$$

Proposition 3.2.2. *Let $1 < p < 2$ and $k \in \mathbb{C} \setminus \{0\}$. Then the operator g_k* is a bounded operator from $L^p(\mathbb{R}^2)$ into $W^{1,\tilde{p}}(\mathbb{R}^2)$ and satisfies*

$$\|g_k * f\|_{W^{s,\tilde{p}}(\mathbb{R}^2)} \leq C|k|^{s-1}\|f\|_{L^p(\mathbb{R}^2)}, \tag{3.13}$$

for $0 \leq s \leq 1$. Here the constant C is uniform for $|k| \geq \epsilon > 0$.

Proof. Since $-\Delta - 2(k, ik) \cdot \nabla = -4(\partial(\overline{\partial} + ik))$, we have

$$g_k * f = -\frac{1}{4}(\overline{\partial} + ik)^{-1}\partial^{-1}f.$$

The result is then obtained by invoking Proposition 3.1.2 and interpolating (3.4) and (3.5). □

The definition of exceptional points (Definition 2.4.5) were based on the solvability of the Lippmann-Schwinger-Faddeev equation (2.24) in $H^1_\delta(\mathbb{R}^n)$. Due to the improved estimate (3.13), we will in the following equivalently consider (2.24) in a usual Sobolev space and in this context adopt the notion of exceptional points.

The absence of exceptional points is the result of the next lemma. We will give the proof of the result, since it is illustrative in relation to the reconstruction for less regular conductivities to follow later.

Lemma 3.2.3. *Let $q = \Delta\gamma^{1/2}/\gamma^{1/2}$ for $\gamma \in W^{2,p}(\Omega), 1 < p < 2$. Then for any $k \in \mathbb{C} \setminus \{0\}$ there exists a unique solution $\psi(\cdot, k)$ to*

$$(-\Delta + q)\psi = 0 \text{ in } \mathbb{R}^2 \tag{3.14}$$

with $\omega(\cdot, k) = \psi(\cdot, k)e^{-ikz} - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$. For k sufficiently large we have the estimate

$$\|\omega(\cdot, k)\|_{W^{s,\tilde{p}}(\mathbb{R}^2)} \leq C|k|^{s-1}\|q\|_{L^p(\mathbb{R}^2)} \tag{3.15}$$

for any $0 \leq s \leq 1$.

Proof. The integral equation for ω is (cf. (2.24))

$$\omega(z, k) + g_k * (q\omega(\cdot, k))(z) = (g_k * q)(z). \tag{3.16}$$

We consider this integral equation in $W^{1,\tilde{p}}(\mathbb{R}^2)$. Since the inclusion $W^{1,\tilde{p}}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ is compact due to Rellich's theorem, multiplication by $q \in L^p(\mathbb{R}^2)$ is bounded from $L^\infty(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$, and the operator g_k* is bounded from $L^p(\mathbb{R}^2)$ into $W^{1,\tilde{p}}(\mathbb{R}^2)$, we deduce that $g_k * (q\cdot)$ is compact in $W^{1,\tilde{p}}(\mathbb{R}^2)$. Hence (3.16) is a Fredholm equation of the second kind in $W^{1,\tilde{p}}(\mathbb{R}^2)$. Thus uniqueness of a solution to (3.16) implies existence.

To prove uniqueness assume $\omega_0 \in W^{1,\tilde{p}}(\mathbb{R}^2)$ solves the homogeneous equation and let $\psi_0 = e^{izk}\omega_0$. Then ψ_0 solves (3.14), and by the reality of q, $\overline{\psi}_0$ is another solution with the same asymptotic behavior. Hence we can without loss of generality assume that ψ_0 is real. Introduce

$$v = \gamma^{1/2}\partial(\gamma^{-1/2}\psi_0)e^{-izk},$$

which is in $L^{\tilde{p}}(\mathbb{R}^2)$. Now since $\gamma^{-1/2}\psi_0$ solves the conductivity equation, a simple calculation using (3.14) and the reality of ψ_0 shows that

$$\bar{\partial}v = \frac{\partial(\gamma^{1/2})}{\gamma^{1/2}}e(z, -k)\overline{v}. \tag{3.17}$$

Theorem 3.1.3 then implies that $v = 0$, since $\gamma^{-1/2}\partial(\gamma^{1/2}) \in L^2(\mathbb{R}^2)$. This gives the equation

$$\partial(\gamma^{-1/2}\psi_0)e^{-izk} = 0,$$

which implies

$$\begin{aligned} 0 &= \overline{\partial(\gamma^{-1/2}\psi_0)e^{-izk}e(z,k)} \\ &= \overline{\partial(\gamma^{-1/2}\psi_0 e^{\overline{izk}})} \\ &= \overline{\partial}(\gamma^{-1/2}\psi_0 e^{-izk}) \end{aligned}$$

for the the function $\gamma^{-1/2}\psi_0 e^{-izk} \in L^{\tilde{p}}(\mathbb{R}^2)$. By the usual Liouville theorem we conclude that $\psi_0 = 0$ and hence $\omega_0 = 0$.

The estimate (3.15) is a direct consequence of (3.16) and (3.13). □

Note that the existence of exponentially growing solutions to the Schrödinger equation relies on the particular form of the potential q, i.e. the existence of a positive function $\gamma^{1/2}$ such that $q = \Delta\gamma^{1/2}/\gamma^{1/2}$. When this is not the case, there will be exceptional points even for small potentials, see **[Nac96, Tsa93]**.

3.2.2. From Λ_q to t. Since $e^{i\overline{kz}}$ is harmonic the identity (2.13) implies that

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\overline{kz}}(\Lambda_q - \Lambda_0)\psi(\cdot,k)d\sigma(z). \tag{3.18}$$

From Theorem 2.4.8 and Lemma 3.2.3 it follows that for any $k \in \mathbb{C} \setminus \{0\}$ the exponentially growing solution $\psi(\cdot,k)|_{\partial\Omega}$ can be constructed from Λ_q by solving the boundary integral equation (2.36). This enables the reconstruction of the scattering transform from the Dirichlet-to-Neumann map and hence we have carried out the first step in the reconstruction.

From the boundary integral equation we have for small k the following useful estimate.

Proposition 3.2.4. *The solution $\psi(z,k), z \in \partial\Omega$ to* (2.36) *satisfies for k sufficiently small the estimate*

$$\|\psi(\cdot,k) - 1\|_{H^{3/2}(\partial\Omega)} \leq C|k|. \tag{3.19}$$

Proof. In **[SMI00]** the estimate $\|\psi(\cdot,k) - 1\|_{H^{1/2}}(\partial\Omega) \leq C|k|$ was obtained by decomposing $S_\zeta = S_0 + \mathcal{H}_\zeta$, where S_0 is the usual single layer operator and $\mathcal{H}_\zeta$ is a convolution operator with a harmonic kernel, and then analyzing the dependency of k in $\mathcal{H}_\zeta$. From this result we get

$$\begin{aligned} &\|\psi(\cdot,k) - 1\|_{H^{3/2}(\partial\Omega)} \\ &\qquad \leq \|e^{izk} - 1\|_{H^{3/2}(\partial\Omega)} + \|S_\zeta(\Lambda_\gamma - \Lambda_1)(\psi(\cdot,k) - 1)\|_{H^{3/2}(\partial\Omega)} \\ &\qquad \leq C|k|, \end{aligned}$$

since $\|S_\zeta(\Lambda_\gamma - \Lambda_1)\|_{\mathcal{B}(H^{1/2}(\partial\Omega),H^{3/2}(\partial\Omega))}$ is uniformly bounded for small ζ. □

We refer to section 4.2.6 for a generalization of this result to higher dimensions, where we also give a more detailed proof. Note that this result shows that $\psi(\cdot, 0)|_{\partial\Omega}$ is well-defined.

3.2.3. From t to γ. This second step is divided into two steps. First the exponentially growing solutions ψ are computed inside Ω from $\mathbf{t}$, next the conductivity is obtained from ψ.

From $\mathbf{t}$ *to* ψ. The following lemma gives the key relation between the scattering transform and the exponentially growing solution through a differential equation in the parameter variable k. We let $\mu(z,k) = \psi(z,k)e^{-izk}$ and define $\overline{\partial}_k = \frac{\partial}{\partial \bar{k}}$.

Lemma 3.2.5. *The mapping $k \mapsto \mu(\cdot, k)$ is differentiable as a mapping from $\mathbb{C} \setminus \{0\}$ into $W^{1,\tilde{p}}_{-\beta}(\mathbb{R}^2)$, $\beta\tilde{p} > 2$, and it satisfies the equation*

$$\overline{\partial}_k \mu(z,k) = \frac{\mathbf{t}(k)}{4\pi\bar{k}} e(z,-k)\overline{\mu(z,k)}. \tag{3.20}$$

Proof. See [**Nac96**]. □

Since $W^{1,\tilde{p}}_{-\beta}(\mathbb{R}^2) \subset C_{\mathrm{loc}}(\mathbb{R}^2)$, we can consider (3.20) pointwise for $z \in \Omega$ fixed.

The equation (3.20) is the equation from which ψ is to be found. The equation is a pseudoanalytic equation, and to solve it we will need the following decay property of $\mathbf{t}$:

Proposition 3.2.6. *For any $s \in (\tilde{p}', \tilde{p})$*

$$\mathbf{t}(k)/\bar{k} \in L^s(\mathbb{R}^2).$$

Proof. A proof can be found in [**Nac96**]. □

As a consequence of Corollary 3.1.4 and Proposition 3.2.6, (3.20) has a unique solution v with $v - 1 \in L^r(\mathbb{R}^2)$ for $r > (\tilde{p}')\tilde{} = p'$. To relate this solution to μ we need a decay condition of $\omega(z,\cdot) = \mu(z,\cdot) - 1$ when $|k|$ grows.

Proposition 3.2.7. *For $\omega(z,\cdot)$ we have the estimate*

$$\sup_z \|\omega(z,\cdot)\|_{L^r(\mathbb{R}^2)} < C$$

for $r > p'$.

Proof. Note that only the large k behavior has to be considered; (3.19) deals with the small k behavior and for k in a compact set away from $k = 0$, $\omega(z,\cdot)$

is clearly uniformly bounded in k. Choose now $2/\tilde{p} < s < 1$. Then for k sufficiently large, (3.15) and the Sobolev embedding theorem implies that

$$\sup_z |\omega(z,\cdot)| \leq C|k|^{s-1}.$$

Hence for $r(1-s) > 2$ and $z \in \mathbb{R}^2$ fixed it follows that $(\mu(z,\cdot)-1) \in L^r(\mathbb{R}^2)$. The result then follows by noting that $r(1-2/\tilde{p}) = r(2-2/p) > 2$ implies $r > p/(p-1) = p'$. □

We can now conclude that for fixed $z \in \Omega$, $\mu(z,\cdot)$ is the unique solution to (3.20) with $\mu(z,\cdot)-1 \in L^r(\Omega), r > p'$.

Moreover, Proposition 3.2.7 shows that this solution must coincide with $\mu(z,\cdot)$. Hence $\mu(z,k)$ and then $\psi(z,k)$ can be obtained from $\mathbf{t}$ by solving (3.20).

From ψ to γ. From the exponentially growing solution $\psi(z,k) = e^{izk}\mu(z,k)$ for $z \in \Omega$, the potential q can be obtained by inserting ψ into (3.14) and then the conductivity γ can be obtained by solving (2.5). However, we can obtain γ directly from ψ as the following result shows:

Corollary 3.2.8. *γ can be obtained from ψ by*

$$\lim_{k\to 0} \psi(z,k) = \gamma^{1/2}(z).$$

Proof. Since $\psi(\cdot,k) - \gamma^{1/2}(\cdot) \in W^{2,p}(\Omega)$ solves the equation

$$(-\Delta + q)(\psi(z,k) - \gamma^{1/2}(z,k)) = 0 \text{ in } \Omega,$$

it can be estimated by

$$\|\psi(\cdot,k) - \gamma^{1/2}\|_{W^{2,p}(\Omega)} \leq C\|\psi(\cdot,k) - \gamma^{1/2}\|_{W^{2-1/p,p}(\partial\Omega)}.$$

Now the result follows by combining (3.19) with the embedding $H^{3/2}(\partial\Omega) \subset W^{2-1/p,p}(\partial\Omega), 1 < p < 2$, i.e.

$$\|\psi(\cdot,k) - \gamma^{1/2}\|_{W^{2-1/p,p}(\partial\Omega)} \leq C\|\psi(\cdot,k) - \gamma^{1/2}\|_{H^{3/2}(\partial\Omega)} \leq C|k|.$$

□

The reconstruction algorithm for conductivities $\gamma \in W^{2,p}(\Omega)$ is now

(1) Solve (2.36) for $\psi(\cdot,k)|_{\partial\Omega}$ for any $k \in \mathbb{C} \setminus \{0\}$.

(2) Compute $\mathbf{t}$ by (3.18).

(3) Solve the $\overline{\partial}_k$-equation for $\mu(z,\cdot)$.

(4) Reconstruct γ by the formula

$$\lim_{k\to 0} \psi(z,k) = \gamma^{1/2}(z).$$

Since each step is governed by uniqueness, this method also solves the uniqueness question for the inverse problem.

3.3. Uniqueness for $\gamma \in W^{1,p}(\Omega)$

In this section we review the uniqueness proof by Brown and Uhlmann for less regular conductivities. We will describe the basic reductions, notions, and ideas and hence give the basis for the reconstruction in the next section. The main result is the following:

Theorem 3.3.1. *Let $\Omega \subset \mathbb{R}^2$ be open, bounded and smooth, and let $\gamma \in W^{1,p}(\Omega)$, $p > 2$. Then γ is uniquely determined by Λ_γ.*

We note that the Sobolev embedding $W^{2,p}(\Omega) \subset W^{1,\tilde{p}}(\Omega),\ 1 < p < 2$, shows that the above theorem extends Theorem 3.2.1.

The idea of the proof of Theorem 3.3.1 is the following. Instead of reducing the conductivity equation to a Schrödinger equation it is reduced to a first order 2×2 matrix equation. Let $u \in H^1(\Omega)$ be a solution to the conductivity equation (1.1) and define the vector valued function (v, w) by

$$\begin{pmatrix} v \\ w \end{pmatrix} = \gamma^{1/2} \begin{pmatrix} \partial u \\ \overline{\partial} u \end{pmatrix}. \tag{3.21}$$

Since

$$\nabla \cdot \gamma \nabla u = 2\partial\gamma\overline{\partial}u + 2\overline{\partial}\gamma\partial u + 4\gamma\partial\overline{\partial}u = 0,$$

one easily checks that

$$(D - Q) \begin{pmatrix} v \\ w \end{pmatrix} = 0, \tag{3.22}$$

where the matrix operator D and the matrix potential Q are defined by

$$D = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} \tag{3.23}$$

with

$$q = -\gamma^{-1/2}\partial\gamma^{1/2}. \tag{3.24}$$

By assumption $q \in L^p(\Omega), p > 2$.

In analogy with the scattering theory for the two-dimensional Schrödinger equation described above, Beals and Coifman [**BC88**] developed a scattering theory for (3.22). For the direct scattering problem we look for a family of functions, which solve the equation (3.22) in the whole plane (with Q extended by zero outside Ω) and have a certain exponential behavior at infinity. More precisely, for each $k \in \mathbb{C}$, we seek solutions $\Psi(\cdot, k)$ of the form

$$\Psi(z,k) = m(z,k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}, \tag{3.25}$$

where m is a 2×2 matrix valued function, which approaches the identity matrix I as $|z| \to \infty$. The solution Ψ is called an exponentially growing solution.

A simple calculation shows that m solves the equation

$$(D_k - Q)m = 0, \tag{3.26}$$

where D_k is the matrix operator

$$D_k A = \begin{pmatrix} \bar{\partial} A_{11} & (\bar{\partial} - ik)A_{12} \\ (\partial + ik)A_{21} & \partial A_{22} \end{pmatrix}$$

and Q denotes multiplication by the matrix potential Q.

Given our assumptions on Q we have the following result from [**BU97**]. We will give the proof since it shows nicely the use of complex analysis and since the intermediate calculations are useful for the numerical considerations later.

Lemma 3.3.2. *Let $Q \in L^p(\mathbb{R}^2; \mathcal{M}_2), p > 2$, be an off-diagonal, hermitian, compactly supported matrix. Then for any $k \in \mathbb{C}$ there is a unique solution $m(\cdot, k)$ with $m(\cdot, k) - I \in L^r(\mathbb{R}^2; \mathcal{M}_2)$ for any any $r > 2$. Moreover, the map $Q \mapsto m - I$ is continuous with respect to the norm topologies on $L^p(\mathbb{R}^2; \mathcal{M}_2)$ and $L^r(\mathbb{R}^2; \mathcal{M}_2)$.*

Furthermore, for any $r > 2p/(p-2)$ we have that

$$\sup_{z \in \mathbb{C}} \|m(z, \cdot) - I\|_{L^r(\mathbb{R}^2; \mathcal{M}_2)} < C \tag{3.27}$$

with C depending on p, r, Q.

Proof. The equation for m is

$$D_k m = \begin{pmatrix} \bar{\partial} m_{11} & (\bar{\partial} - ik)m_{12} \\ (\partial + ik)m_{21} & \partial m_{22} \end{pmatrix} = \begin{pmatrix} q m_{21} & q m_{22} \\ \bar{q} m_{11} & \bar{q} m_{12} \end{pmatrix}.$$

Now to find the first column of m, $(m_{11}, m_{21})^T$, note that

$$m_\pm = m_{11} \pm \overline{m}_{21} e(z, -k) \tag{3.28}$$

solves (cf. (3.17))

$$\bar{\partial} m_\pm = \pm q e(z, -k) \overline{m}_\pm. \tag{3.29}$$

Since $q \in L^p(\mathbb{R}^2)$ is compactly supported $q \in L^s(\mathbb{R}^2)$, $1 \leq s \leq p$ and hence Corollary 3.1.4 gives the existence of a unique $m_\pm$ with $m_\pm - 1 \in L^r(\mathbb{R}^2)$

for any $r > 2$. The second column of m is now found from (3.30). That m depends continuously on Q is a consequence of Lemma 3.1.5.

The estimate (3.27) is due to Li-Yeng Sung; a proof can be found in [**BU97**]. We note that this fact relies on $q \in L^p(\Omega), p > 2$. $\square$

Note that unlike the result of Lemma 3.2.3, the proof of existence of exponentially growing solutions to (3.22) is also valid when Q is not defined from a conductivity. Note further that (3.3) implies that $m \in C^\alpha(\mathbb{R}^2; \mathcal{M}_2)$, $\alpha < 1 - 1/p$.

Due to the symmetry in D_k and Q, the uniqueness in Lemma 3.3.2 implies that

$$m_{11}(z,k) = \overline{m_{22}(z,\overline{k})}, \qquad m_{21}(z,k) = \overline{m_{12}(z,\overline{k})}, \tag{3.30}$$

so it suffices to consider only one column of m. We will occasionally do so.

Define now in analogy with (3.12) the scattering transform of the potential Q by

$$S(k) = \frac{i}{\pi} \int_{\mathbb{C}} \begin{pmatrix} 0 & e^{-iz\overline{k}} q(z) \Psi_{22}(z,k) \\ -e^{i\overline{zk}} \overline{q}(z) \Psi_{11}(z,k) & 0 \end{pmatrix} d\mu(z). \tag{3.31}$$

S relates for large k to the Fourier transform of q

$$S(k) \sim i2 \begin{pmatrix} 0 & \hat{q}(2k_1, 2k_2) \\ -\hat{\overline{q}}(-2k_1, k_2) & 0 \end{pmatrix} \qquad \text{as } k \to \infty. \tag{3.32}$$

Since Ψ solves (3.22), Q is supported in Ω, and $\partial e^{i\overline{zk}} = \overline{\partial} e^{-iz\overline{k}} = 0$, integration by parts in (3.31) shows that

$$S(k) = \frac{i}{2\pi} \int_{\partial\Omega} \begin{pmatrix} 0 & e^{-iz\overline{k}} \nu(z) \Psi_{12}(z,k) \\ -e^{i\overline{zk}} \overline{\nu}(z) \Psi_{21}(z,k) & 0 \end{pmatrix} d\sigma(z). \tag{3.33}$$

In (3.33) ν is the complex normal at the boundary, i.e. if $(\nu_1(z), \nu_2(z))$ denotes the outer unit normal at $z \in \partial\Omega$, then $\nu(z) = \nu_1(z) + i\nu_2(z)$ and $\overline{\nu} = \nu_1(z) - i\nu_2(z)$.

The scattering transform S is the key intermediate object in the solution of the inverse problem. Indeed, the uniqueness proof can decomposed into three steps. The first step is to see that Λ_γ determines S, the second step is to determine Q from S, and the third step is to determine γ from Q.

The first step is treated by the following lemma. Let $\gamma_i \in W^{1,p}(\Omega), i = 1, 2$, define Q_i by (3.23), and let $\Psi^{(i)}$ be the exponentially growing solution to (3.22). Then

Lemma 3.3.3. *If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then $\Psi^{(1)}(\cdot, k) = \Psi^{(2)}(\cdot, k)$ for any in $\mathbb{R}^2 \setminus \Omega, k \in \mathbb{C}$.*

Proof. See [**BU97**]. □

This lemma says by (3.33) in particular that Λ_γ determines S.

The next step in the uniqueness proof is to prove injectivity of the nonlinear map $Q \mapsto S$. Consider $m(z,\cdot)$ as a function of the dual variable $k \in \mathbb{C}$. Then we have the following useful relation between m and S.

Lemma 3.3.4. *The scattering transform $S \in L^2(\mathbb{R}^2;\mathcal{M}_2)$. Furthermore, for fixed $z \in \mathbb{R}^2$, the map $k \mapsto m(z,k)$ is a differentiable map from $\mathbb{C}$ into $L^r_{-\beta}(\mathbb{R}^2;\mathcal{M}_2)$, and we have the equation*

$$\overline{\partial}_k m(z,k) = m(z,\overline{k})\Lambda_k(z)S(k), \tag{3.34}$$

where

$$\Lambda_k(z) = \begin{pmatrix} e(z,\overline{k}) & 0 \\ 0 & e(z,-k) \end{pmatrix}.$$

Proof. This is a combination of results from [**BC88**] and [**BU97**]. □

The equation (3.34) is not a pseudoanalytic equation in the sense of (3.6), but it can be rewritten in a suitable way.

Lemma 3.3.5. *Let $z \in \mathbb{C}$ be fixed and let*

$$m^{\pm}(z,k) = m_{11}(z,k) \pm \overline{m_{12}(z,\overline{k})}. \tag{3.35}$$

Then we have the equation

$$\overline{\partial}_k m^{\pm}(z,k) = \pm S_{21}(k)e(z,-k)\overline{m^{\pm}(z,k)}. \tag{3.36}$$

Proof. This is a consequence of (3.30) and (3.34). □

Now since $S \in L^2(\mathbb{R}^2;\mathcal{M}_2)$, Liouville's theorem for pseudoanalytic functions states that (3.36) has at most one solution subject to the condition $m^{\pm}(z,\cdot) - 1 \in L^r(\mathbb{R}^2)$ (cf. (3.27)). Hence $m(z,k), z, k \in \mathbb{C}$ is uniquely determined by S. Now to obtain Q from m we can either use the equation (3.22) directly or use the formula

$$Q(z) = \lim_{k_0\to\infty} \frac{1}{\pi} \int_{\{k:|k-k_0|<1\}} D_k m(z,k)\,d\mu(k) \tag{3.37}$$

from [**BU97**]. This shows that S determines Q.

The last step is to determine γ from Q. Once again we invoke the generalized Liouville theorem to conclude that $\gamma^{1/2}$ is the unique solution to (3.24) with $\gamma^{1/2} \sim 1$. This ends our sketch of the uniqueness proof for conductivities $\gamma \in W^{1,p}(\Omega),\ p > 2$.

3.4. Reconstruction of $\gamma \in C^{1+\epsilon}(\overline{\Omega})$

In this section we will turn the uniqueness proof in section 3.3 into a reconstruction algorithm. The main result is

Theorem 3.4.1. *Let Ω be open, bounded and smooth, and let $\gamma \in C^{1+\epsilon}(\overline{\Omega})$ for some $\epsilon > 0$. Then γ can be reconstructed uniquely from Λ_γ.*

We note that compared to the Theorem 3.3.1 we assume in Theorem 3.4.1 slightly more regularity. We need this extra smoothness only when solving the pseudoanalytic equation (3.36) constructively. We will return to this later.

In the proof of Theorem 3.4.1 we follow closely the method described in the previous section, and we will see how each step can be made constructive. In particular we give a method for reconstruction of the exponentially growing solutions on the boundary of Ω. This gives a constructive way of obtaining the scattering transform $S(k)$ from Λ_γ. Next we will consider the solvability of the $\overline{\partial}_k$-equation (3.36). To be precise, we consider the $\overline{\partial}_k$-equation for the function $\tilde{m}$ given by Lemma 3.3.2 as the unique solution to $D_k\tilde{m} = -Q^T\tilde{m}$ with $\tilde{m} \sim I$. It turns out that $\tilde{m}^+$ (defined in (3.35)) satisfies the pseudoanalytic equation

$$\overline{\partial}_k \tilde{m}^+(z,k) = \overline{S_{21}(-k)}e(z,-k)\overline{\tilde{m}^+(z,k)} \tag{3.38}$$

with the asymptotic condition $\tilde{m}^+ \sim 1$. From this equation we can reconstruct $\tilde{m}^+$ from S. Moreover, we will see that the conductivity γ can be reconstructed from $\tilde{m}^+$ by

$$\gamma^{1/2}(z) = \mathrm{Re}(\tilde{m}^+(z,0)). \tag{3.39}$$

The outline of the reconstruction method is then:

(1) Determine Ψ on $\partial\Omega$ (see Theorem 3.4.4 below).
(2) Define the scattering transform S by (3.33).
(3) Solve (3.38) for $\tilde{m}^+(z,\cdot), z \in \Omega$.
(4) Recover γ from (3.39).

3.4.1. A boundary relation. In this section we give an explicit characterization of the Cauchy data for the first order system (3.22), in case the potential Q comes from a conductivity. This characterization will enable us to construct Ψ on $\partial\Omega$.

To fix notation let $s : [0, |\partial\Omega|] \to \partial\Omega$ be an arclength parameterization of $\partial\Omega$ and let for any $f \in C^1(\partial\Omega)$, $\partial_\tau f = d/dt(f(s(t)))$ be the derivative along the boundary. Let further $C_0(\partial\Omega) = \{f \in C(\partial\Omega) \ : \ \int_{\partial\Omega} f d\sigma = 0\}$, and define on $C_0(\partial\Omega)$

$$(\partial_\tau^{-1} f)(s(t)) = \int_0^t f(s(t'))dt'.$$

Note that $(\partial_\tau^{-1}\partial_\tau f)(s(t)) = f(s(t)) - f(s(0))$ for $f \in C^1(\partial\Omega)$.

When $\gamma \in C^{1+\epsilon}(\overline{\Omega})$ it is natural to measure the smoothness of solutions to the conductivity equation in Hölder spaces. Assume $u \in C^{2+\epsilon}(\overline{\Omega})$ solves (1.1) for some $f \in C^{2+\epsilon}(\partial\Omega)$. Then $(v,w) = \gamma^{1/2}(\partial u, \overline{\partial} u)$ solves (3.22). Decomposing ∂ and $\overline{\partial}$ at the boundary in the normal direction $\nu = (\nu_1, \nu_2)$ and the tangential direction $\tau = (-\nu_2, \nu_1)$ gives

$$\begin{aligned} 2v = 2\partial u = (1,-i)\cdot\nabla u &= ((\nu_1 - i\nu_2)\nu + (-\nu_2 - i\nu_1)\tau)\cdot\nabla u \\ &= (v_1 - iv_2)\Lambda_\gamma - (\nu_1 - i\nu_2)i\partial_\tau, \\ 2w = 2\overline{\partial} u &= (\nu_1 + i\nu_2)\Lambda_\gamma + (\nu_1 + i\nu_2)i\partial_\tau. \end{aligned}$$

Equivalently

$$\begin{pmatrix} v \\ w \end{pmatrix}\Bigg|_{\partial\Omega} = \frac{1}{2}\begin{pmatrix} \overline{\nu} & -i\overline{\nu} \\ \nu & i\nu \end{pmatrix}\begin{pmatrix} \Lambda_\gamma f \\ \partial_\tau f \end{pmatrix}, \text{ on } \partial\Omega. \tag{3.40}$$

Inverting (3.40) gives

$$\begin{pmatrix} \Lambda_\gamma f \\ \partial_\tau f \end{pmatrix} = \begin{pmatrix} \nu & \overline{\nu} \\ i\nu & -i\overline{\nu} \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}\Bigg|_{\partial\Omega}. \tag{3.41}$$

Now since $f \in C^{2+\epsilon}(\partial\Omega)$, we find by the fundamental theorem of calculus that

$$0 = \int_{\partial\Omega} \partial_\tau f d\sigma = i\int_{\partial\Omega} (\nu v - \overline{\nu} w)d\sigma.$$

Hence $(\nu v - \overline{\nu} w)|_{\partial\Omega} \in C_0^{1+\epsilon}(\partial\Omega)$ and from (3.41) we arrive at the relation

$$iH_\gamma(\nu v|_{\partial\Omega} - \overline{\nu} w|_{\partial\Omega}) = (\nu v|_{\partial\Omega} + \overline{\nu} w|_{\partial\Omega}),$$

where the operator $H_\gamma = \Lambda_\gamma \partial_\tau^{-1}$ is defined on $C_0^{1+\epsilon}(\partial\Omega)$. This relation motivates the definition of the space

$$\begin{aligned} \mathcal{BR} = \{(h_1,h_2) \in C^{1+\epsilon}(\partial\Omega) \times C^{1+\epsilon}(\partial\Omega) : (\nu h_1 - \overline{\nu} h_2) \in C_0^{1+\epsilon}(\partial\Omega) \\ iH_\gamma(\nu h_1 - \overline{\nu} h_2) = \nu h_1 + \overline{\nu} h_2\}. \end{aligned} \tag{3.42}$$

A pair of functions $(h_1,h_2) \in \mathcal{BR}$ is said to satisfy the boundary relation.

For $Q \in C^\epsilon(\mathbb{R}^2)$ it follows as a consequence of Proposition 3.1.1 that the natural Cauchy data for the system (3.22) is the set

$$\mathcal{C}_Q = \{(v|_{\partial\Omega}, w|_{\partial\Omega}) : (v,w) \in C^{1+\epsilon}(\overline{\Omega}) \times C^{1+\epsilon}(\overline{\Omega}),\ (D-Q)(v,w)^T = 0\}.$$

The boundary value of any solution to (3.22) is in $\mathcal{C}_Q$, and we just saw that if the solution is defined through (3.21), then on the boundary the boundary relation is satisfied. However, as the following theorem shows, (3.42) is in fact a complete characterization of $\mathcal{C}_Q$:

Lemma 3.4.2. *If $Q \in C^\epsilon(\overline{\Omega})$ is given by* (3.24)*, then $\mathcal{C}_Q = \mathcal{BR}$*

Proof. First we show that $\mathcal{BR} \subset \mathcal{C}_Q$. Let $(h_1, h_2) \in \mathcal{BR}$ and let $u \in C^{2+\epsilon}(\overline{\Omega})$ be the unique solution to the Dirichlet problem

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0, & \text{in } \Omega, \\ u = i\partial_\tau^{-1}(\nu h_1 - \overline{\nu} h_2), & \text{on } \partial\Omega. \end{cases}$$

Define a solution (v, w) to (3.22) by the relation (3.21) with u from above. Then $(v, w)|_{\partial\Omega} = (h_1, h_2)$ by (3.40) and (3.42), i.e. $(h_1, h_2) \in \mathcal{C}_Q$.

Next we see $\mathcal{C}_Q \subset \mathcal{BR}$: let $(h_1, h_2) \in \mathcal{C}_Q$ and let $(v, w) \in C^{1+\epsilon}(\overline{\Omega}) \times C^{1+\epsilon}(\overline{\Omega})$ be a solution to (3.22) with Cauchy data (h_1, h_2). Since Q is of the form (3.24), we have the compatibility relation

$$\overline{\partial}(\gamma^{-1/2} v) = \partial(\gamma^{-1/2} w),$$

which ensures the existence of a $u \in C^{2+\epsilon}(\overline{\Omega})$ such that

$$\gamma^{-1/2} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \partial u \\ \overline{\partial} u \end{pmatrix}.$$

It is easy to check that u is a solution of the conductivity equation in the form $2\partial u \overline{\partial} \gamma + 2\overline{\partial} u \partial \gamma + 4\gamma \partial \overline{\partial} u = 0$. Now relation (3.40) with $f = u|_{\partial\Omega}$ shows that $(h_1, h_2) \in \mathcal{BR}$. □

3.4.2. Construction of $\Psi(\cdot, k)|\partial\Omega$. In this section we show how to reconstruct the trace on $\partial\Omega$ of the exponentially growing solutions Ψ defined by (3.25). Due to (3.30) it suffices to consider only the first column of Ψ. Note that Ψ_{11} is analytic outside Ω while Ψ_{21} is anti-analytic outside Ω. Moreover, they have a prescribed behavior at infinity and their traces on $\partial\Omega$ satisfy the boundary relation (3.42). We will prove that these relations characterize the trace of (Ψ_{11}, Ψ_{21}) uniquely.

Introduce

$$\mathfrak{g}_k(z) = \frac{1}{\pi} \frac{e^{-ikz}}{z},$$

a Green's kernel for $\overline{\partial}$ which takes into account exponential growth at infinity. Define the single layer potentials $\mathcal{S}_k$ and $\overline{\mathcal{S}_k}$ as boundary integral operators by

$$\mathcal{S}_k f(z) = p.v. \int_{\partial\Omega} f(\zeta) \mathfrak{g}_k(\zeta - z) d\zeta, \quad \overline{\mathcal{S}_k} f(z) = p.v. \int_{\partial\Omega} f(\zeta) \overline{\mathfrak{g}_k(\zeta - z)} d\overline{\zeta}. \tag{3.43}$$

Since the integral kernels have the same singularity as the one treated in Proposition 3.1.6, $\mathcal{S}_k$, $\overline{\mathcal{S}_k} \in \mathcal{B}(C^{m+\alpha}(\partial\Omega))$ for $0 < \alpha < 1$ and $m \in \mathbb{Z}_+$.

The following result gives an equation on the boundary $\partial\Omega$ for the exponentially growing solutions. This equation encodes in some sense the exponential growth.

Lemma 3.4.3. *Let $r > 2$ and assume that the functions v, w satisfy $(e^{-izk}v - 1)$, $e^{-izk}w \in L^r(\mathbb{R}^2)$, and that $\overline{\partial}v, \partial w \in C^\alpha(\mathbb{R}^2)$, $0 < \alpha < 1$, are compactly supported in Ω. Then on $\partial\Omega$*

$$\begin{aligned} (I - i\mathcal{S}_k)v &= 2e^{izk}, \\ (I + i\overline{\mathcal{S}_k})w &= 0. \end{aligned} \tag{3.44}$$

Proof. Let $z \in \mathbb{C} \setminus \overline{\Omega}$. Then since $\overline{\partial}v \in C^\alpha(\mathbb{R}^2)$ is compactly supported in Ω, Proposition 3.1.1 implies $v \in C^{1+\alpha}_{\text{loc}}(\mathbb{R}^2)$. Moreover, $v(z)e^{-izk} - 1 \in L^r(\mathbb{R}^2)$ and (3.2) shows that

$$\begin{aligned} v(z)e^{-izk} - 1 &= \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\overline{\partial}(v(\zeta)e^{-i\zeta k} - 1)}{z - \zeta} d\mu(\zeta) \\ &= \frac{1}{\pi} \int_{\Omega} \frac{\overline{\partial}(v(\zeta)e^{-i\zeta k})}{z - \zeta} d\mu(\zeta) \\ &= \frac{1}{2\pi i} \int_{\partial\Omega} \frac{v(\zeta)e^{-i\zeta k}}{z - \zeta} d\zeta \end{aligned}$$

where the last equality follows by integration by parts. This is equivalent to

$$v(z) + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{v(\zeta)e^{-i(\zeta - z)k}}{\zeta - z} d\zeta = e^{izk}, \tag{3.45}$$

and the equation for v now follows by letting $z \in \mathbb{C} \setminus \overline{\Omega}$ approach the boundary and then applying (3.11).

To prove the result for w, we note that $we^{-izk}e(z,k) = we^{\overline{izk}} \in L^r(\mathbb{R}^2)$. As above we then get

$$w(z) - \frac{1}{2\pi i} \int_{\partial\Omega} \frac{w(\zeta)\overline{e^{-i(\zeta - z)k}}}{\overline{\zeta} - \overline{z}} d\overline{\zeta} = 0, \quad z \in \mathbb{C} \setminus \overline{\Omega}, \tag{3.46}$$

and again the result follows by invoking (3.11). $\square$

The next result shows that the conditions (3.42) and (3.44) are a complete characterization of the traces of the exponentially growing solutions.

Theorem 3.4.4. *The system of equations on $\partial\Omega$*

$$\begin{pmatrix} (I - i\mathcal{S}_k) & 0 \\ 0 & (I + i\overline{\mathcal{S}_k}) \\ (iH_\gamma - 1)\nu & -(iH_\gamma + 1)\overline{\nu} \end{pmatrix} \Phi = \begin{pmatrix} 2e^{izk} \\ 0 \\ 0 \end{pmatrix}, \tag{3.47}$$

has in $\{\Phi \in C^{1+\epsilon}(\partial\Omega) \times C^{1+\epsilon}(\partial\Omega) : (\nu\Phi_1 - \overline{\nu}\Phi_2) \in C_0^{1+\epsilon}(\partial\Omega)\}$ the unique solution $\Phi = (\Psi_{11}, \Psi_{21})|_{\partial\Omega}$.

Proof. Since $(\Psi_{11}, \Psi_{21})|_{\partial\Omega} \in \mathcal{C}_Q$ and satisfies the assumptions of Lemma 3.4.3, it is a solution to (3.47).

Let $h \in \{(h_1, h_2) \in C^{1+\epsilon}(\partial\Omega) \times C^{1+\epsilon}(\partial\Omega) : (\nu h_1 - \overline{\nu} h_2) \in C_0^{1+\epsilon}(\partial\Omega)\}$ be another solution to (3.47). Then

$$(iH_\gamma - 1)\nu h_1 - (iH_\gamma + 1)\overline{\nu} h_2 = 0$$

shows that $h \in \mathcal{BR}$. Then by Lemma 3.4.2 we may extend h inside Ω to a solution (v, w) to the system (3.22), and guided by (3.45) and (3.46) we extend h in $\mathbb{C} \setminus \overline{\Omega}$ to (v, w) by

$$\begin{aligned} v(z) &= -\frac{1}{2i}\int_{\partial\Omega} h_1(\zeta)\mathfrak{g}_k(\zeta - z)d\zeta + e^{izk}, \\ w(z) &= \frac{1}{2i}\int_{\partial\Omega} h_2(\zeta)\overline{\mathfrak{g}_k(\zeta - z)}d\overline{\zeta}. \end{aligned}$$

We will prove that defined this way, (v, w) is a solution to (3.22) and that $ve^{-izk} - 1, we^{-izk} \in L^r(\mathbb{R}^2),\ r > 2$. Then the result follows from the uniqueness in Lemma 3.3.2.

By construction (v, w) solves (3.22) in Ω and in $\mathbb{C} \setminus \overline{\Omega}$, so we have to show that the equation holds as we cross $\partial\Omega$. Since the support of Q is strictly inside Ω, v is analytic and w is anti-analytic in a neighborhood of the boundary. Hence for the equation to hold it suffices to have continuity across the boundary and then invoke Morera's theorem (see for instance [**Ahl53**, p. 98]). We shall show the continuity of v, for w similar reasoning works. Let z approach some point $z_0 \in \partial\Omega$ from outside. Then using (3.11) we get

$$\lim_{z \to z_0^+} v(z) = -\left(-\frac{h_1(z_0)}{2} + \frac{1}{2i}\mathcal{S}_k h_1(z_0)\right) + e^{iz_0 k}.$$

Now use (3.47) to conclude $\lim_{z \to z_0^+} v(z) = h_1(z_0)$. The continuity of v from inside follows by the construction.

That $ve^{-izk} - 1 \in L^r(\mathbb{R}^2)$ follows since

$$v(z)e^{-izk} - 1 = -\frac{1}{2\pi i}\int_{\partial\Omega} \frac{h_1(\zeta)e^{-ik\zeta}}{\zeta - z}d\zeta = O\left(\frac{1}{|z|}\right), \text{ as } |z| \to \infty.$$

This proves the theorem. ☐

This characterization gives a method for finding (Ψ_{11}, Ψ_{21}) on $\partial\Omega$ and hence by (3.33) a method for finding the scattering transform.

3.4.3. From $S(k)$ to γ. In this section we will see how γ can be reconstructed from $S(k)$. More precisely we will solve the equation (3.38) for $\tilde{m}$, and then finally find γ by the formula (3.39).

First the pseudoanalytic equation (3.38) for $\tilde{m}^+$ will be derived.

Lemma 3.4.5. *Let for some $p > 2$, $Q \in L^p(\mathbb{R}^2; \mathcal{M}_2)$ be an off-diagonal, hermitian, compactly supported matrix and let $\tilde{Q} = -Q^T$. Denote by S and $\tilde{S}$ the associated scattering transforms. Then*

$$\tilde{S}(k) = \overline{S(-k)}. \tag{3.48}$$

Proof. In [**BC88**] it was noted that

$$\tilde{S}(k) = S(-\overline{k})^T;$$

a proof can be found in [**BBR01**]. Furthermore, by (3.30) it follows that

$$\begin{aligned} S_{12}(k) &= \frac{i}{\pi} \int q(z) e(z, -\overline{k}) m_{22}(z, k) d\mu(z) \\ &= \overline{\frac{-i}{\pi} \int \overline{q}(z) e(z, \overline{k}) m_{11}(z, \overline{k}) d\mu(z)} \\ &= \overline{S_{21}(\overline{k})}, \end{aligned}$$

which shows the claim. □

The pseudoanalytic equation for $\tilde{m}$ is (3.36) with $\tilde{S}_{21}(k) = \overline{S_{21}(-k)}$ instead of S_{21}. This substitution shows that (3.38) holds.

When $\gamma \in C^{1+\epsilon}$ the special solution m and the scattering transform S are slightly more regular than what is stated in Lemma 3.3.2. This extra regularity is what we need for (3.38) to be uniquely solvable.

Lemma 3.4.6. *Assume $Q \in C^{\epsilon}(\mathbb{R}^2; \mathcal{M}_2)$, $0 < \epsilon < 1$, is off-diagonal, hermitian and compactly supported. Then the scattering transform of $S \in L^s(\mathbb{R}^2; \mathcal{M}_2)$ for any $s > 4/(2+\epsilon)$. Furthermore, the equation (3.38) has the unique solution $\tilde{m}^+(z, \cdot) - 1$ in $L^r(\mathbb{R}^2) \cap C^{\alpha}(\mathbb{R}^2)$ for $r > 4/\epsilon$ and $\alpha < (1+\epsilon)/2$.*

Proof. The $L^s(\mathbb{R}^2)$ property of S can be found in [**BU96, BBR01**]. Then the unique solvability of (3.38) in $L^r(\mathbb{R}^2)$ follows from Corollary 3.1.4, since $S_{12} \in L^s(\mathbb{R}^2)$, $s > 4/(2+\epsilon), \epsilon > 0$, and $\tilde{m}^+(z, \cdot) - 1 \in L^r(\mathbb{R}^2)$ for any $r > 2$ by (3.27). The Hölder continuity of $\tilde{m}^+$ follows from (3.3). □

This lemma is where the assumption $\gamma \in C^{1+\epsilon}(\mathbb{R}^2)$ is used. If q is only in $L^p(\Omega)$ for some $p > 2$, then a result from [**BU97**] shows that $S \in L^2(\mathbb{R}^2; \mathcal{M}_2)$. This does imply uniqueness of a solution to (3.38) (cf. Theorem

3.1.3), but seems insufficient for solvability of the equation in a suitable Lebesgue space.

By the previous lemma we find $\tilde{m}^+(z,k)$ for fixed $z,k \in \mathbb{C}$. To reconstruct γ we use the formula [**BBR01**, Proposition 4.2]

$$\gamma^{1/2}(z) = \tilde{m}_{11}(z,0) + \overline{\tilde{m}_{21}(z,0)},$$

which follows from Corollary 3.1.4 since

$$\begin{aligned}\overline{\partial}(\tilde{m}_+(z,0) - \gamma^{1/2}(z)) &= -\overline{q}\overline{(\tilde{m}_+(z,0) - \gamma^{1/2}(z))},\\ (\tilde{m}_+(z,0) - \gamma^{1/2}(z)) &\in L^r(\mathbb{R}^2).\end{aligned}$$

Since γ is real, we rewrite the formula using (3.30) and (3.35)

$$\begin{aligned}\gamma^{1/2}(z) &= \tilde{m}_{11}(z,0) + \overline{\tilde{m}_{21}(z,0)}\\ &= \operatorname{Re}\tilde{m}_{11}(z,0) + \operatorname{Re}\overline{\tilde{m}_{21}(z,0)}\\ &= \operatorname{Re}\tilde{m}_{11}(z,0) + \operatorname{Re}\tilde{m}_{12}(z,0)\\ &= \operatorname{Re}\tilde{m}^+(z,0).\end{aligned}$$

This ends the reconstruction of conductivities $\gamma \in C^{1+\epsilon}(\overline{\Omega})$.

3.5. Comparison of the two reconstruction algorithms

We have now two algorithms available for the reconstruction of a conductivity $\gamma \in C^{1+\epsilon}(\overline{\Omega}) \cap W^{2,p}(\Omega)$, $\epsilon > 0, p > 1$. Both methods are based on the existence of exponentially growing solutions, the construction of a scattering transform from the boundary data, and the solution of a $\overline{\partial}_k$-equation. Moreover, the conductivity is in both cases obtained as a low frequency limit. In this section we will investigate the similarities of the two methods further. We will see that the method presented in section 3.4 in some sense can be seen as a direct generalization of the method presented in section 3.2.

As noted in the proof of Lemma 3.4.2 it is possible to go back and forth between solutions to the conductivity equation and solutions to (3.22). Using this idea Barceló, Barceló and Ruiz proved the existence of exponentially growing solutions to the conductivity equation [**BBR01**]. Moreover, they showed that the scattering transform S can be found using these exponentially growing solutions. Their result is

Lemma 3.5.1. *Let $\gamma \in W^{1,p}(\Omega), p > 2$. Then for any $k \in \mathbb{C} \setminus \{0\}$ there is a unique solution*

$$u(z,k) = e^{izk}(\frac{1}{ik} + \tilde{\omega}(z,k)) \tag{3.49}$$

to $\nabla \cdot \gamma \nabla u = 0$ *in* $\mathbb{R}^2$ *such that*

$$\tilde{\omega}(\cdot, k) \in W^{1,r}(\mathbb{R}^2), \quad 2 < r < \infty.$$

The norm is estimated by

$$\begin{aligned} &\|\tilde{\omega}(\cdot, k)\|_{W^{1,r}(\mathbb{R}^2)} \\ &\leq C(1 + \frac{1}{k})(\|\gamma^{-1/2} m_{11} - 1\|_{L^r(\mathbb{R}^2)} + \|m_{11}\overline{\partial}\gamma^{-1/2} + \gamma^{-1/2} q m_{21}\|_{L^{\tilde{r}}(\mathbb{R}^2)}), \end{aligned} \tag{3.50}$$

where C *is independent of* k.

Assume further that $\gamma \in C^{1+\epsilon}(\overline{\Omega})$ *is such that* $\gamma = 1$ *near* $\partial\Omega$. *Then*

$$\frac{1}{\overline{k}}(S_{21}^{(1)}(k) - S_{21}^{(2)}(k)) = \frac{1}{2}\int_{\partial\Omega} \overline{u_1(z, -k)}(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})u_2(\cdot, k)d\sigma(z), \tag{3.51}$$

where $S_{21}^{(i)}$ *and* u_i *are the scattering transform and special solution corresponding to* γ_i, $i = 1, 2$.

Proof. We will review the construction of the solutions. For a proof of (3.51) we refer to [**BBR01**, Proposition 3.1].

Assume there is a solution u to the conductivity equation of the form (3.49) and let $(v, w) = \gamma^{1/2}(\partial u, \overline{\partial} u)$ be the corresponding solutions to (3.22). Then

$$e^{-izk} v - 1 = (\gamma^{1/2} - 1) + \gamma^{1/2}(\partial + ik)\tilde{\omega} \in L^r(\mathbb{R}^2),$$

and by the uniqueness in Theorem 3.3.1 we see that $e^{-izk} v = m_{11}$. This calculation motivates the definition

$$\tilde{\omega} = (\partial + ik)^{-1}(\gamma^{-1/2} m_{11} - 1).$$

The $W^{1,r}(\mathbb{R}^2)$ property and (3.50) now follow by Proposition 3.1.2, since

$$\overline{\partial}(\gamma^{-1/2} m_{11} - 1) = \overline{\partial}(\gamma^{-1/2}) m_{11} + \gamma^{-1/2} q m_{21} \in L^r(\mathbb{R}^2)$$

for any $1 \leq r \leq 2$. ☐

Specializing (3.51) to the case $\gamma_1 = 1$ in Ω we have that

$$S_{21}(k) = \frac{-i}{2}\int_{\partial\Omega} e^{i\overline{zk}}(\Lambda_\gamma - \Lambda_1)u(\cdot, k)d\sigma(z), \tag{3.52}$$

since for this particular conductivity $u_1(z, k) = \frac{1}{ik}e^{izk}$ and $S^{(1)} = 0$. In this formula $u(z, k)$ is the special solution of Lemma 3.5.1 corresponding to the conductivity γ. This formula for the scattering transform S_{21} is similar to the formula (3.18) for $\mathbf{t}$, and in fact it enables us to relate the two quantities in case γ is sufficiently smooth. We denote as usual by ψ the exponentially growing solution to the Schrödinger equation.

Theorem 3.5.2. *Let $\gamma \in C^{1+\epsilon}(\overline{\Omega}) \cap W^{2,p}(\Omega)$ for $p > 1,\ \epsilon > 0$, and assume $\gamma = 1$ near $\partial\Omega$. Then*

$$iku(z,k) = \gamma^{-1/2}(z)\psi(z,k)$$

and the scattering transforms satisfy

$$\mathbf{t}(k) = -2kS_{21}(k). \tag{3.53}$$

Proof. Since the function

$$\begin{aligned} iku(z,k) &= e^{izk}(1 + ik\tilde{\omega}(z,k)) \\ &= e^{izk}\gamma^{-1/2}(\gamma^{1/2} + \gamma^{1/2}ik\tilde{\omega}(z,k)) \\ &= e^{izk}\gamma^{-1/2}(1 + (\gamma^{1/2} - 1) + \gamma^{1/2}ik\tilde{\omega}(z,k)) \end{aligned}$$

is a solution to the conductivity equation with $(\gamma^{1/2} - 1) + \gamma^{1/2}ik\tilde{\omega}(\cdot,k) \in W^{1,r}(\Omega)$, and

$$\gamma^{-1/2}(z)\psi(z,k) = e^{izk}\gamma^{-1/2}(1 + \omega(z,k)),$$

is another solution to the conductivity equation with $\omega(\cdot,k) \in W^{1,\tilde{p}}(\mathbb{R}^2)$, by the uniqueness of exponentially growing solutions (Lemma 3.2.3 and Lemma 3.5.1), we see that they must agree.

Furthermore, (3.18) and (3.52) implies by the assumption that $\gamma = 1$ on $\partial\Omega$ that

$$\begin{aligned} S_{21}(k) &= \frac{-i}{2}\int_{\partial\Omega} e^{i\overline{zk}}(\Lambda_\gamma - \Lambda_1)u(\cdot,k)d\sigma(z) \\ &= \frac{-1}{2k}\int_{\partial\Omega} e^{i\overline{zk}}(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k)d\sigma(z) \\ &= \frac{-1}{2k}\mathbf{t}(k). \end{aligned}$$

□

Equation (3.53) relates the coefficients in the pseudoanalytic equations (3.20) and (3.38) for $\gamma \in C^{1+\epsilon}(\overline{\Omega}) \cap W^{2,p}(\Omega),\ p > 1$. This shows that the $\overline{\partial}_k$-equations solved as the third step in the two reconstruction methods are essentially identical.

An interesting question is now whether the exponentially growing solutions u from Lemma 3.5.1 satisfy a boundary integral equation similar to (2.36). An affirmative answer is given in the following result.

Theorem 3.5.3. *Let $\gamma \in W^{1,p}(\Omega)$ for a $p > 2$ and assume that $\gamma = 1$ near $\partial\Omega$. Then for any $k \in \mathbb{C} \setminus \{0\}$ the trace of the exponentially growing solution $u(\cdot,k)|_{\partial\Omega}$*

is the unique solution to

$$u(z,k) = \frac{1}{ik}e^{izk} - S_k(\Lambda_\gamma - \Lambda_1)u(\cdot,k), \qquad z \in \partial\Omega. \tag{3.54}$$

Moreover, the operator $(I + S_k(\Lambda_\gamma - \Lambda_1))$ *is invertible in* $H^{1/2}(\partial\Omega)$.

Proof. Let $\{\gamma_n\}_{n\in\mathbb{N}} \subset W^{2,r}(\Omega), 1/r = 1/p + 1/2$ be a sequence converging to γ in $W^{1,p}(\Omega)$. Denote by ψ_n the exponentially growing solution to the Schrödinger equation with potential $\gamma_n^{-1/2}\Delta\gamma_n^{1/2}$ and by u, u_n the exponentially growing solutions to the conductivity equation with conductivity γ, γ_n respectively. Since m depends continuously on γ (Lemma 3.3.2), (3.50) implies that so does $\tilde{\omega}$. Hence it follows that for $k \neq 0$

$$\frac{\gamma_n^{-1/2}}{ik}\psi_n(\cdot,k) = u_n(\cdot,k) \to u(\cdot,k) \text{ in } H^{1/2}(\partial\Omega).$$

This result and the continuity of the Dirichlet-to-Neumann map with respect to the conductivity (cf. (2.12)) now implies that

$$S_k(\Lambda_{\gamma_n} - \Lambda_1)u_n(\cdot,k) \to S_k(\Lambda_\gamma - \Lambda_1)u(\cdot,k) \text{ in } H^{1/2}(\partial\Omega),$$

which shows that $u(\cdot,\zeta)|_{\partial\Omega}$ satisfies (3.54). Uniqueness follows from the uniqueness in Lemma 3.5.1, since by Theorem 2.4.10 any solution to (3.54) can be extended to an exponentially growing solution to the conductivity equation.

Invertibility of $(I + S_k(\Lambda_\gamma - \Lambda_1))$ in $H^{1/2}(\partial\Omega)$ follows from the compactness of $S_k(\Lambda_\gamma - \Lambda_1)$ (see Lemma 2.4.11) and the uniqueness above. □

This theorem validates that the computation of the scattering transform S can be done by first computing the function $u(\cdot,k)|_{\partial\Omega}$ and then S_{21} by (3.52). With this modification the algorithm is

(1) Solve (3.54) for $u(\cdot,k)|_{\partial\Omega}$ for any $k \in \mathbb{C} \setminus \{0\}$.

(2) Compute S_{21} by (3.52).

(3) Solve (3.38) for $\tilde{m}^+(z,\cdot)$ for any $z \in \mathbb{R}^2$.

(4) Reconstruct γ by

$$\mathrm{Re}(\tilde{m}(z,0)) = \gamma^{1/2}(z).$$

We see now that this new procedure is a direct generalization of Nachman's method to the class of less regular conductivities.

3.6. Numerical implementation of the algorithm

In this section we describe a preliminary implementation of the algorithm described in section 3.4. We will test the algorithm on a radial conductivity defined on the ball $\Omega = B(0,1)$. We choose the conductivity to be constant 1 near the boundary. This example is chosen because we then have a fast and reliable algorithm available for the computation of the Dirichlet-to-Neumann map from the conductivity. Furthermore, as we will see, symmetry in γ implies symmetry in S, and this will be extremely useful from a computational point of view.

We will first describe our test example and see how the Dirichlet-to-Neumann map can be computed. Second we will describe an approximate method for computing the scattering transform. Next we will focus on the numerical solution method for pseudoanalytic equations. Then we see how symmetry in the conductivity gives symmetry in the scattering transform, and finally we will show some numerical results.

3.6.1. Test example and computation of the Dirichlet-to-Neumann map. Our test example is the radial conductivity

$$\gamma(z) = \begin{cases} (1 + 10F(|z|)), & -\rho < |z| < \rho, \\ 1, & |z| \geq \rho, \end{cases}$$

where

$$F(t) = |t^2 - \rho^2|^{1.1} |t^2 - \rho^2/4|^{1.1} \cos(\frac{\pi}{\rho} t),$$

and $\rho = 3/4$. This defines $\gamma \in C^{1.1}(B(0,1))$, with $\gamma - 1$ supported on $B(0,3/4)$. We note that γ is smooth away from the $C^{1.1}$ singularities $\rho, \rho/2$. A plot of the test conductivity can be seen in Figure 1.

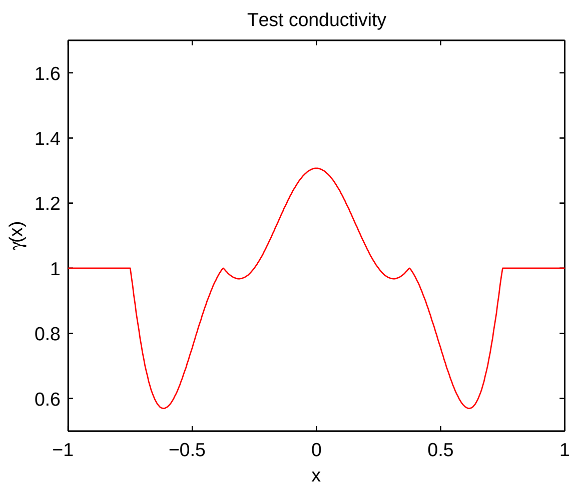

Figure 1. Profile of the test conductivity.

For this conductivity we will compute the Dirichlet-to-Neumann map. As noted by **[Syl92]** in the case of a radial conductivity on $B(0,1)$, the Dirichlet-to-Neumann map has eigenfunctions $e^{in\theta},\ n \in \mathbb{Z}$, i.e.

$$\Lambda_\gamma e^{in\theta} = \lambda_n e^{in\theta}.$$

In particular

$$\Lambda_1 e^{in\theta} = |n| e^{in\theta},\ n \in \mathbb{Z}.$$

Thus the Dirichlet-to-Neumann map can be represented by the eigenvalues λ_n and the difference $\Lambda_\gamma - \Lambda_1$ can be represented by $\lambda_n - |n|$. In **[SMI00]** a direct method for the computation of the eigenvalues is proposed. The idea is to approximate a radial conductivity from below and above by piecewise constant conductivities and then compute by explicit formulas the eigenvalues for the Dirichlet-to-Neumann associated the approximate conductivities. This gives upper and lower bounds for the eigenvalues λ_n, which are then approximated by taking the average of the bounds. We have used this method in the computation of the eigenvalues for our specific example. We refer to **[SMI00]** for further details concerning this algorithm.

Figure 2 is a logarithmic plot of the eigenvalues $\lambda_n - |n|$ for the specific test example. It shows that the the difference is exponentially decreasing.

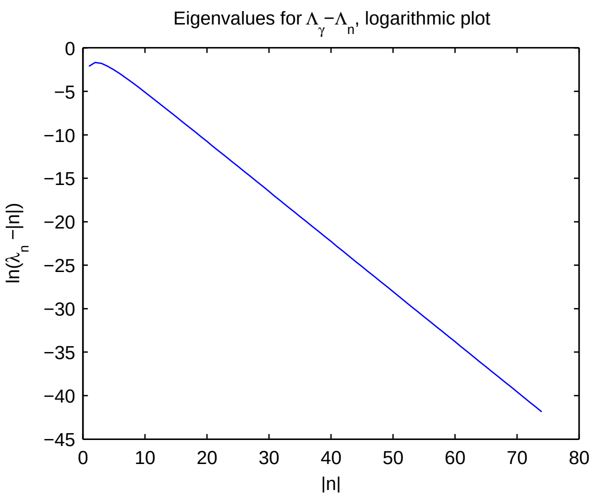

Figure 2. The eigenvalues of $\Lambda_\gamma - \Lambda_1$.

3.6.2. Computing the scattering transform from boundary data. The equation (3.47) is a complete characterization of $\Psi|_{\partial\Omega}$. To compute $\Psi|_{\partial\Omega}$ from this equation numerically, a suggestion could be to discretize the boundary operators involved as a matrix A and the right hand side in the equation as a vector b, and then consider the linear system $Ax = b$. Then a candidate for a solution could be found using a numerical linear solver. Having found the boundary values $\Psi|_{\partial\Omega}$ the scattering transform can be computed from (3.33) by numerical integration. This idea has in preliminary tests caused numerical difficulties, due to the fact that when the equation (3.47) is solved numerically, the exponentially growing diagonal part will dominate the solution. However, the computation of S by (3.33) involves only the off-diagonal entries of $\Psi|_{\partial\Omega}$, which are small and dominated by computational errors. In some sense this makes the computation of S by (3.33) based on information obtained from (3.47) hard, and the practical value of (3.47) needs further investigation.

To overcome this difficulty we suggest a different method for the computation of an approximation of S from Λ_γ based on the formula (3.52). In the implementation of Nachman's method **[SMI00]** the non-physical scattering transform is not computed by solving the boundary integral equation (2.36) and then inserting the boundary values into (3.18). Instead a first order approximation of $\mathbf{t}$ is computed by inserting the approximation e^{izk} for ψ in (3.18). Following this idea we suggest the approximation

$$S_{21}^{\exp}(k) = \frac{-1}{2k} \int_{\partial\Omega} e^{i\overline{zk}} (\Lambda_\gamma - \Lambda_1) e^{izk} d\sigma(z). \tag{3.55}$$

We will use this formula for our computation of the scattering transform.

We note that (3.19) and

$$\|e^{izk} - 1\|_{H^{3/2}(\partial\Omega)} < C|k|$$

easily imply that for $|k|$ sufficiently small

$$\|\psi(\cdot, k) - e^{izk}\|_{H^{3/2}(\partial\Omega)} < C|k|.$$

Since $\mathrm{Ran}(\Lambda_\gamma - \Lambda_1) \subset H_0^{1/2}(\partial\Omega)$ this shows that for $k > 0$

$$\begin{aligned} |S_{21}(k) - S_{21}^{\exp}(k)| &\leq \left| \frac{C}{k} \int_{\partial\Omega} (e^{i\overline{zk}} - 1)(\Lambda_\gamma - \Lambda_1)(\psi(z,k) - e^{izk}) d\sigma(z) \right| \\ &\leq C|k|. \end{aligned}$$

Hence we can expect the approximation to be accurate for small values of k.

$S_{21}^{\exp}$ is implemented by decomposing e^{izk} in a Fourier basis on $e^{in\theta}$, which makes the application of $\Lambda_\gamma - \Lambda_1$ trivial. The formula is then (cf. **[SMI00**, formula (41)**]**

$$S_{21}^{\exp}(k) = \frac{-1}{2k} \sum_{n=0}^{\infty} \frac{(-1)^n |k|^{2n}}{(n!)^2} (\lambda_n - n). \tag{3.56}$$

In the implementation this series is simply cut off at a sufficiently large n $(= 75)$.

3.6.3. Solving pseudoanalytic equations numerically. In the algorithm we have proposed, a pseudoanalytic equation appears both in the forward problem (3.29) and in the inverse problem (3.38). In our numerical test solving (3.29) will be the crucial step in the computation of the exact S_{21} from the known potential, while solving (3.38) is an essential step in the algorithm for reconstruction of the conductivity. Both equations can be transformed into Fredholm equations of the second kind having the generic form

$$(I - \bar{\partial}^{-1}(T(\cdot, p)\bar{\cdot}))m(v, p) = 1, \tag{3.57}$$

where v is the variable (z, k, respectively), p is the parameter (k, z, respectively), $T : \mathbb{C} \times \mathbb{C} \mapsto \mathbb{C}$ is the multiplier in the pseudoanalytic equation ($\pm q(z)e(z, -k)$, $\overline{S_{21}(-k)}e(z, -k)$, respectively) and

$$\bar{\partial}^{-1}(T(\cdot, p)\ \bar{\cdot}))m(\cdot, p) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{T(v', p)}{v - v'} \overline{m}(v', p) d\mu(v').$$

In the following we will suppress the dependency on the parameter p.

The first problem in the numerical realization of (3.57) is that the equation is an equation in an unbounded domain, so to make it practicable we will have to truncate in a suitable way. The first truncation is done on the domain of integration for the convolution operator. In the forward problem (3.29) this is not a issue since q is compactly supported, but in the inverse problem this cut-off will introduce a systematic error. Lemma 3.1.5 ensures, however, that if the domain of integration is sufficiently large then the error can be neglected. We note that choosing a smooth cut-off instead may improve the numerical results.

Assuming that the domain of integration in (3.57) is bounded we will now make a second truncation by considering the equation as an integral equation in a bounded domain Ω, i.e.

$$(I - \mathbf{R}_{\Omega}\bar{\partial}^{-1}(T\ \bar{\cdot}))m = 1 \text{ in } \Omega, \tag{3.58}$$

where $\mathbf{R}_{\Omega}$ is the restriction operator to Ω. The following lemma states that the restricted equation is uniquely solvable, and that the solution coincides with the solution to (3.57) on Ω.

Lemma 3.6.1. *Assume that $T \in L^p(\mathbb{R}^2), p > 2$, is compactly supported in the bounded domain Ω. Let m be the unique solution to* (3.57) *with $m - 1 \in L^r(\mathbb{R}^2)$, $r > 2$. Then* (3.58) *has the unique solution $m|_{\Omega}$ in $L^r(\Omega)$.*

Proof. It is clear that $m|_\Omega$ is a solution to (3.58) in $L^r(\Omega)$. Uniqueness follows from the uniqueness of the solution to (3.57), since any solution m_0 to (3.58) can be extended beyond Ω by

$$m_0(v) = \frac{1}{\pi}\int_\Omega \frac{T(v')}{v-v'}\overline{m}_0(v')d\mu(v'), \qquad v \in \mathbb{R}^2 \setminus \overline{\Omega}$$

as a solution to (3.57). □

Note that the uniqueness of a solution to (3.58) gives invertibility in $L^r(\Omega)$ of $(I - \mathbf{R}_\Omega \overline{\partial}^{-1}(T\,\bar{\cdot}))$, since $\mathbf{R}_\Omega\overline{\partial}^{-1}(T\,\bar{\cdot})$ is compact.

We now need a fast and reliable implementation of (3.58). To serve this purpose we adapt a method developed by Vainikko [**Vai93, Vai00**] for solving Lippmann-Schwinger type equations. This method has been used successfully in the numerical computation of **t** from a known q in the implementation of Nachman's algorithm [**MS01**].

Assume that $\Omega = [-1,1]^2$. Let n be a positive integer and let

$$\mathbb{Z}_n^2 = \{(z_1,z_2) \in \mathbb{Z}^2 \mid -2^{n-1} \le z_j \le 2^{n-1}, j = 1,2\}.$$

Fix the discretization level $h = 2^{1-n}$. Then $h\mathbb{Z}_n^2$ will be a uniform grid on Ω consisting of N^2 points with $N = (2^n+1)$. The grid approximation of a function f on Ω is defined by

$$f_{j,h} = f(jh),\ j \in \mathbb{Z}_n^2.$$

Discretizing (3.58) by the piecewise constant collocation method (see [**Vai93**, section 5.3]) yields

$$m_{j,h} + (\mathbf{T}_h\overline{m})_{j,h} = 1, \qquad j \in \mathbb{Z}_n^2$$

where

$$(\mathbf{T}_h\phi)_{j,h} = \sum_{\substack{k\in\mathbb{Z}_n^2\\ j\neq k}} \mathfrak{g}_{j-k,h}(T_{k,h}\overline{\phi_{k,h}}),$$

and $\mathfrak{g}(z) = 1/(\pi z)$. The convolution operator can be implemented numerically as a matrix directly, which would give the complexity $\mathcal{O}(N^4)$ for one application of T_h. Instead of doing so we implement the convolution operator using fast Fourier transform, i.e. by computing as matrices

$$(\mathbf{T}_h\overline{\phi})_{\cdot,h} = \mathrm{iFFT}(\mathrm{FFT}(g_{\cdot,h}).*\mathrm{FFT}(T_h\overline{\phi_{\cdot,h}}))$$

where $.*$ denotes pointwise multiplication of matrices. This method gives a complexity $\mathcal{O}(N^2\log N)$ for one application of the convolution operator. Having implemented the convolution operator, the equation can be solved numerically using an iterative algorithm like GMRES [**SS86**].

From the general convergence analysis of algorithms for solving weakly singular Fredholm equations of the second kind [**Vai93**, Theorem 5.1] we can deduce that the convergence rate for the particular implementation described here is $\mathcal{O}(h)$. In our particular implementation we have the coefficient $N = 7$. This choice gives a good accuracy with reasonable computation time.

3.6.4. Symmetry in the scattering transform. The following result shows how radial symmetry in the conductivity implies symmetry in the scattering transform. This result is an analogue of a result for the non-physical scattering transform **t**, see [**SMI00**, Theorem 3.3].

Lemma 3.6.2. *Let $\gamma \in W^{1,p}(\Omega)$, $p > 2$ and assume $\gamma(z) = \gamma(e^{i\theta} z)$ for some angle θ. Then $e^{i\theta} S_{21}(e^{i\theta} k) = S_{21}(k)$ and $e^{-i\theta} S_{12}(e^{i\theta} k) = S_{12}(k)$.*

Proof. First note that $\gamma(z) = \gamma(e^{i\theta} z)$ implies

$$q(z) = -\frac{\partial \gamma^{1/2}(z)}{\gamma^{1/2}(z)} = -\frac{\partial(\gamma^{1/2}(e^{i\theta} z))}{\gamma^{1/2}(e^{i\theta} z)} = e^{i\theta} q(e^{i\theta} z).$$

Next note that since the transformation $z \mapsto e^{i\theta} z$ implies the change $\bar{z} \mapsto e^{-i\theta}\bar{z}$, the exponentially growing solution $(\Psi_{11}(e^{i\theta} z, k), e^{-2i\theta}\Psi_{21}(e^{i\theta} z, k))$ satisfies

$$\begin{aligned}
\bar{\partial}\Psi_{11}(e^{i\theta} z, k) &= e^{-i\theta}(\bar{\partial}\Psi_{11})(e^{i\theta} z, k) \\
&= e^{-i\theta} q(e^{i\theta} z)\Psi_{21}(e^{i\theta} z, k) \qquad (3.59)\\
&= q(z) e^{-2i\theta}\Psi_{21}(e^{i\theta} z, k) \\
\partial(e^{-2i\theta}\Psi_{21}(e^{i\theta} z, k)) &= e^{-i\theta}(\partial\Psi_{21})(e^{i\theta} z, k) \\
&= e^{-i\theta}\bar{q}(e^{i\theta} z)\Psi_{11}(e^{i\theta} z, k) \qquad (3.60)\\
&= \bar{q}(z)\Psi_{11}(e^{i\theta} z, k).
\end{aligned}$$

Now since the vector $(\Psi_{11}(z, e^{i\theta} k), \Psi_{21}(z, e^{i\theta} k))$ also solves (3.59) and (3.60) and has the same asymptotic behavior when $|z| \to \infty$, we deduce by the uniqueness in Lemma 3.3.2 that

$$\begin{aligned}
\Psi_{11}(e^{i\theta} z, k) &= \Psi_{11}(z, e^{i\theta} k) \\
\Psi_{21}(e^{i\theta} z, k) &= e^{2i\theta}\Psi_{21}(z, e^{i\theta} k)
\end{aligned}$$

which implies

$$\begin{aligned}
m_{11}(e^{i\theta} z, k) &= m_{11}(z, e^{i\theta} k) \\
m_{21}(e^{i\theta} z, k) &= e^{2i\theta} m_{21}(z, e^{i\theta} k).
\end{aligned}$$

Hence

$$\begin{aligned} e^{i\theta}S_{21}(e^{i\theta}k) &= \frac{-i}{\pi}\int_{\mathbb{R}^2} \overline{e^{-i\theta}q(z)}e(z,e^{i\theta}k)m_{11}(z,e^{i\theta}k)d\mu(z) \\ &= \frac{-i}{\pi}\int_{\mathbb{R}^2} \overline{q(e^{i\theta}z)}e(e^{i\theta}z,k)m_{11}(e^{i\theta}z,k)d\mu(z) \\ &= \frac{-i}{\pi}\int_{\mathbb{R}^2} \overline{q}(z')e(z',k)m_{11}(z',k)d\mu(z') \\ &= S_{21}(k). \end{aligned}$$

For S_{12} a similar calculation can be carried out. □

This result is useful when dealing with radial conductivities since one only has to compute the scattering transform for real values of the parameter k and then obtain

$$S_{21}(k) = e^{-i\arg(k)}S_{21}(|k|), \qquad k \in \mathbb{C}. \tag{3.61}$$

A similar identity can be found for $S_{21}^{\exp}$.

3.6.5. Numerical results.

Computing the scattering transform. We compute the scattering transform both by solving the forward problem $Q \mapsto S_{21}$ and by the inverse problem $\Lambda_\gamma \mapsto S_{21}$. In the first method we start by solving the pseudoanalytic equation (3.29) for $m_\pm$, then solving the linear systems (3.28), and finally integrating in (3.31). In the implementation of this method we use the method for solving pseudoanalytic equations numerically outlined above. The second method is the more realistic method of computing the scattering transform from the Dirichlet-to-Neumann map. This method relies on the approximation (3.55) computed by (3.56).

In Figure 3 the scattering transform S_{21} is displayed, and in Figure 4 it is displayed together with the approximation $S_{21}^{\exp}$. Moreover, the error $S_{21}(k) - S_{21}^{\exp}$ is displayed. Both functions are computed for a range of real positive k. In Figure 4 the axis is truncated at $k = 30$, since $S_{21}^{\exp}$ becomes highly inaccurate beyond $k = 25$. We note that below this value the approximation seems reasonable accurate.

Reconstructing the conductivity. The reconstruction of the conductivity from the scattering transform is done by solving (3.38) for $\tilde{m}^+(z,k)$ for a suitable range of real z and then obtaining $\gamma(z) = (\operatorname{Re}\tilde{m}^+(z,0))^2$. The first reconstruction is done from the true scattering transform S_{21} (Figure 3), which is known for real k and extended to the complex plane by (3.61). The result of this inversion is displayed in Figure 5 together with the true conductivity. Moreover, we have plotted the relative error $|\gamma(x) - \gamma_{\text{rec}}(x)|/|\gamma(x)|$. We see that the reconstruction is very accurate where γ is smooth, but the

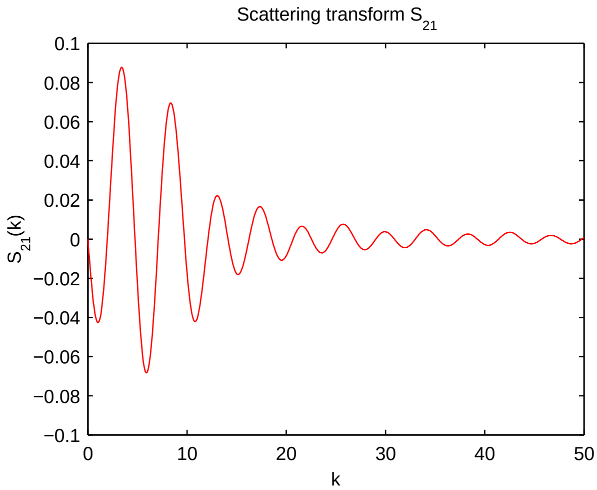

Figure 3. The true scattering transform.

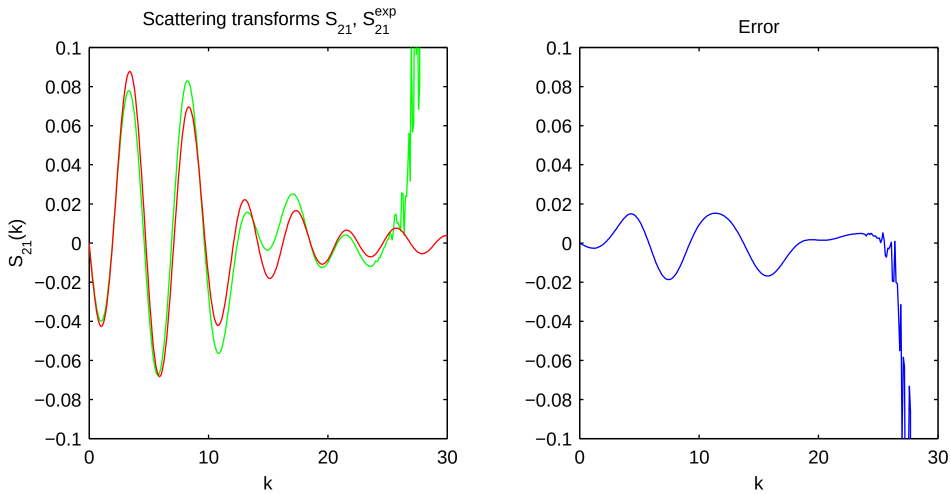

Figure 4. The scattering transforms S_{21} (red) and S_{21}^{exp} (green).

singularities are smoothed out. This is no surprise, since we from (3.32) do expect the singularities of γ to be represented in the large k behavior of S_{21}, but this behavior is discarded in the numerical implementation.

The second reconstruction is based on the approximation S_{21}^{exp}. As can be seen from Figure 4, $S_{21}^{\mathrm{exp}}(k)$ is unreliable for $k > 25$. Hence we will have to truncate this function before doing the inversion. To illustrate how delicate the choice of cut-off parameter is, we have chosen to truncate at the different values $k = 10, 15, 20, 25, 28$ and then reconstruct from the truncated S_{21}^{exp}. In Figure 6 – 11 the reconstructions are displayed together with the true conductivity. Furthermore, we have for comparison displayed the reconstructed conductivity based on the same truncation of the true S_{21}. We see from the figures that when the cut-off is small, then the inversion gives a low frequency approximation of the conductivity. The quality of these low frequency approximations obtained from S_{21}^{exp} are comparable to those obtained from S_{21}. When the cut-off parameter increases more detailed reconstructions appears. These reconstructions both capture some features in the original conductivity and introduce certain artifacts. When the cut-off is taken beyond $k = 25$ the error in S_{21}^{exp} dominates.

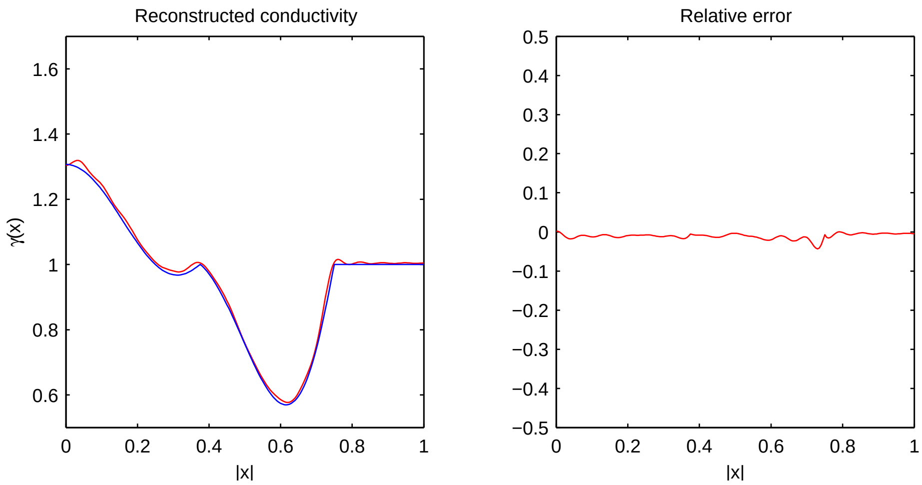

Figure 5. The reconstructed conductivities. Blue curve is the true conductivity and red curve is the conductivity reconstructed from the true $S_{21}(k)$, $k < 50$.

3.6.6. Conclusion. In this section we have seen a numerical implementation of the reconstruction algorithm for conductivities in $C^{1+\epsilon}(\Omega)$. We have tested the algorithm on synthetic, noiseless data. The conclusions are as follows:

Concerning the computation of the scattering transform from the Dirichlet-to-Neumann map, the implementation based on solving (3.47) to obtain Ψ_{21} and computing S_{21} by (3.33) seems to be unstable. Hence we have suggested a different method, which computes the approximation S_{21}^{exp}. This approximation shows good accuracy for small $k < 25$ and gives fair low frequency reconstructions, however, if more detailed images are needed, we need a better method for the computation of an approximate scattering

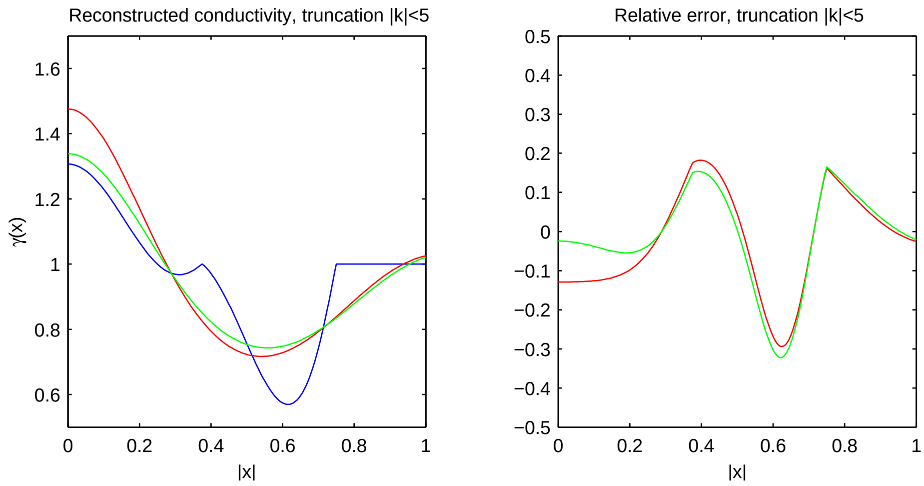

Figure 6. The reconstructed conductivities. Green curve is the conductivity reconstructed from $S_{21}^{\text{exp}}(k)$, $k < 5$, red curve is the conductivity reconstructed from the true S_{21}, $k < 5$, and blue curve is the true conductivity.

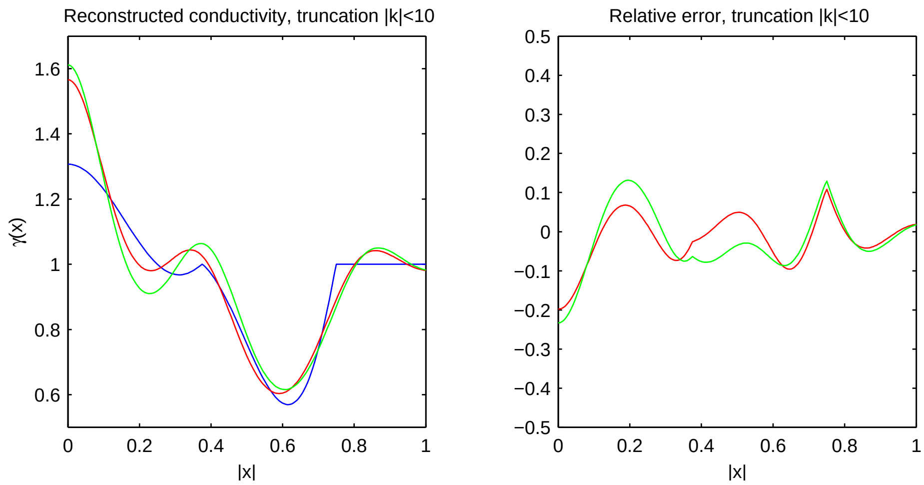

Figure 7. The reconstructed conductivities. Green curve is the conductivity reconstructed from $S_{21}^{\text{exp}}(k)$, $k < 10$, red curve is the conductivity reconstructed from the true S_{21}, $k < 10$, and blue curve is the true conductivity.

transform. An important issue in this implementation concerns the choice of cut-off parameter for truncating the scattering transform before the inversion is done. We have seen that this choice has a great impact on the

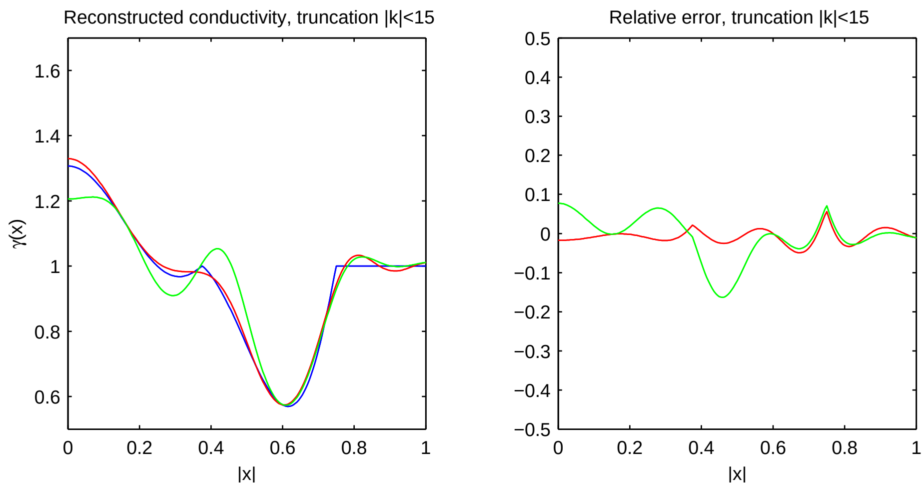

Figure 8. The reconstructed conductivities. Green curve is the conductivity reconstructed from $S_{21}^{\exp}(k)$, $k < 15$, red curve is the conductivity reconstructed from the true S_{21}, $k < 15$, and blue curve is the true conductivity.

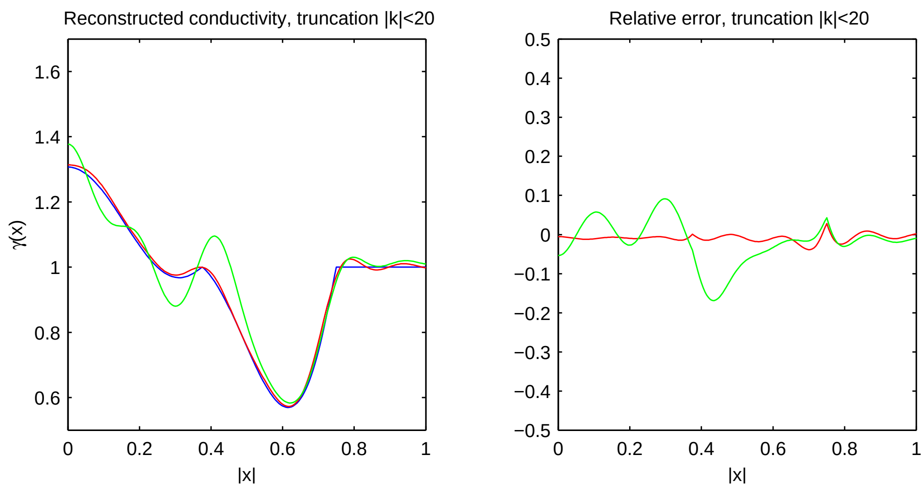

Figure 9. The reconstructed conductivities. Green curve is the conductivity reconstructed from $S_{21}^{\exp}(k)$, $k < 20$, red curve is the conductivity reconstructed from the true S_{21}, $k < 20$, and blue curve is the true conductivity.

reconstructions. In presence of noise this method of cutting of the scattering transform may be seen as a non-linear regularization strategy **[MS01]**.

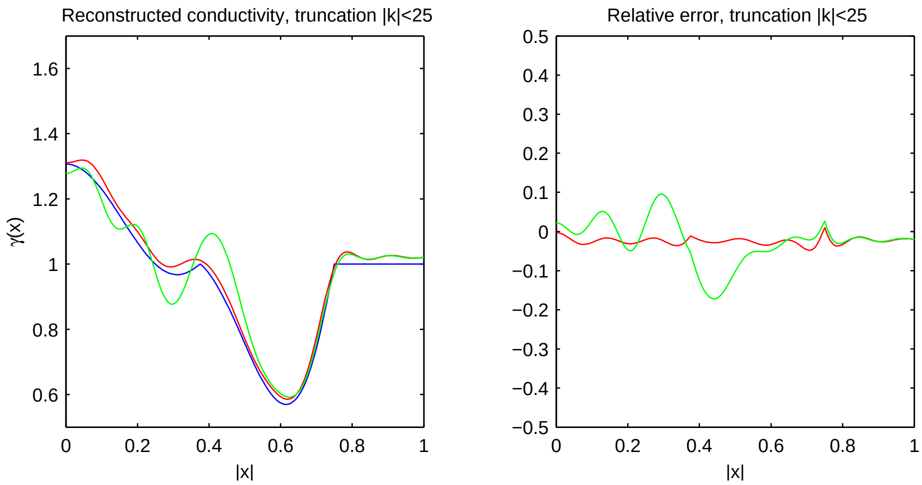

Figure 10. The reconstructed conductivities. Green curve is the conductivity reconstructed from $S_{21}^{\text{exp}}(k)$, $k < 25$, red curve is the conductivity reconstructed from the true S_{21}, $k < 25$, and blue curve is the true conductivity.

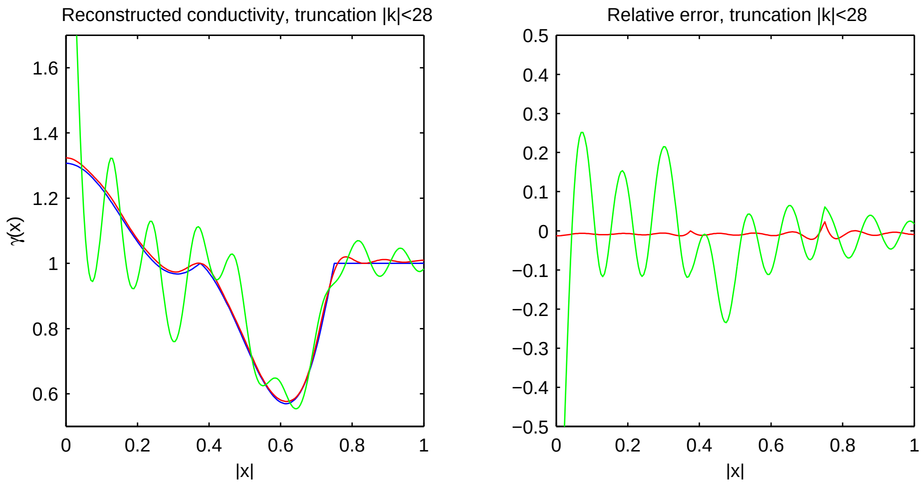

Figure 11. The reconstructed conductivities. Green curve is the conductivity reconstructed from $S_{21}^{\text{exp}}(k)$, $k < 28$, red curve is the conductivity reconstructed from the true S_{21}, $k < 28$, and blue curve is the true conductivity.

The implementation has shown that the direct and inverse scattering transform (the numerical solver for pseudoanalytic equations) are numerically stable operations, even when performed by the fast method based on FFT. The fast implementation of a solver for (3.57) may be useful in other applications as well, for instance in the numerical solution of the non-linear Davey-Stewartson equations. This is a project for further studies. Concerning computation speed and accuracy we note that the method in [**Vai00**] is actually a multi-grid method, which can give more accuracy at a low cost. We do not exploit this feature in the current implementation.

3.7. Notes

The results in section 3.1 are classical. A good reference for pseudoanalytic equations is [**Vek62**]. The formula (3.11) goes back to Plemelj in 1908.

Both Nachman's method and Brown-Uhlmann's method rely on the $\bar{\partial}$-method in inverse scattering. This method was initially introduced in the study of some one-dimensional non-linear evolution equations, which could be linearized by the scattering transform [**BC81, BC82**]. The method was later developed and extended to higher dimensional problems by a number of people. For an general introduction to the $\bar{\partial}$-method we refer to the review papers [**BC85, BC89**].

For the two-dimensional inverse conductivity problem local uniqueness for small $\gamma \in W^{3,\infty}(\Omega)$ was proved in [**SU86**]; the general question remained open until 1996, when Nachman [**Nac96**] gave the uniqueness proof and a reconstruction algorithm outlined in section 3.2. Based on this method Liu [**Liu97**] proved a conditional stability estimate of the form

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C\lambda(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}).$$

Here the constant C depends on an a priori bound of $\|\gamma_i\|_{W^{2,p}(\Omega)}$, $1 < p < 2$, and λ is a real function with

$$\lambda(t) \leq |\log(t)|^{-\delta}$$

for $0 < \delta < 2/p'$ and t sufficiently small. Furthermore, the algorithm has been tried out numerically (see [**SMI00, SMI01a, SMI01b**]). We note that a scattering theory for the two-dimensional Schrödinger problem similar to the theory outlined in section 3.2 was considered by Boite, Leon, Manna and Pempinelli in [**BLMP87**] and Tsai in [**Tsa93**].

The inverse problem for the Schrödinger equation (when the potential is not defined from a conductivity) concerning injectivity of the map $q \mapsto \Lambda_q$ is open. There are partial results due to Sun and Uhlmann concerning generic uniqueness [**SU91**] and the recovery of the singularities of the potential [**SU93**].

In 1997 Brown and Uhlmann ([**BU97**]) improved the uniqueness result for the class of less regular conductivities $\gamma \in W^{1,p}(\Omega)$, $p > 2$. Based on their method Barceló, Barceló and Ruiz proved conditional stability

[**BBR01**]. Indeed, if $\gamma_i \in C^{1+\epsilon}(\overline{\Omega})$ and $\|\gamma_i\|_{C^{1+\epsilon}(\overline{\Omega})} < M$, then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C\lambda(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\mathcal{B}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))}).$$

Here the constant C depends on M and λ is a real function with

$$\lambda(t) \leq |\log(t)|^{-\delta}$$

for some $\delta > 0$ and t sufficiently small. Following the same method Francini proved uniqueness for conductivities $\gamma = \gamma_R + i\gamma_I$, with γ_R, γ_I real and γ_I is small [**Fra00**].

The motivation of Beals and Coifman in [**BC88**] for constructing a scattering theory for (3.22) was the applicability of this theory in the analysis of certain non-linear evolution equations. The crucial observation is that the transformation $q \mapsto S$ linearizes the non-linear equations, a fact that makes the evolution trivial. Among the results in [**BC88**] is that $q \in \mathcal{S}(\mathbb{R}^2)$ implies $S \in \mathcal{S}(\mathbb{R}^2)$ with norm equality $\|q\|_{L^2(\mathbb{R}^2)} = \|S\|_{L^2(\mathbb{R}^2)}$. This shows that the map $q \mapsto S$ is isometric on $\mathcal{S}(\mathbb{R}^2)$ equipped with the $L^2(\mathbb{R}^2)$ norm. Due to the non-linearity of the map a density argument does not directly extend the result to $L^2(\mathbb{R}^2)$. Further results concerning the scattering theory for (3.22) can be found in [**Sun94a**],[**Sun94b**],[**Sun94c**]. The motivation is again the solvability of a non-linear evolution equation. There the analysis is given for $Q \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

Chapter 4

Uniqueness and reconstruction in higher dimensions

In this chapter we will focus on the questions of uniqueness and reconstruction for the inverse conductivity problem in dimensions $n \geq 3$. We will first review the proof of uniqueness for sufficiently regular conductivities. Then we will present a new result concerning absence of small exceptional points for a merely bounded conductivity. Next we will review the two known reconstruction algorithms, including one based on the so-called $\overline{\partial}$-method of inverse scattering, and finally we will propose a new method for the reconstruction of almost constant conductivities.

4.1. Uniqueness for the inverse conductivity problem

The uniqueness results in this section are all well-known. We give the proofs anyway, since they show the importance of having exponentially growing solutions.

The following theorem due to Sylvester and Uhlmann **[SU87]** answers the uniqueness question for the inverse problem for the Schrödinger operator. Recall from (2.10) the definition of the set of Cauchy data $\mathcal{C}_q$ for $(-\Delta + q)$.

Theorem 4.1.1. *Let $n \geq 3$ and assume $q_i \in L^\infty(\Omega)$. Then $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ implies $q_1 = q_2$ in Ω.*

Proof. Let u_i satisfy $(-\Delta + q_i)u_i = 0$ in Ω. By assumption $(u_1, \partial_\nu u_1)|_{\partial\Omega} \in \mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ and then integration by parts shows that

$$\begin{aligned}\int_\Omega (q_1 - q_2)(x)u_1(x)u_2(x)dx = &= \langle u_1, \partial_\nu u_2\rangle - \langle u_2, \partial_\nu u_1\rangle \\ &= \int_\Omega (q_2 - q_2)(x)\tilde{u}_1(x)u_2(x)dx \\ &= 0,\end{aligned}$$

where $\tilde{u}_1$ solves $(-\Delta + q_2)\tilde{u}_1 = 0$ in Ω with $(\tilde{u}_1, \partial_\nu \tilde{u}_1)|_{\partial\Omega} = (u_1, \partial_\nu u_1)|_{\partial\Omega}$. Take now two parameters $\zeta_1, \zeta_2 \in \mathcal{V}$ defined by

$$\begin{aligned}\zeta_1 &= \frac{1}{2}(\xi + \kappa k_\perp + \kappa' ik), \\ \zeta_2 &= \frac{1}{2}(\xi - \kappa k_\perp + \kappa' ik),\end{aligned}$$

where $\xi \in \mathbb{R}^n$ is arbitrary, $k_\perp, k \in \mathbb{R}^n$ are chosen such that $\xi \cdot k_\perp = \xi \cdot k = k_\perp \cdot k = 0$, and $\kappa, \kappa' \in \mathbb{R}$ satisfy $\kappa'^2 = \kappa^2 + |\xi|^2$ (cf. (2.19)). This choice is possible since $n \geq 3$. Then for κ sufficiently large, Theorem 2.4.3 guarantees the existence of exponentially growing solutions

$$u_i(x, \zeta_i) = e^{ix\cdot\zeta}(1 + \omega_i(x, \zeta_i))$$

corresponding to the potential q_i and parameter $\zeta_i,\ i = 1, 2$.

Since (2.26) implies that

$$\begin{aligned}&\|e^{ix\cdot\xi} - u_1(\cdot, \zeta_1)u_2(\cdot, \zeta_2)\|_{L^1(\Omega)} \\ &\quad \leq C(\|\omega_1(\cdot, \zeta_1)\|_{L^2_\delta(\Omega)}\|\omega_2(\cdot, \zeta_2)\|_{L^2_\delta(\Omega)} + \|\omega_1(\cdot, \zeta_1)\|_{L^2_\delta(\Omega)} + \|\omega_2(\cdot, \zeta_2)\|_{L^2_\delta(\Omega)}) \\ &\quad \leq \frac{C}{|\zeta|} \\ &\quad \leq \frac{C}{\kappa}\end{aligned}$$

it follows by taking $\kappa \to \infty$ that

$$\begin{aligned}0 &= \lim_{\kappa\to\infty} \int_\Omega (q_1 - q_2)(x)u_1(x, \zeta_1)u_2(x, \zeta_2)dx \\ &= \int_\Omega (q_1 - q_2)(x)e^{ix\cdot\xi}dx \\ &= (\widehat{q_1 - q_2})(\xi).\end{aligned}$$

Since ξ is arbitrary the result now follows by inverting the Fourier transform. $\square$

Note that in two dimensions it is impossible to choose the two parameters ζ_1 and ζ_2 as above. This is essentially why the uniqueness proof in this case has to be based on different methods.

Now we can apply the previous result to settle the uniqueness question for the inverse conductivity problem.

Corollary 4.1.2. *Let $\gamma_i \in W^{2,\infty}(\Omega)$. Then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $\gamma_1 = \gamma_2$.*

Proof. From Lemma 2.1.1 it follows that $\gamma = \gamma_1 = \gamma_2$ and $\partial_\nu \gamma_1 = \partial_\nu \gamma_2$ on $\partial\Omega$. Hence with $q_i = \Delta\gamma_i^{1/2}/\gamma_i^{1/2}$ we have $\Lambda_{q_1} = \Lambda_{q_2}$ by (2.9), which implies $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. We conclude by Theorem 4.1.1 that $q = q_1 = q_2$ in Ω. The conductivities γ_i thus satisfy the boundary value problem

$$(-\Delta + q)\gamma_i^{1/2} = 0 \text{ in } \Omega, \qquad \gamma_i^{1/2} = \gamma^{1/2}|_{\partial\Omega} \text{ on } \partial\Omega,$$

which has a unique solution $\gamma^{1/2} = \gamma_1^{1/2} = \gamma_2^{1/2}$, since the potential is defined from a conductivity. □

4.2. Absence of exceptional points near zero for $\gamma \in L^\infty_+(\Omega)$

From Corollary 2.4.6 we know that for any potential there are no exceptional points outside a sufficiently large ball. In this section we consider the other extreme, i.e. the possibility of having small exceptional points. We assume without loss of generality that $\Omega = B_a$, $\gamma \in L^\infty_+(B_a)$ and $\gamma = 1$ near ∂B_a. Then we will show that the boundary integral equation (2.36) is solvable in $H^{1/2}(\partial B_a)$ for any $\zeta \in \mathcal{V}$ sufficiently small, and hence that such a ζ is not exceptional for γ. Moreover, we prove that for γ sufficiently close to 1, there are no exceptional points satisfying $|\zeta| \leq 1$. We emphasize here that the main achievement is that we only assume that $\gamma \in L^\infty_+(\Omega)$, i.e. no further regularity is assumed for the conductivity.

Recall from (2.29) and (2.18) the Green's functions G_0 and G_ζ for the Laplacian and define the harmonic function

$$H_\zeta(x) = G_\zeta(x) - G_0(x). \tag{4.1}$$

The single layer potential S_ζ defined in (2.28) can then be decomposed as

$$S_\zeta = S_0 + \mathcal{H}_\zeta,$$

where S_0 is the usual single layer potential and

$$\mathcal{H}_\zeta \phi(x) = \int_{\partial B_a} H_\zeta(x-y)\phi(y)d\sigma(y), \qquad x \in \partial B_a. \tag{4.2}$$

This decomposition motivates a splitting of the operator

$$(I + S_\zeta(\Lambda_\gamma - \Lambda_1)) = (I + S_0(\Lambda_\gamma - \Lambda_1)) + \mathcal{H}_\zeta(\Lambda_\gamma - \Lambda_1) \tag{4.3}$$

into a free part independent of ζ and a perturbation. For the free part we have the result.

Lemma 4.2.1. *Assume $\gamma \in L^\infty_+(B_a)$ satisfies $\gamma = 1$ near ∂B_a. Then the boundary integral operator $I + S_0(\Lambda_\gamma - \Lambda_1)$ is invertible in $H^{1/2}(\partial B_a)$.*

Proof. The operator $S_0(\Lambda_\gamma - \Lambda_1)$ is compact in $H^{1/2}(\partial B_a)$ by Lemma 2.4.11, and hence injectivity of $I + S_0(\Lambda_\gamma - \Lambda_1)$ implies invertibility by Fredholm's alternative.

Assume now that $h \in H^{1/2}(\partial B_a)$ and

$$(I + S_0(\Lambda_\gamma - \Lambda_1))h = 0. \tag{4.4}$$

Let v be the unique harmonic function in Ω with $v|_{\partial\Omega} = h$ and let $w = \mathbf{S}_0(\Lambda_\gamma - \Lambda_1)h \in H^1(B_a)$, where $\mathbf{S}_0$ denotes the standard single layer potential. Then $v + w$ is harmonic in B_a and has trace $h + S_0(\Lambda_\gamma - \Lambda_1)h = 0$. It follows that $v + w = 0$ in B_a. Hence from the jump in the normal derivative of the single layer potential (see (2.34)) it follows from taking the trace of $\partial_\nu(v + w)$ from inside Ω that

$$\begin{aligned} 0 &= \Lambda_1 h + \frac{1}{2}(\Lambda_\gamma - \Lambda_1)h + K_0'(\Lambda_\gamma - \Lambda_1)h \\ &= \frac{1}{2}(\Lambda_\gamma + \Lambda_1)h + K_0'(\Lambda_\gamma - \Lambda_1)h. \end{aligned} \tag{4.5}$$

Since on ∂B_a

$$\begin{aligned} \partial_{\nu_x} G_0(x-y) &= \frac{x}{|x|} \cdot \nabla_x \frac{1}{n\omega_n |x-y|^{n-2}} \\ &= \frac{x}{|x|} \cdot \frac{-(n-2)(x-y)}{n\omega_n |x-y|^n} \\ &= \frac{-(n-2)}{2a} \frac{|x|^2 + |y|^2 - 2x\cdot y}{n\omega_n |x-y|^n} \\ &= \frac{-(n-2)}{2a} G_0(x-y) \end{aligned}$$

we get by (4.4)

$$K_0'(\Lambda_\gamma - \Lambda_1)h = \frac{-(n-2)}{2a} S_0(\Lambda_\gamma - \Lambda_1)h = \frac{(n-2)}{2a} h. \tag{4.6}$$

Inserting (4.6) into (4.5) gives

$$0 = (\Lambda_\gamma + \Lambda_1)h + \frac{(n-2)}{a} h$$

which implies $h = 0$, since

$$0 = \langle (\Lambda_\gamma + \Lambda_1)h + \frac{(n-2)}{a} h, h \rangle \geq \frac{(n-2)}{a} \|h\|^2_{L^2(\partial B_a)}.$$

$\square$

Next we will investigate the dependence of the perturbation in (4.3) on ζ. We need the following lemma concerning properties of the harmonic function H_ζ:

Lemma 4.2.2. *Let $\zeta = \kappa(k_\perp + ik) \in \mathcal{V}$ with $\kappa \in \mathbb{R}$, $k_\perp, k \in \mathbb{R}^n$, $k_\perp \cdot k = 0$ (cf.* (2.19)*), and let $\mathbf{R}$ be a real orthogonal $n \times n$ matrix with* $\det(\mathbf{R}) = 1$. *Then the harmonic function $H_\zeta = G_\zeta - G_0$ satisfies*

$$H_\zeta(x) = \kappa^{n-2} H_{k_\perp + ik}(\kappa x), \tag{4.7}$$

$$H_\zeta(x) = H_{\mathbf{R}^T\zeta}(\mathbf{R}^T x). \tag{4.8}$$

Proof. From (2.20) it follows that

$$\begin{aligned} H_\zeta(x) &= G_\zeta(x) - G_0(x) \\ &= e^{ix\cdot\zeta} g_\zeta(x) - G_0(x) \\ &= e^{i\kappa x\cdot(k_\perp + ik)} \kappa^{n-2} g_{(k_\perp+ik)}(\kappa x) - \kappa^{n-2} G_0(\kappa x) \\ &= \kappa^{n-2} H_{(k_\perp+ik)}(\kappa x). \end{aligned}$$

Further,

$$\begin{aligned} H_\zeta(\mathbf{R}x) &= G_\zeta(\mathbf{R}x) - G_0(\mathbf{R}x) \\ &= e^{i(\mathbf{R}x)\cdot\zeta} g_\zeta(\mathbf{R}x) - G_0(|\mathbf{R}x|) \\ &= e^{ix\cdot(\mathbf{R}^T\zeta)} g_{\mathbf{R}^{-1}\zeta}(x) - G_0(|x|) \\ &= H_{\mathbf{R}^T\zeta}(x). \end{aligned}$$

The result now follows since $\det(\mathbf{R}) = 1$ together with the reality of $\mathbf{R}$ implies $\mathbf{R}^T = \mathbf{R}^{-1}$. $\square$

From the properties of H_ζ it is now straightforward to derive estimates for $\mathcal{H}_\zeta$ defined by (4.2).

Lemma 4.2.3. *The integral operator $\mathcal{H}_\zeta$ satisfies for any $\zeta \in \mathcal{V}$ and $0 < |\zeta| \leq 1$ the estimates*

$$\|\mathcal{H}_\zeta\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \leq C|\zeta|^{n-2}, \tag{4.9}$$

$$\|\mathcal{H}_\zeta\|_{\mathcal{B}(H_0^{1/2}(\partial B_a), H^{1/2}(\partial B_a))} \leq C|\zeta|^{n-1}, \tag{4.10}$$

where the constant C depends only on the radius a.

Proof. From (4.8) we can without loss of generality assume that $\zeta = \kappa(e_1 + ie_2)$ with $e_1 = (1, 0, \cdots, 0), e_2 = (0, 1, 0, \cdots, 0)$. Then an application of (4.7) gives

$$\begin{aligned}\mathcal{H}_{\kappa(e_1+ie_2)}f(x) &= \int_{\partial B_a} H_{\kappa(e_1+ie_2)}(x-y)f(y)d\sigma(y) \\ &= \kappa^{n-2}\int_{\partial B_a} H_{e_1+ie_2}(\kappa(x-y))f(y)d\sigma(y). \qquad (4.11)\end{aligned}$$

Define the smooth function $H(x) = H_{e_1+ie_2}(x) - H_{e_1+ie_2}(0)$ and set

$$\mathcal{K}_\kappa f(x) = \int_{\partial B_a} H(\kappa(x-y))f(y)d\sigma(y), \qquad x \in \partial B_a. \qquad (4.12)$$

Then

$$\mathcal{H}_{\kappa(e_1+ie_2)}f(x) = \kappa^{n-2}\mathcal{K}_\kappa f(x) + \kappa^{n-2}H_{e_1+ie_2}(0)\int_{\partial B_a} f(y)d\sigma(y).$$

In particular

$$\mathcal{H}_{\kappa(e_1+ie_2)}f(x) = \kappa^{n-2}\mathcal{K}_\kappa f(x)$$

for $f \in H_0^{1/2}(\partial B_a)$.

Since $H(0) = 0$ and H is smooth, there is a constant C depending only on a such that $\operatorname{supp}_{|x|\leq 2a}|H(x)| \leq C|x|$. Then for $0 \leq \kappa \leq 1$

$$\int_{\partial B_a}\int_{\partial B_a} |H(\kappa(x-y))|^2 d\sigma(y)d\sigma(x) \leq C\kappa^2,$$

which implies

$$\|\mathcal{K}_\kappa\|_{\mathcal{B}(L^2(\partial B_a))} \leq C\kappa. \qquad (4.13)$$

Furthermore, since H is a smooth function, $\operatorname{supp}_{|x|\leq 2a}|\nabla H(x)| \leq C$ for some constant $C > 0$. Hence by taking derivatives in (4.12) we find that

$$\|\mathcal{K}_\kappa\|_{\mathcal{B}(H^1(\partial B_a))} \leq C\kappa. \qquad (4.14)$$

Thus the estimate

$$\|\mathcal{K}_\kappa\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \leq C\kappa \qquad (4.15)$$

can be obtained for $0 \leq \kappa \leq 1$ from (4.13) and (4.14) by interpolation. □

The solvability of the boundary integral equation for small ζ can now be proved using a standard perturbation argument.

Theorem 4.2.4. *Assume $\gamma \in L_+^\infty(B_a)$ satisfies $\gamma = 1$ near ∂B_a. Then the boundary integral equation* (2.36) *is uniquely solvable in $H^{1/2}(\partial B_a)$ for $\zeta \in \mathcal{V}$ sufficiently small. Furthermore, if $\|\gamma - 1\|_{L^\infty(\mathbb{R}^n)}$ is sufficiently small, then solvability holds for all $|\zeta| \leq 1$.*

Proof. Let

$$A_{\gamma,0} = S_0(\Lambda_\gamma - \Lambda_1), \qquad A_{\gamma,\zeta} = S_\zeta(\Lambda_\gamma - \Lambda_1).$$

Then by Lemma 4.2.1 the operator $I + A_{\gamma,0}$ is invertible in $H^{1/2}(\partial\Omega)$. Thus

$$\begin{aligned} I + A_{\gamma,\zeta} &= I + A_{\gamma,0} + (A_{\gamma,\zeta} - A_{\gamma,0}) \\ &= (I + A_{\gamma,0})(I + (I + A_{\gamma,0})^{-1}(A_{\gamma,\zeta} - A_{\gamma,0})) \end{aligned}$$

is invertible if

$$\|(I + A_{\gamma,0})^{-1}(A_{\gamma,\zeta} - A_{\gamma,0})\|_{\mathcal{B}(H^{1/2}(\partial B_a))} < 1.$$

From the decomposition $S_\zeta = S_0 + \mathcal{H}_\zeta$ it now follows from (4.9) that for a fixed $\gamma \in L^\infty_+(B_a)$

$$\begin{aligned} &\|(I + A_{\gamma,0})^{-1}(A_{\gamma,\zeta} - A_{\gamma,0})\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \\ &\quad = \|(I + A_{\gamma,0})^{-1}\mathcal{H}_\zeta(\Lambda_\gamma - \Lambda_1)\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \\ &\quad \le C|\zeta|^{n-2}\|(I + A_{\gamma,0})^{-1}\|_{\mathcal{B}(H^{1/2}(\partial B_a))}\|\Lambda_\gamma - \Lambda_1\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \\ &\quad < 1 \end{aligned} \tag{4.16}$$

for ζ sufficiently small.

If $\gamma - 1$ is sufficiently small then by Lemma 2.4.11, $I + A_{\gamma,0}$ can be inverted by a Neumann series which gives the estimate

$$\|(I + A_{\gamma,0})^{-1}\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \le \frac{1}{1 - \|S_0(\Lambda_\gamma - \Lambda_1)\|_{\mathcal{B}(H^{1/2}(\partial B_a))}}.$$

Hence from (4.16) and Lemma 2.4.11 we conclude that if ζ and γ satisfy the estimate

$$0 < |\zeta|^{n-2}\frac{C\|\gamma - 1\|_{L^\infty(B_a)}}{1 - C\|S_0\|_{\mathcal{B}(H^{1/2}(\partial B_a))}\|\gamma - 1\|_{L^\infty(B_a)}} < 1,$$

with C being the constant from Lemma 2.4.11, then ζ is not exceptional for γ. Furthermore, we conclude that if $\|\gamma - 1\|_{L^\infty(B_a)}$ is sufficiently small, then there are no exceptional points $\zeta \in \mathcal{V}$ with $|\zeta| \le 1$. □

From the invertibility of the boundary integral equation we get the following estimate of the exponentially growing solutions for small ζ.

Lemma 4.2.5. *Assume $\gamma \in L^\infty_+(B_a)$ satisfies $\gamma = 1$ near ∂B_a. Then for $\zeta \in \mathcal{V}$ sufficiently small*

$$\|\psi(\cdot, \zeta) - 1\|_{H^{3/2}(\partial B_a)} \le C|\zeta|. \tag{4.17}$$

Proof. Let $A_{\gamma,\zeta} = S_\zeta(\Lambda_\gamma - \Lambda_1)$. Then by Theorem 4.2.4 the operator $I + A_{\gamma,\zeta}$ is invertible for ζ near zero. Since $(\Lambda_\gamma - \Lambda_1)1 = 0$ we have

$$\psi(x, \zeta) - 1 = (I + A_{\gamma,\zeta})^{-1}(e^{ix\cdot\zeta} - 1). \tag{4.18}$$

Now $\|(I + A_{\gamma,\zeta})^{-1}\|_{\mathcal{B}(H^{1/2}(\partial B_a))}$ is uniformly bounded for small $|\zeta|$ and thus

$$\begin{aligned}\|\psi(\cdot,\zeta) - 1\|_{H^{1/2}(\partial B_a)} &\leq C\|e^{ix\cdot\zeta} - 1\|_{H^{1/2}(\partial B_a)} \\ &\leq C|\zeta|,\end{aligned}$$

where the last estimate can be proved by writing the Taylor expansion for $e^{i\zeta\cdot x}$ around zero and then using interpolation similarly to the proof of Lemma 4.2.3. To get (4.17) we note that $A_{\gamma,\zeta}$ is uniformly bounded from $H^{1/2}(\partial B_a)$ to $H^{3/2}(\partial B_a)$ for $|\zeta| < 1$. This implies that

$$\begin{aligned}&\|\psi(\cdot,\zeta) - 1\|_{H^{3/2}(\partial B_a)} \\ &\quad = \|e^{ix\cdot\zeta} - 1\|_{H^{3/2}(\partial B_a)} + \|A_{\gamma,\zeta}\|_{\mathcal{B}(H^{1/2}(\partial B_a),H^{3/2}(\partial B_a))}\|\psi(\cdot,\zeta) - 1\|_{H^{1/2}(\partial B_a)},\end{aligned}$$

which gives the result. □

The next step would now be to prove solvability of (2.36) for a general $\gamma \in L^\infty_+(B_a)$ and $\zeta \in \mathcal{V}$ without the smallness assumptions. This would require a decay estimate for S_ζ, when $|\zeta| \to \infty$, which does not seem to be available. However, when the conductivity is sufficiently regular, we know from Corollary 2.4.6 that there are no exceptional points for large ζ. On the other hand Theorem 4.2.4 shows the absence of exceptional points for small ζ. Balancing these two statements gives the following result.

Corollary 4.2.6. *Let $\gamma \in W^{2,\infty}(B_a)$ with $\gamma = 1$ near ∂B_a. Then there is a $\delta > 0$ depending only on the radius a, such that if with $\|\gamma - 1\|_{W^{2,\infty}(B_a)} < \delta$, then there are no exceptional points for γ.*

Proof. From Theorem 4.2.4 it follows that there is a δ_1, such that when $\|\gamma - 1\|_{L^\infty(B_a)} < \delta_1$ there are no exceptional points $\zeta \in \mathcal{V}$ with $|\zeta| \leq 1$. Furthermore, from Theorem 2.4.3 there is a constant δ_2 such that for $\|q\|_{L^\infty(B_a)} \leq \|\gamma - 1\|_{W^{2,\infty}(B_a)} < \delta_2$ there are no exceptional points with $\zeta \in \mathcal{V}$ with $|\zeta| \geq 1$. The result now follows by choosing $\delta = \min(\delta_1, \delta_2)$. □

4.3. Reconstruction of potentials and conductivities

In this section we will focus on the reconstruction issue of the inverse conductivity problem in higher dimensions. The methods we will outline all rely on the reduction of the conductivity equation to a Schrödinger equation, and hence we will generally assume that the conductivity $\gamma \in W^{2,\infty}(\Omega)$. First we will define the higher dimensional non-physical scattering transform and recall the $\overline{\partial}_\zeta$-equation satisfied by the exponentially growing solutions. Next we will review the two reconstruction algorithms available in the literature. Finally we turn our attention to the case when the conductivity is smooth and close to a constant. The absence of exceptional points for such conductivities enables us to give a new reconstruction algorithm, which can be seen as a direct analogue of the algorithm presented in section 3.2.

4.3.1. Scattering transform and the $\overline{\partial}_\zeta$-equation. Let $\zeta \in \mathcal{V}$ and assume it is not exceptional. Define for $\zeta \in \mathcal{V}$, which is not exceptional, and $\xi \in \mathbb{R}^n$ the non-physical scattering transform of the potential q by

$$\mathbf{t}(\xi, \zeta) = \int_\Omega e^{-ix\cdot(\xi+\zeta)} q(x)\psi(x, \zeta) dx, \tag{4.19}$$

where $\psi(x, \zeta)$ is the unique solution to (2.25). To turn the integral into an integral on the boundary we define for any $\xi \in \mathbb{R}^n$

$$\mathcal{V}_\xi = \{\zeta \in \mathcal{V} \ : \ (\xi + \zeta)^2 = |\xi|^2 + 2\xi \cdot \zeta = 0\}.$$

Then $e^{-ix\cdot(\xi+\zeta)}$ is harmonic if and only if $\zeta \in \mathcal{V}_\xi$, and when $\zeta \in \mathcal{V}_\xi$ is not exceptional we get by (2.13) the formula

$$\mathbf{t}(\xi, \zeta) = \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)}(\Lambda_q - \Lambda_0)\psi(\cdot, \zeta) d\sigma(x) \quad \xi \in \mathbb{R}^n, \zeta \in \mathcal{V}_\xi. \tag{4.20}$$

We note that for fixed $\xi \in \mathbb{R}^n$, the set $\mathcal{V}_\xi$ consists of those $\zeta \in \mathcal{V}$ with real part in the hyper plane $|\operatorname{Re}(\zeta) + \xi| = |\operatorname{Re}(\zeta)|$ and imaginary part in the hyper plane $\operatorname{Im}\zeta \cdot \xi = 0$. Hence $\mathcal{V}_\xi$ is a variety of complex dimensions $n-2$. Note that in two dimensions the choice $\xi = (-2k_1, 2k_2),\ k_1, k_2 \in \mathbb{R}$, implies that $\mathcal{V}_\xi = \{(k, ik), (\overline{k}, -i\overline{k})\},\ k = k_1 + ik_2 \in \mathbb{C}$, and that the scattering transform evaluated at this ξ and $\zeta = (k, ik)$ has exactly the form (3.12).

The usefulness of the scattering transform in relation to the inverse boundary value problem is that it can be computed from the Dirichlet-to-Neumann map as just like in two dimensions. For $\xi \in \mathbb{R}^n$ and $\zeta \in \mathcal{V}_\xi$ not exceptional, we can first solve (2.36) to find $\psi(\cdot, \zeta)|_{\partial\Omega}$, and then compute $\mathbf{t}(\xi, \zeta)$ from Λ_q by (4.20).

Consider now for fixed $x \in \mathbb{R}^n$, $\mu(x, \cdot)$ as a function on $\mathcal{V}$. In two dimensions $\mathcal{V}$ is parameterized by one complex variable, and the absence of exceptional points implies that $\mu(x, \cdot)$ can be seen as a function of one complex variable defined everywhere in the plane. This brings into play the classical tools from complex analysis. In higher dimensions the issue is more complicated, since $\mathcal{V}$ is a variety of complex dimension $n-1$, and there there may be exceptional points. Therefore $\mu(x, \cdot)$ is locally a function of $n-1$ complex variables and has singularities at exceptional points. However, away from the singularities μ is differentiable:

Let $\zeta \in \mathcal{V}$ and let $\mathcal{W}_\zeta = \{w \in \mathbb{C}^n : w \cdot \overline{\zeta} = 0\}$. Using a local coordinate chart on $\mathcal{V}$ it can be seen that $\mathcal{W}_\zeta$ is the tangent space to $\mathcal{V}$ in the point ζ.

Then for $f \in C^1(\mathcal{V})$ we define the $\overline{\partial}_\zeta$-derivative in the point ζ and direction $w \in \mathcal{W}_\zeta$ by

$$w \cdot \overline{\partial}_\zeta f(\zeta) = \sum_{j=1}^{n} w_j \overline{\partial}_{\zeta_j} f(\zeta).$$

Similarly, for $\xi \in \mathbb{R}^n, \zeta \in V_\xi$ we denote by $\mathcal{W}_{\xi,\zeta} = \{w \in \mathbb{C}^n : w \cdot \overline{\zeta} = w \cdot \xi = 0\}$ the tangent space to $\mathcal{V}_\xi$ in ζ, and then we define for $\xi \in \mathbb{R}^n$ and $f \in C^1(\mathcal{V}_\xi)$ the derivative in the point $\zeta \in \mathcal{V}_\xi$ in direction $w \in \mathcal{W}_{\xi,\zeta}$ by

$$w \cdot \overline{\partial}_\zeta f(\zeta) = \sum_{j=1}^{n} w_j \overline{\partial}_{\zeta_j} f(\zeta).$$

For μ and $\mathbf{t}$ it is now possible to state the following $\overline{\partial}_\zeta$-equations.

Lemma 4.3.1. *Assume that $\zeta \in \mathcal{V}$ is not exceptional. Then for $w \in \mathcal{V}_\zeta$ we have*

$$w \cdot \overline{\partial}_\zeta \mu(x,\zeta) = \frac{-1}{(2\pi)^{n-1}} \int_{B_\zeta} e^{ix\cdot\xi} \mathbf{t}(\xi,\zeta)\mu(x,\zeta+\xi)(w\cdot\xi)d\sigma(\xi), \tag{4.21}$$

where $B_\zeta = \{\xi \in \mathbb{R}^n : (\xi+\zeta)^2 = 0\}$ is the ball in the plane $\xi \cdot \operatorname{Im}\zeta = 0$ centered at $c = -\operatorname{Re}\zeta$ with radius $r = |\operatorname{Re}\zeta|$.

Furthermore, for $\xi \in \mathbb{R}^n, \zeta \in \mathcal{V}_\xi$ and $w \in \mathcal{W}_{\xi,\zeta}$ we have

$$w \cdot \overline{\partial}_\zeta \mathbf{t}(\xi,\zeta) = \frac{-1}{(2\pi)^{n-1}} \int_{B_\zeta} \mathbf{t}(\xi-\eta,\zeta+\eta)\mathbf{t}(\eta,\zeta)(w\cdot\eta)d\sigma(\eta). \tag{4.22}$$

for any $\xi \in \mathbb{R}^n, \zeta \in \mathcal{V}_\xi$.

Proof. See any of the references **[NA84, BC85, NK87, Nac88]** for the higher dimensional $\overline{\partial}_\zeta$-equations; the formulation here is taken from **[Nac88]**. □

Note that integrating by parts in (2.18) shows that $G_{\zeta+\xi}(x) = G_\zeta(x)$ for $\xi \in \mathbb{R}^n$ with $(\xi+\zeta)^2 = 0$, so from (2.25) we have that ζ is exceptional if and only if $\zeta + \xi \in \mathcal{V}$ is exceptional for $\xi \in \mathbb{R}^n$. Hence $\mu(x,\zeta+\xi)$ in the right hand side of (4.21) is well-defined on the domain of integration. The same argument shows that the scattering transform in the right hand side of (4.21) and (4.22) is well-defined and can be evaluated from boundary measurements by (4.20).

The $\overline{\partial}$-equation (4.22) for $\mathbf{t}$ is in fact an implicit characterization of the admissible set of scattering data, i.e. $\mathbf{t}$ is the scattering transform of some q if and only if it solves (4.22) **[BC85]**. In presence of noisy data it could be useful to project the scattering transform onto the admissible set defined by that equation before solving the inverse scattering problem of computing q from $\mathbf{t}$.

4.3.2. Reconstruction of potentials and conductivities. There are two methods available for reconstruction of the potential q from its scattering transform $\mathbf{t}$. Since (4.19) and (2.26) implies that

$$|\hat{q}(\xi) - \mathbf{t}(\xi, \zeta)| \leq \frac{C}{|\zeta|}, \tag{4.23}$$

we can directly reconstruct

$$\hat{q}(\xi) = \lim_{\substack{|\zeta| \to \infty \\ \zeta \in \mathcal{V}_\xi}} \mathbf{t}(\xi, \zeta), \tag{4.24}$$

and then q by inverting the Fourier transform. This method is already implicitly in the proof of Theorem 4.1.1 in **[SU87]** and given explicitly in **[NSU88, Nov88]**. This reconstruction procedure seems, however, impractical since it relies only on the large (complex) frequency information in Λ_γ. We note again that in two dimensions it is impossible to use (4.24) for reconstruction, since the set $\mathcal{V}_\xi$ consists of two isolated points only .

The second reconstruction method is based on the $\overline{\partial}_\zeta$-equation (4.22). This idea was suggested in a different context in **[NA84, LN87]**; a rigorous treatment was given by Nachman in **[Nac88]**. The method is based on a higher dimensional version of the generalized Cauchy formula, which gives a formula for $\hat{q}$ in terms of $\mathbf{t}$, $\overline{\partial}_\zeta \mathbf{t}$ and the asymptotic value (4.24) of $\mathbf{t}$. A suitable version due to Hatziafratis **[Hat86]** of this so-called Bochner-Martinelli formula gives the existence of a differential form K on $\mathcal{V}_\xi$ such that for $\xi \in \mathbb{R}^n,\ R > 0,\ f \in C^1(\mathcal{V}_\xi \setminus B(0, R))$ and $R' > R$

$$\begin{aligned} f(\zeta) = \int_{z \in \mathcal{V}_\xi, |z| = R'} f(z) K(z, \zeta) - \int_{z \in \mathcal{V}_\xi, |z| = R} f(z) K(z, \zeta) \\ + \int_{z \in \mathcal{V}_\xi, |z| > R} \bar{\partial}_z f(z) \wedge K(z, \zeta). \end{aligned} \tag{4.25}$$

For the exact definition of the form $K(z, \zeta)$ we refer to **[Nac88]**.

To avoid the appearance of exceptional points, the reconstruction formula is based only on a range of large $\zeta \in \mathcal{V}_\xi$, where by Theorem 2.4.6 it is known that there are no exceptional points. Choose $R > 0$ such that there are no exceptional points with $|\zeta| \geq R$ and consider this formula with $f(\zeta) = \mathbf{t}(\xi, \zeta)$. For $R' \to \infty$ the estimate (4.23) can be seen to imply that

$$\lim_{R \to \infty} \int_{z \in \mathcal{V}_\xi, |z| = R'} \mathbf{t}(\xi, z) K(z, \zeta) = \hat{q}(\xi).$$

The resulting formula is then

$$\hat{q}(\xi) = \mathbf{t}(\xi,\zeta) + \int_{z\in\mathcal{V}_\xi,|z|=R} \mathbf{t}(\xi,z)K(z,\zeta) - \int_{z\in\mathcal{V}_\xi,|z|>R} \bar{\partial}_z \mathbf{t}(\xi,z) \wedge K(z,\zeta), \tag{4.26}$$

and since the right hand side is known from the boundary measurements, this formula can be interpreted as a reconstruction formula for $\hat{q}$. We note that contrary to the method described in section 3.2 for the two-dimensional problem, the method (4.26) is not based on solving a $\bar{\partial}_\zeta$-equation.

The algorithm for reconstructing the conductivity $\gamma \in W^{2,\infty}(\Omega)$ now consists of the steps

(1) Solve the boundary integral equation (2.36) for $\psi(\cdot,\zeta)|_{\partial\Omega}$ for $\zeta \in \mathcal{V}$ sufficiently large.

(2) Calculate the scattering transform $\mathbf{t}$ by (4.20) and $\bar{\partial}_\zeta \mathbf{t}$ by (4.22) for $\xi \in \mathbb{R}^n$, $\zeta \in \mathcal{V}_\xi$.

(3) Calculate $\hat{q}$ by (4.24) or (4.22) and then q by inverting the Fourier transform.

(4) Solve (2.5) with $\gamma^{1/2}|_{\partial\Omega} = 1$ for $\gamma^{1/2}$.

4.3.3. Reconstruction of a conductivity close to constant. In this section we will show how to reconstruct a conductivity $\gamma \in C^\infty(\overline{\Omega})$ close to a constant from boundary data. This is a well-known result, however, the interesting point here is that the algorithm to be given is a direct analogue of the reconstruction algorithm for the two-dimensional problem given in section 3.2. The idea is first to compute the scattering transform from boundary measurements, then to solve the $\bar{\partial}$-equation (4.21) for μ, and finally to reconstruct the conductivity from μ by taking $\zeta \to 0$. A difficulty in dimension $n \geq 3$ is the possibility of having exceptional points. In order to avoid this we will assume that γ is sufficiently close to a constant such that there are no exceptional points by Corollary 4.2.6. The smallness assumption will also together with the smoothness enable us to solve (4.21). We will in this section without loss of generality assume that $\Omega = B_a$ and $\gamma = 1$ near $\partial\Omega$.

The next result concerns the solvability of (4.21).

Lemma 4.3.2. *For $q \in C^\infty(\overline{\Omega})$ sufficiently small and $x \in \Omega$ fixed, there is a unique solution $\mu(x,\cdot) \in C(\mathcal{V})$ to* (4.21) *with*

$$\lim_{|\zeta|\to\infty} \mu(x,\zeta) = 1. \tag{4.27}$$

Proof. We refer to [**BC85**, Theorem 4]. □

The proof of this theorem is based on the fact that smoothness in q is equivalent to decay of $\mathbf{t}$ and that smallness in q implies smallness of $\mathbf{t}$. Beals and Coifman do actually give a method for obtaining the unique solution μ to (4.21) subject to the asymptotic condition (4.27). Hence from the scattering transform $\mathbf{t}(\xi, \zeta)$, the exponentially growing solution $\psi(x, \zeta) = e^{ix\cdot\zeta}\mu(x, \zeta)$ can be computed.

Next we will show that a conductivity can be reconstructed from the exponentially growing solutions by taking the parameter ζ to zero. This is a consequence of the estimate on ∂B_a for the solution to the boundary integral equation.

Theorem 4.3.3. *Let $\zeta \in \mathcal{V}$ be sufficiently small and let $\psi(x, \zeta)$ be the unique exponentially growing solution. Then*

$$\|\psi(\cdot, \zeta) - \gamma^{1/2}\|_{H^2(B_a)} \leq C|\zeta|.$$

Proof. Since $\gamma = 1$ near $\partial\Omega$ we have by Lemma 4.2.5 that for ζ near zero

$$\|\psi(x, \zeta) - \gamma^{1/2}\|_{H^{3/2}(\partial B_a)} = \|\psi(x, \zeta) - 1\|_{H^{3/2}(\partial B_a)} \leq C|\zeta|.$$

Since $\psi(x, \zeta) - \gamma^{1/2}(x)$ is the unique solution to $(-\Delta + q)(\psi(x, \zeta) - \gamma^{1/2}) = 0$ in B_a elliptic regularity implies that

$$\|\psi(\cdot, \zeta) - \gamma^{1/2}(\cdot)\|_{H^2(B_a)} \leq C\|\psi(\cdot, \zeta) - \gamma^{1/2}(\cdot)\|_{H^{3/2}(\partial B_a)} \leq C|\zeta|.$$

$\square$

To sum up the proposed reconstruction algorithm for conductivities close to constant we outline the main steps:

(1) Solve (2.36) for $\psi(\cdot, \zeta), \zeta \in \mathcal{V}$.

(2) Calculate the scattering transform $\mathbf{t}$ by (4.20) for $\xi \in \mathbb{R}^n,\ \zeta \in \mathcal{V}_\xi$.

(3) Solve (4.21) for $\mu(x, \cdot),\ x \in \Omega$.

(4) Reconstruct γ from

$$\gamma^{1/2}(x) = \lim_{|\zeta| \to 0} \psi(x, \zeta).$$

We note that a generalization of this algorithm to conductivities, which are not close to one, requires the solution of two problems. First we would have to show absence of exceptional points for this class conductivities, next we would have to show solvability of (4.21).

4.4. Notes

In higher dimensions a major breakthrough in the theory of inverse boundary value problems was the paper **[SU87]**. There Sylvester and Uhlmann proved the existence of exponentially growing solutions and that Λ_γ determines $\gamma \in C^\infty(\overline{\Omega})$ when Ω is a smooth domain. The proof of Theorem 4.1.1 using exponentially growing solutions in (2.13) as presented here was given by Nachman, Sylvester, and Uhlmann [**NSU88**] for smooth domains. The smoothness of the boundary was then relaxed by Nachman [**Nac88**] to cover $C^{1,1}$ domains and Alessandrini [**Ale88**] to Lipschitz domains. We note that by using (2.47), Theorem 4.1.1 can be seen to be valid for $q \in L^p(\Omega),\ p > n/2$ when Ω is a Lipschitz domain.

The a priori assumption on γ for uniqueness to hold has since been relaxed beyond $W^{2,\infty}(\Omega)$ by a number of authors. The method of proof seems in all of the works to be based on the construction of exponentially growing solutions. Brown [**Bro96**] showed existence of exponentially growing solutions to the Schrödinger equation with a more singular potential coming from a conductivity. This gave uniqueness for conductivities in $\gamma \in C^{3/2+\epsilon}(\overline{\Omega})$. In a recent paper by Päivärinta, Panchenko, and Uhlmann [**PPU**] uniqueness was obtained for $\gamma \in C^{3/2}(\overline{\Omega})$. In their approach the conductivity equation is reduced to a Schrödinger equation with a magnetic potential instead of an electric potential, again the uniqueness proof is based on the construction of exponentially growing solutions. Finally in a recent preprint Brown and Torres [**BT02**] proved uniqueness for conductivities $\gamma \in W^{3/2,p}(\Omega), p > 2n$ improving the uniqueness result slightly. However, there seems to be no reason to believe that this result is optimal.

The solvability of the boundary integral equation for small ζ and $\gamma \in L^\infty(\Omega)$ is based on the joint work [**CKS02**]. The technique of proof is analogous to the two-dimensional case analyzed by Siltanen, Mueller, and Isaacson [**SMI00**]. Concerning the absence of exceptional points for small potentials, we note that Corollary 4.2.6 can also be deduced from (2.47) by taking $r = n/2$. A similar result was obtained by Beals and Coifman [**BC85**] in dimension $n = 3$ assuming that $q \in L^{3/2-\epsilon}(\mathbb{R}^3) \cap L^{3/2+\epsilon}(\mathbb{R}^3)$. For the inverse conductivity problem it seems natural to conjecture that if $\gamma \in W^{2,\infty}(\Omega)$ and $q = \Delta\gamma^{1/2}/\gamma^{1/2}$, then there are no exceptional points for q. This would extend the result in two dimensions (Theorem 3.2.3) to higher dimensions.

The reconstruction method outlined in section 4.3.2 is due to Nachman [**Nac88**] and Novikov [**Nov88**] independently. The method is based on the so-called $\bar\partial$-approach to inverse scattering, which was introduced by Beals and Coifman [**BC81**], [**BC82**] in the study of integrable evolution equations. The theory was developed further by a number of authors, see [**ABYF83, NA84, BC85, Nac88, NK87, New89**].

The idea of solving (4.21) for $\mu(x,\zeta)$ is not new. In [**BC85**] a small potential q is reconstructed by first solving (4.21) and then by plugging $\omega = \mu - 1$ into the equation (2.17). A similar approach is suggested in

[**NA84, NK87, Nov88**]. However, the idea of reconstruction the conductivity by taking the parameter to zero seems to be new. The estimates for small ζ (Lemma 4.2.5) and the direct reconstruction of γ close to constant are based on [**CKS02**].

Concerning stability for the higher dimensional problem we mention that Alessandrini [**Ale88, Ale90**] proved a conditional stability result using exponentially growing solutions. He proved that if $\gamma_i \in W^{2,\infty}(\Omega)$ with

$$0 < \theta < \gamma_i, \quad \|\gamma_i\|_{W^{2,\infty}(\Omega)} < M,$$

then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C\lambda(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\mathcal{B}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega)}),$$

where $\lambda(t) = |\log t|^{-\delta}$ for $0 < t < 1$, $\delta < 1$ depends on n, s, and C depends on M. Mandache showed by example recently in [**Man01**] that this stability result is optimal.

Chapter 5

Local uniqueness, stability, and reconstruction from two boundary measurements

This chapter concerns the question of inferring knowledge of a conductivity or a potential from a finite number of boundary measurements. When the boundary data is limited we cannot hope to recover an arbitrary coefficient, so we will have to restrict the set of admissible coefficients.

In the classical paper **[CDJ63]**, Cannon, Douglas and Jones considered the inverse conductivity problem for a class of conductivities homogeneous in one direction. More precisely let $\Omega = D \times [0, a] \subset \mathbb{R}^n$ for a bounded and smooth $D \subset \mathbb{R}^{n-1}$ and $a > 0$, and assume that the smooth conductivity γ is independent of the cylindrical variable, i.e. $\gamma(x', z) = \gamma(x', 0),\ x' \in D,\ 0 < z \leq a$. Then the result is that γ can be recovered from only one boundary measurement.

The fundamental result of this chapter can in some sense be seen as a generalization of the result of Cannon, Douglas and Jones to the Schrödinger equation and to a different geometry. Let $B_a = B(0, a) \subset \mathbb{R}^3$ be the ball centered at zero with radius $0 < a < \infty$, and let the potential q be homogeneous in the radial direction, i.e. $q \in L^2(\partial B_a)$ is extended inside B_a by

$q(x) = q(a\hat{x})$, where $\hat{x} = x/|x|$. We call such a potential a spherical potential. Consider the boundary value problem

$$(-\Delta + q)u = 0 \text{ in } B_a, \qquad u = f \text{ on } \partial B_a. \tag{5.1}$$

Assuming that zero is not a Dirichlet eigenvalue of $(-\Delta + q)$, the elliptic boundary value problem (5.1) has a unique weak solution $u \in H^1(B_a)$ for any $f \in H^{1/2}(\partial B_a)$ (see section 5.2 for further details on the existence and regularity of solutions). Moreover, the unique solution admits a normal derivative at the boundary $\partial_\nu u$ as defined in section 2.2.

The inverse problem can then be stated: assume that zero is not a Dirichlet eigenvalue for $(-\Delta + q)$ in B_a. Let I be a finite index set and let

$$\{(u_i, \partial_\nu u_i)|_{\partial B_a} \,:\, (-\Delta + q)u_i = 0\}_{i \in I} \subset H^{1/2}(\partial B_a) \times H^{-1/2}(\partial B_a)$$

be the data. Now, is q uniquely determined from this set of data, and in that case, how can it be reconstructed?

The main result to be presented in this chapter is that a sufficiently small spherical potential can be found from only two boundary measurements, i.e. by first applying $f_1 = 1$ on ∂B_a and measuring $\Lambda_q(1)$, then applying $f_2 = \Lambda_q 1$ and measuring $\Lambda_q^2(1)$. Note that for the class of spherical potentials reconstructing q on ∂B_a is equivalent to reconstructing q everywhere. Hence this inverse problem is fundamentally different from the inverse problems studied in Chapter 3 and Chapter 4.

In the following section we state the exact results and outline the method of proof. Then in the following sections the proofs of the theorems will be given. In section 5.2 we consider the regularity properties of solutions to (5.1). These results are well-known but included here for completeness. Then in section 5.3 we derive a fundamental equation relating the data and the potential, and in section 5.4 we see how this equation can be solved. This gives existence and uniqueness. Stability is then proved in section 5.5, and finally in section 5.6 we give an application to the inverse conductivity problem. The results presented in this chapter are based on a joint work with Horia Cornean [**CK02**].

5.1. Outline of the method and results

Our first result is an equation relating the data and the potential q. Let $-\Delta_D$ be the Friedrichs extension of $(-\Delta)|_{C_0^\infty(B_a)}$, which is selfadjoint on the domain $\mathcal{D}(-\Delta_D) = H_0^1(B_a) \cap H^2(B_a)$. For $q \in L^2(B_a)$ we can define $(-\Delta_D + q)$ as a selfadjoint operator on the same domain, and when zero is not an eigenvalue of this operator, $R_q = (-\Delta_D + q)^{-1}$ exists and is a smoothing operator of degree two, see section 5.2 below. Let now for $q \in L^2(B_a)$, M_q be the multiplication operator $M_q : \phi \mapsto q\phi$ defined in

$L^2(B_a)$ on the domain $\mathcal{D}(M_q) = \{\phi \in L^2(B_a) \mid q\phi \in L^2(B_a)\}$. Let further $\rho_0 : H^s(B_a) \to H^{s-1/2}(\partial B_a),\ s > 1/2$, be the usual trace operator. A fundamental result relating Λ_q and q is then

Proposition 5.1.1. *Let $q \in L^2(B_a)$ be spherical and assume that zero is not an eigenvalue of $(-\Delta_D + q)$. Then in $L^2(\partial B_a)$ we have*

$$q = \Lambda_q^2(1) + \frac{3}{a}\Lambda_q(1) + \frac{2}{a}\rho_0 \partial_r R_q M_q R_q(q). \tag{5.2}$$

The equation (5.2) is the fundamental equation in the solution of the inverse problem. Define the non-linear operator

$$F(p) = \frac{2}{a}\rho_0 \partial_r R_p M_p R_p(p) \tag{5.3}$$

for any spherical potential q for which zero is not an eigenvalue of $(-\Delta_D + q)$. Then the forward problem is defined by the operator

$$I - F : q \mapsto q_0 = \Lambda_q^2(1) + \frac{3}{a}\Lambda_q(1)$$

and the inverse problem concerns the inversion of this map. We will show that when q is sufficiently small then $I - F$ can be inverted. To control the size of the potentials, we introduce the set W_λ of spherical potentials q, which satisfy $\|q\|_{L^2(\partial B_a)} < \lambda$. This bound implies the norm bound on the whole domain $\|q\|_{L^2(B_a)} < (1/3a^3)^{1/2}\lambda$. In Lemma 5.2.4 below we show that when λ_1 is sufficiently small and $q \in W_{\lambda_1}$, then zero is not an eigenvalue of $(-\Delta_D + q)$, and therefore (5.2) holds.

In the inversion procedure the equation (5.2) is the equation to be solved. Assume that $q_0 = \Lambda_q^2(1) + \frac{3}{a}\Lambda_q(1)$ is given. Define the non-linear operator $T : W_{\lambda_1} \to L^2(\partial B_a)$ by

$$T(p) = q_0 + F(p). \tag{5.4}$$

We see from (5.2) that q is a fixed point for T. Now, to solve the inverse problem we will prove that for a $\lambda_0 \leq \lambda_1$ sufficiently small, T has a unique fixed point in the set W_{λ_0}, and moreover, that this fixed point can be found by iteration. The following theorem states the result:

Theorem 5.1.2. *There is a constant λ_0 depending only on the radius a, such that if $q \in W_{\lambda_0}$ and $q_0 = \Lambda_q^2(1) + \frac{3}{a}\Lambda_q(1)$, then* (5.4) *has a unique solution. Moreover, this solution can be found as the $L^2(\partial B_a)$ limit of the convergent sequence $\{T^n(0)\}_{n\in\mathbb{N}} \subset L^2(\partial B_a)$, i.e.*

$$q = \lim_{n\to\infty} T^n(0).$$

Not only can q be reconstructed uniquely but we can also easily prove the following stability result:

Theorem 5.1.3. *Let λ_0 be given from Theorem 5.1.2 and assume that $q_1, q_2 \in W_{\lambda_0}$. Let $q_0^{(i)} = \Lambda_{q_i}^2(1) + \frac{3}{a}\Lambda_{q_i}(1)$ for $i = 1, 2$ be the given data. Then the stability estimate holds*

$$\|q_1 - q_2\|_{L^2(\partial B_a)} \leq 2\|q_0^{(1)} - q_0^{(2)}\|_{L^2(\partial B_a)}.$$

The simplicity in the proofs given below relies heavily on the assumption that the potential is spherical. A very interesting open problem along this line is to determine a layered potential from boundary measurements. In the two-layered case the potential is assumed to have the form $q(x) = q_1(\hat{x}) + q_2(\hat{x})\chi_{B_{a_1}}(x)$ in B_a, where $0 < a_1 < a$. The problem is then to determine q_1, q_2 and a_1 from only a finite number of measurements. The solution of this problem and its generalization to a finite number of layers may lead to a layer stripping approach for solving the general problem with $q \in L^2(B_a)$. This is a project for further studies.

5.2. Regularity of solutions

In this section we collect some results concerning perturbations of $-\Delta_D$ and inversion of such perturbation operators. Then we use these results to prove regularity properties of solutions to boundary value problems. The results in this section do not require q to be spherical and the domain to be the sphere B_a.

We start out by defining $-\Delta_D + q$ rigorously as a selfadjoint operator:

Lemma 5.2.1. *For $q \in L^2(B_a)$ the operator $(-\Delta_D + q)$ in $L^2(B_a)$ is selfadjoint on the domain $\mathcal{D}(-\Delta_D + q) = \mathcal{D}(-\Delta_D) = H_0^1(B_a) \cap H^2(B_a)$.*

Furthermore, $(-\Delta_D + q)$ has discrete spectrum, and if zero is not an eigenvalue of $(-\Delta_D + q)$, then $R_q = (-\Delta_D + q)^{-1}$ is in $\mathcal{B}(L^2(B_a), H^2(B_a))$ and in $\mathcal{B}(H^{-1}(B_a), H^1(B_a))$.

Proof. We will prove that the multiplication operator $M_q : \phi \mapsto q\phi$ is operator bounded by $(-\Delta_D)$ with bound less than 1, and then apply the Kato-Rellich theorem (see [**Kat66**, Theorem 4.3]).

Since for $f \in L^\infty(B_a)$

$$\|qf\|_{L^2(B_a)} \leq \|q\|_{L^2(B_a)}\|f\|_{L^\infty(B_a)}$$

it follows for $\phi \in \mathcal{D}(-\Delta_D)$ and $\mu > 0$ that with $R_\mu = (-\Delta_D + \mu)^{-1}$

$$\begin{aligned}
&\|q\phi\|_{L^2(B_a)} \\
&\qquad \leq \|qR_\mu(-\Delta_D + \mu)\phi\|_{L^2(B_a)} \\
&\qquad \leq \|q\|_{L^2(B_a)}\|R_\mu\|_{\mathcal{B}(L^2(B_a),L^\infty(B_a))}\|(-\Delta_D + \mu)\phi\|_{L^2(B_a)} \\
&\qquad \leq \|q\|_{L^2(B_a)}\|R_\mu\|_{\mathcal{B}(L^2(B_a),L^\infty(B_a))}(\|-\Delta_D\phi\|_{L^2(B_a)} + \mu\|\phi\|_{L^2(B_a)}). \qquad (5.5)
\end{aligned}$$

Let $G_\mu(x,y)$ denote the Dirichlet Green's function for $(-\Delta_D + \mu)$ defined for fixed $x \in B_a$ as the unique solution to

$$(-\Delta_D + \mu)G_\mu(x,y) = \delta(x-y) \text{ in } B_a, \qquad G_\mu(x,y) = 0 \text{ on } \partial B_a.$$

Let further $\Phi_\mu(x,y) = e^{-\sqrt{\mu}|x-y|}(4\pi|x-y|)^{-1}$ be the standard fundamental solution. Since $\Phi_\mu(x,y) - G_\mu(x,y)$ is harmonic on B_a and hence bounded, $G_\mu(x,y)$ is positive near the singularity x. By the maximum principle applied to $G_\mu(x,y)$ in $B_a \setminus B(x,\epsilon),\ x \in B_a, \epsilon > 0$ then shows, that $G_\mu(x,y)$ is positive everywhere. Another application of the maximum principle to $\Phi_\mu(x,y) - G_\mu(x,y)$ gives for $x, y \in B_a,\ x \neq y$ the estimate

$$G_\mu(x,y) < \frac{e^{-\sqrt{\mu}|x-y|}}{4\pi|x-y|}.$$

This shows that

$$\begin{aligned}
\|R_\mu\|_{\mathcal{B}(L^2(B_a),L^\infty(B_a))} &= \sup_{f\in L^2(B_a),\ \|f\|_{L^2(B_a)}=1} \left\| \int_{B_a} G_\mu(x,y)f(y)dy \right\|_{L^\infty(B_a)} \\
&\leq \sup_{x\in B_a} \left(\int_{B_a} |G_\mu(x,y)|^2 dy \right)^{1/2} \\
&\leq \sup_{x\in B_a} \frac{1}{4\pi} \left(\int_{B_a} \frac{e^{-2\sqrt{\mu}|x-y|}}{|x-y|^2} dy \right)^{1/2} \\
&\leq \left(\int_0^{2a} e^{-2\sqrt{\mu}r} dr \right)^{1/2} \\
&\leq C\mu^{-1/4},
\end{aligned}$$

and by (5.5) we find that $\phi \mapsto q\phi$ is $(-\Delta_D)$-bounded, i.e. for $\phi \in \mathcal{D}(-\Delta_D)$

$$\|q\phi\|_{L^2(B_a)} \leq b\|(-\Delta_D)\phi\|_{L^2(B_a)} + c\|\phi\|_{L^2(B_a)}, \tag{5.6}$$

where $b = C\mu^{-1/4}\|q\|_{L^2(B_a)}, c = C\mu^{3/4}\|q\|_{L^2(B_a)}$. The choice $\mu > C\|q\|^4_{L^2(B_a)}$ implies that $b < 1$, and then it follows from the Kato-Rellich theorem that $-\Delta_D + q$ is well-defined in $L^2(B_a)$ and selfadjoint on $\mathcal{D}(-\Delta_D)$.

It is well-known that $(-\Delta_D + q)$ has a purely discrete spectrum. When zero is not an eigenvalue the inverse R_q exists as a bounded operator on $L^2(B_a)$. The smoothing property of R_q is an easy consequence of the mapping properties of $(-\Delta_D)^{-1}$. $\square$

To solve the problem (5.1) we apply the standard procedure of transforming the problem into a problem with a source term and zero boundary condition. This reduction relies on well-known properties of solutions to the Laplace equation:

Proposition 5.2.2. *For any $f \in H^{1/2+s}(\partial B_a)$, $s \geq 0$, there is a unique solution $v \in H^{1+s}(B_a)$ to*

$$-\Delta v = 0 \text{ in } B_a, \qquad v = f \text{ on } \partial B_a. \tag{5.7}$$

Proof. See for instance **[GT83]** for a proof. □

We can now prove the result

Proposition 5.2.3. *Assume zero is not an eigenvalue of $(-\Delta_D + q)$. Then for $s = 0, 1$ there is for any $f \in H^{1/2+s}(\partial B_a)$ a unique solution $u \in H^{1+s}(B_a)$ to* (5.1).

Proof. Let $v \in H^{1+s}(B_a)$ solve (5.7) and introduce $w = u - v$. Then w solves $(-\Delta + q)w = -(qv)$ in B_a and vanishes on ∂B_a, and formally we define $w = u - v = -R_q(qv)$. For $f \in H^{1/2+s}(\partial B_a)$ Proposition 5.2.2 gives that $v \in H^{1+s}(B_a)$. Hence the result follows from the mapping properties of R_q given in Lemma 5.2.1 provided that $qv \in H^{-1+s}(B_a)$. For $s = 1$ the Sobolev embedding $H^2(B_a) \subset L^\infty(B_a)$ gives that $qv \in L^2(B_a)$. For $s = 0$ the Sobolev embedding $H^1(B_a) \subset L^6(B_a)$ and Hölder's inequality implies that $qv \in L^{3/2}(B_a)$. By the Sobolev embedding $qv \in H^{-1/2}(B_a) \subset H^{-1}(B_a)$ we get the result.

Uniqueness follows from the injectivity of R_q. □

It is well-known that the spectrum of $(-\Delta_D)$ is positive. When q is small this property is inherited by $(-\Delta_D + q)$. In particular zero cannot be an eigenvalue. We will give the easy proof of this claim.

Lemma 5.2.4. *There is a constant C_a depending only on a such that if $\lambda_1 = (C_a \|(-\Delta_D)^{-1}\|_{\mathcal{B}(L^2(B_a),H^2(B_a))})^{-1}$ and $q \in W_{\lambda_1}$, then zero is not an eigenvalue of $(-\Delta_D + q)$.*

Proof. To prove the result, we write in $\mathcal{D}(-\Delta_D)$

$$(-\Delta_D + q) = (1 + q(-\Delta_D)^{-1})(-\Delta_D), \tag{5.8}$$

and hence formally

$$R_q = (-\Delta_D + q)^{-1} = (-\Delta_D)^{-1}(1 + q(-\Delta_D)^{-1})^{-1}.$$

Since $(-\Delta_D)^{-1}$ is a smoothing operator of degree two, the result follows if $(1 + q(-\Delta_D)^{-1})$ is invertible in $L^2(B_a)$.

The inverse $(1 + q(-\Delta_D)^{-1})$ is given by a convergent Neumann series if

$$\|q(-\Delta_D)^{-1}\|_{\mathcal{B}(L^2(B_a),L^2(B_a))} < 1.$$

By the estimates

$$\begin{aligned}\|q(-\Delta_D)^{-1}\|_{\mathcal{B}(L^2(B_a),L^2(B_a)))} &\leq \|q\|_{L^2(B_a)}\|(-\Delta_D)^{-1}\|_{\mathcal{B}(L^2(B_a),L^\infty(B_a))}\\ &\leq a\|q\|_{L^2(\partial B_a)}\|(-\Delta_D)^{-1}\|_{\mathcal{B}(L^2(B_a),L^\infty(B_a))}\\ &\leq C_a\|q\|_{L^2(\partial B_a)}\|(-\Delta_D)^{-1}\|_{\mathcal{B}(L^2(B_a),H^2(B_a))},\end{aligned}$$

where the last estimate follows from the Sobolev embedding theorem and C_a is a constant depending on a, the result is obtained when $\|q\|_{L^2(\partial B_a)} < \lambda_1 = (C_a\|(-\Delta_D)^{-1}\|_{\mathcal{B}(L^2(B_a),H^2(B_a))})^{-1}$. □

Note that from the convergent Neumann series we can get the estimate

$$\|R_q\|_{\mathcal{B}(L^2(B_a),H^2(B_a))} \leq (\|(-\Delta_D)^{-1}\|^{-1}_{\mathcal{B}(L^2(B_a),H^2(B_a))} - C_a\|q\|_{L^2(B_a)})^{-1}, \tag{5.9}$$

provided $\|q\|_{L^2(B_a)} < (C_a\|(-\Delta_D)^{-1}\|_{\mathcal{B}(L^2(B_a),H^2(B_a))})^{-1}$.

5.3. An equation on the boundary

In this section we derive the equation (5.2) for q. The idea is to establish a relation between $\Lambda_q^2(f)$ and the second order derivative $\partial_r^2 u|_{\partial B_a}$ of the solution to (5.1), and then use the partial differential equation for u to express $\partial_r^2 u|_{\partial B_a}$ as a sum of lower order terms in the special case $f = 1$.

The details will be given in two lemmas:

Lemma 5.3.1. *Assume zero is not an eigenvalue of* $(-\Delta_D + q)$*. Then for any* $f \in H^{3/2}(B_a)$ *we have the relation*

$$\Lambda_q^2(f) - \rho_0\partial_r^2 u = \frac{2}{a}\Lambda_0(f) - \frac{1}{a}\Lambda_q f - \frac{2}{a}\rho_0\partial_r R_q M_q R_q(qv), \tag{5.10}$$

where u, v *are solutions to* (5.1) *and* (5.7) *respectively.*

Note that when $f \in H^{3/2}(\partial B_a)$, then the right hand side in (5.10) is in $H^{1/2}(\partial B_a)$ and hence the $H^{-1/2}(\partial B_a)$ singularity of the left hand side cancels.

Proof. Introduce the operator $A = (\frac{x}{a} \cdot \nabla)$ and note that $\rho_0 A = \rho_0\partial_r$ on the boundary ∂B_a. Hence $\rho_0 Au = \Lambda_q f \in H^{1/2}(\partial B_a)$, where $u \in H^2(B_a)$ is the solution to (5.1) according to Proposition 5.2.3. To get an equation for Au we apply A to $(-\Delta + q)u$ in the distributional sense. By the definition of A we get for $\phi \in \mathcal{D}(B_a)$ that

$$\langle A(qu), \phi\rangle = -\frac{3}{a}\langle qu, \phi\rangle - \langle qu, A\phi\rangle.$$

Now since q is spherical, an integration in spherical coordinates gives that

$$\begin{aligned}\langle qu, A\phi\rangle &= \int_{B_a} quA\phi \\ &= \int_0^{2\pi}\int_0^{\pi/2} q(\hat{x})\int_0^a u(x)\frac{r^3}{a}\partial_r\phi(x)dr\sin\psi d\psi d\theta \\ &= -\int_0^{2\pi}\int_0^{\pi/2} q(\hat{x})\int_0^a \left(\frac{r^3}{a}\partial_r u(x) + u(x)\frac{3r^2}{a}\right)\phi(x)dr\sin\psi d\psi d\theta \\ &= -\langle qAu, \phi\rangle - \frac{3}{a}\langle qu, \phi\rangle,\end{aligned}$$

which shows that

$$Aqu = qAu.$$

Moreover, an easy calculation shows that

$$[A, -\Delta] = \frac{2}{a}\Delta.$$

Hence Au solves the boundary value problem

$$(-\Delta + q)Au = -\frac{2}{a}\Delta u = -\frac{2}{a}qu \text{ in } B_a, \qquad Au = \Lambda_q f \text{ on } \partial B_a.$$

Let $\tilde{u} \in H^1(B_a)$ be the unique solution to $(-\Delta + q)\tilde{u} = 0$ in B_a with $\tilde{u}|_{\partial B_a} = \Lambda_q f$. The function $w = \tilde{u} - Au \in H_0^1(B_a)$ is then a solution to $(-\Delta + q)w = \frac{2}{a}qu$, and since $q \in L^2(B_a)$ and $u \in H^2(B_a) \subset L^\infty(B_a)$ by the Sobolev embedding theorem, the product $qu \in L^2(B_a)$. Hence we can invert the equation to find

$$\tilde{u} - Au = \frac{2}{a}R_q(qu) \in H^2(B_a). \tag{5.11}$$

Apply now $\rho_0\partial_r$ to this equation and note that

$$\begin{aligned}\rho_0\partial_r\tilde{u} &= \Lambda_q^2(f), \\ \rho_0\partial_r Au &= \rho_0\partial_r\frac{r}{a}\partial_r u = \rho_0\partial_r^2 u + \frac{1}{a}\Lambda_q f.\end{aligned} \tag{5.12}$$

Furthermore, recall from the proof of Proposition 5.2.3 that u can be expressed in terms of the solution to the free problem by $u = -R_q(qv) + v$. Hence

$$\begin{aligned}R_q(qu) &= R_q(M_q v - M_q R_q(qv)) \\ &= v - u - R_q M_q R_q(qv).\end{aligned}$$

By applying $\rho_0\partial_r$ to this expression and combining this with (5.11) and (5.12) we find

$$\Lambda_q^2(f) - \rho_0\partial_r^2 u - \frac{1}{a}\Lambda_q f = \frac{2}{a}(\Lambda_0(f) - \Lambda_q f) - \frac{2}{a}\rho_0\partial_r R_q M_q R_q(qv)$$

from which the result follows. □

For the special choice of boundary field $f = 1$ we are able to remove the $\rho_0\partial_r^2 u$ term from (5.10).

Lemma 5.3.2. *Let $u_1 \in H^2(B_a)$ be the solution to* (5.1) *with $f = 1$. Then*

$$\rho_0\partial_r^2 u_1 = q - \frac{2}{a}\Lambda_q 1. \tag{5.13}$$

Note that $\rho_0\partial_r^2 u_1$ a priori is no better than $H^{-1/2}(\partial B_a)$, but the lemma implies that $\rho_0\partial_r^2 u_1 \in L^2(\partial B_a)$.

Proof. Write the Laplace operator in spherical coordinates

$$\Delta = \partial_r^2 + \frac{2}{r}\partial r + \frac{1}{r^2}L_{\theta,\phi}$$

where

$$L_{\theta,\phi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{(\partial\phi)^2}.$$

Hence

$$\begin{aligned}
0 &= \rho_0(-\Delta + q)u_1 \\
&= \rho_0(-\partial_r^2 - \frac{2}{r}\partial r - \frac{1}{r^2}L_{\theta,\phi} + q)u_1 \\
&= -\rho_0\partial_r^2 u_1 - \frac{2}{a}\Lambda_q 1 + q,
\end{aligned}$$

since $\rho_0 L_{\theta,\phi}u_1 = L_{\theta,\phi}\rho_0 u_1 = L_{\theta,\phi}1 = 0$. □

Proof of Proposition 5.1.1. The result follows from (5.10) and (5.13) since $v = 1$ is the unique harmonic function with $v|_{\partial B_a} = 1$ and therefore $\Lambda_0(1) = 0$. □

We note that when q is sufficiently small, it follows from Lemma 5.2.4 that (5.2) holds.

5.4. Solving the boundary integral equation

In this section the proof of Theorem 5.1.2 will be given. The first step is the following lemma concerning the contraction properties of the operator F defined in (5.3):

Lemma 5.4.1. *There is a $\lambda_2 > 0$ depending only on the radius a such that if $p, r \in W_{\lambda_2}$, then*

$$\|F(p) - F(r)\|_{L^2(\partial B_a)} \leq \frac{1}{2}\|p - r\|_{L^2(\partial B_a)}.$$

Proof. From Lemma 5.2.4 we can find λ_1 such that if $q \in W_{\lambda_1}$, then $R_q = (-\Delta_D + q)^{-1} \in \mathcal{B}(L^2(B_a), H^2(B_a))$. Then for $\lambda \leq \lambda_1$ and $p, r \in W_\lambda$ the operator F is well-defined, and we have

$$\begin{aligned} F(p) - F(r) &= \frac{2}{a}\rho_0 \partial_r (R_p M_p R_p p - R_r M_r R_r r) \\ &= \frac{2}{a}\rho_0 \partial_r (R_p M_p R_p (p - r) + R_p M_p (R_p - R_r) r \\ &\qquad + R_p M_{p-r} R_r r + (R_p - R_r) M_r R_r r). \end{aligned}$$

Since the trace theorem for this particular geometry has the explicit form

$$\|u\|_{L^2(\partial B(0,a))} \leq (\frac{2}{a} + 1)\|u\|_{H^1(B(0,a))},$$

we see by the triangle inequality that

$$\begin{aligned} &\|F(p) - F(r)\|_{L^2(\partial B_a)} \leq \|\frac{2}{a}\rho_0 \partial_r (R_p M_p R_p p - R_r M_r R_r r)\|_{L^2(\partial B_a)} \\ &\quad \leq C_a \|R_p M_p R_p p - R_r M_r R_r r\|_{H^2(B_a)} \\ &\quad \leq C_a (\|R_p M_p R_p (p - r)\|_{H^2(B_a)} + \|R_p M_p (R_p - R_r) r\|_{H^2(B_a)} \\ &\qquad + \|R_p M_{p-r} R_r r\|_{H^2(B_a)} + \|(R_p - R_r) M_r R_r r\|_{H^2(B_a)}) \\ &\quad \leq C_a \Bigg(\|R_p - R_r\|_{\mathcal{B}(L^2(B_a), H^2(B_a))} \sup_{z \in W_\lambda} \left(\|R_z\|_{\mathcal{B}(L^2(B_a), H^2(B_a))} \right) \\ &\qquad \cdot \sup_{z \in W_\lambda} \left(\|z\|^2_{L^2(B_a)} \right) \\ &\qquad + \|p - r\|_{L^2(B_a)} \sup_{z \in W_\lambda} \left(\|R_z\|^2_{\mathcal{B}(L^2(B_a), H^2(B_a))} \right) \sup_{z \in W_\lambda} \left(\|z\|_{L^2(B_a)} \right) \Bigg). \end{aligned}$$

The resolvent identity $R_p - R_r = -R_p M_{p-r} R_r$ now implies

$$\|R_p - R_r\|_{\mathcal{B}(L^2(B_a), H^2(B_a))} \leq \|r - p\|_{L^2(B_a)} \sup_{z \in W_\lambda} \left(\|R_z\|^2_{\mathcal{B}(L^2(B_a), H^2(B_a))} \right).$$

Hence

$$\|F(p) - F(r)\|_{L^2(B_a)} \leq C_a C(\lambda) \|p - r\|_{L^2(B_a)},$$

where

$$C(\lambda) = \max_{k=0,1} \sup_{z \in W_\lambda} \left(\|R_z\|^{2+k}_{\mathcal{B}(L^2(B_a),H^2(B_a))} \right) \sup_{z \in W_\lambda} \left(\|z\|^{1+k}_{L^2(B_a)} \right).$$

Since by (5.9) implies that $\|R_z\|_{\mathcal{B}(L^2(B_a),H^2(B_a))}$ is uniformly bounded for $z \in W_\lambda, \lambda < \lambda_1$, the result now follows by taking $\lambda_2 < \lambda_1$ such that $C(\lambda_2) \leq C_a^{-1}/2$. □

Proof of Theorem 5.1.2. Let λ_2 be given from Lemma 5.4.1. For the uniqueness, assume $q_1, q_2 \in W_{\lambda_2}$ are fixed points for T. Then from Lemma 5.4.1 we have

$$\begin{aligned} \|q_1 - q_2\|_{L^2(\partial B_a)} &= \|T(q_1) - T(q_2)\|_{L^2(\partial B_a)} \\ &= \|F(q_1) - F(q_2)\|_{L^2(\partial B_a)} \\ &\leq \frac{1}{2}\|q_1 - q_2\|_{L^2(\partial B_a)} \end{aligned}$$

which implies $\|q_1 - q_2\|_{L^2(\partial B_a)} = 0$.

Concerning the convergence of $\{T^n(0)\}_{n\in\mathbb{N}}$ we note that if $q_0 = 0$, then the existence of a fixed point follows by Schauder's fixed point theorem [**Eva98**, section 9.2.2], since F (and hence T) is a contraction in W_{λ_2} by Lemma 5.4.1. In case $q_0 \neq 0$ we cannot immediately use a general fixed point theorem, so we will have to rely on the smallness assumption. For $q \in W_{\lambda_1}$ it follows from (5.2) that

$$\begin{aligned} &\|\Lambda_q^2(1) + \frac{3}{a}\Lambda_q 1\|_{L^2(\partial B_a)} \\ &\qquad\qquad \leq C_a(1 + \|R_q\|_{\mathcal{B}(L^2(B_a),H^2(B_a))}\|^2 \|q\|_{L^2(\partial B_a)})\|q\|_{L^2(\partial B_a)}, \end{aligned}$$

where C_a is a constant depending only on a. Hence there is a constant $\lambda_0 \leq \lambda_2 \leq \lambda_1$ such that if $q \in W_{\lambda_0}$, then $q_0 = T(0) = \Lambda_q^2(1) + \frac{3}{a}\Lambda_q 1 \in W_{\lambda_2/2}$. An induction argument now shows that if

$$\|T^{m-1}(0)\|_{L^2(\partial B_a)} \leq \sum_{k=1}^{m-1} \left(\frac{1}{2}\right)^k \lambda_2,$$

which in particular implies that $T^{m-1}(0) \in W_{\lambda_2}$, then by Lemma 5.4.1

$$\begin{aligned} \|T^m(0)\|_{L^2(\partial B_a)} &\leq \|F(T^{m-1}(0)) - F(T^{m-2}(0))\|_{L^2(\partial B_a)} + \|T^{m-1}(0)\|_{L^2(\partial B_a)} \\ &= \sum_{k=1}^{m} \left(\frac{1}{2}\right)^m \lambda_2. \end{aligned}$$

By the same type of argument we find that

$$\begin{aligned}\|T^{m+1}(0) - T^m(0)\|_{L^2(\partial B_a)} &\le \frac{1}{2}\|T^m(0) - T^{m-1}(0)\|_{L^2(\partial B_a)} \\ &\le \frac{1}{2^{m+1}}\|T(0)\|_{L^2(\partial B_a)}\end{aligned}$$

and hence

$$\begin{aligned}\|T^{m+k}(0) - T^m(0)\|_{L^2(\partial B_a)} &\le \sum_{j=1}^{k} \|T^{m+j}(0) - T^{m+(j-1)}(0)\|_{L^2(\partial B_a)} \\ &\le \sum_{j=1}^{k} \frac{1}{2^{m+j+1}}\|T(0)\|_{L^2(\partial B_a)}.\end{aligned}$$

From this expression it follows that $\{T^n(0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\partial B_a)$, and hence the sequence converges to a $p \in W_{\lambda_2}$. Since

$$\begin{aligned}\|p - T(p)\|_{L^2(\partial B_a)} &\le \lim_{n\to\infty}(\|p - T^n(0)\|_{L^2(\partial B_a)} + \|T^n(0) - T(p)\|_{L^2(\partial B_a)}) \\ &\le \lim_{n\to\infty}\frac{3}{2}\|p - T^n(0)\|_{L^2(\partial B_a)} \\ &= 0,\end{aligned}$$

we see that p is indeed a fixed point for T. □

5.5. Stability

Proof of Theorem 5.1.3. In the inversion procedure we find q_1 and q_2 as unique fixed points for the operators $T^{(1)}, T^{(2)}$ respectively, where

$$T^{(i)}(p) = q_0^{(i)} + F(p),\ i = 1, 2,$$

for $F(p) = \frac{2}{a}\rho_0\partial_r R_p M_p R_p p$. Hence

$$\begin{aligned}\|q_1 - q_2\|_{L^2(\partial B_a)} &= \|T^{(1)}(q_1) - T^{(2)}(q_2)\|_{L^2(\partial B_a)} \\ &\le \|q_0^{(1)} - q_0^{(2)}\|_{L^2(\partial B_a)} + \|F(q_1) - F(q_2)\|_{L^2(\partial B_a)} \\ &\le \|q_0^{(1)} - q_0^{(2)}\|_{L^2(\partial B_a)} + \frac{1}{2}\|q_1 - q_2\|_{L^2(\partial B_a)},\end{aligned}$$

since $q_1, q_2 \in W_\lambda$. From this inequality the stability result follows. □

5.6. Application to the conductivity problem

We would like to apply the results obtained concerning the spherical potentials to the inverse conductivity problem. To fix notation we denote by S_λ the set

$$S_\lambda = \{\gamma \in H^2(B_a) \cap L^\infty_+(B_a) \ :\ \gamma^{-1/2}\Delta\gamma^{1/2} \in W_\lambda\}.$$

Note that since q is assumed to be spherical in the definition of S_λ, then from the equation $(-\Delta + q)\gamma^{1/2} = 0$ it can be that γ is actually depending on both the radial and spherical variables. The following lemma shows that S_λ is not empty.

Lemma 5.6.1. *For $\lambda < \lambda_0$ sufficiently small the set S_λ is not empty.*

Proof. For any harmonic function v the function

$$u = -(-\Delta_D + q)^{-1}(vq) + v$$

satisfies $(-\Delta + q)u = 0$. Let $v = c > 0$ and note that

$$u \geq -c\|R_q\|_{\mathcal{B}(L^2(B_a),H^2(B_a))}\|q\|_{L^2(B_a)} + c.$$

Now $q \in W_\lambda$ implies that $M = \|R_q\|_{\mathcal{B}(L^2(B_a),H^2(B_a))}\|q\|_{L^2(B_a)} < 1$, and hence

$$u \geq c(1 - M) > 0,$$

which shows that $u \in S_\lambda$. □

Now the the following corollary to Theorem 5.1.2 can be obtained:

Corollary 5.6.2. *Let $\gamma \in S_{\lambda_0}$ be a conductivity and let $\gamma, \partial_\nu\gamma$ be given at the boundary ∂B_a. Then γ can be reconstructed from the two boundary measurements $\Lambda_\gamma(\gamma^{-1/2})$ and $\Lambda_\gamma(1/2\gamma^{-3/2}\partial_\nu\gamma + \gamma^{-1}\Lambda_\gamma(\gamma^{-1/2}))$.*

Proof. Since $\gamma \in S_{\lambda_0}$ we have the potential $q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}} \in W_{\lambda_0}$. Moreover, since $\gamma, \partial_\nu\gamma$ are known on ∂B_a we know from the boundary measurements by (2.9) the quantities

$$\begin{aligned}
\Lambda_q 1 &= \gamma^{-1/2}\left(\frac{1}{2}\partial_\nu\gamma + \Lambda_\gamma\right)(\gamma^{-1/2}) \\
\Lambda_q^2(1) &= \gamma^{-1/2}\left(\frac{1}{2}\partial_\nu\gamma + \Lambda_\gamma\right)\left(\frac{1}{2}\gamma^{-3/2}\partial_\nu\gamma + \gamma^{-1}\Lambda_\gamma(\gamma^{-1/2})\right).
\end{aligned}$$

Hence q can be reconstructed according to Theorem 5.1.2 and then γ can be found by solving the equation $(-\Delta + q)\gamma^{1/2} = 0$ with the known boundary data $\gamma^{1/2}|_{\partial B_a}$. □

5.7. Notes

Another way of restricting the class of potentials in the inverse conductivity problem with a finite number of measurements is by assuming that the conductivity has the form $\gamma(x) = 1 + \chi_D(x)$ in the domain Ω, where χ_D is the characteristic function on the inclusion $D \subset \overline{D} \subset \Omega$. The problem is then to determine the inclusion D. There are in the literature uniqueness results available for certain classes of domains including polygons in the plane, convex polyhedra in three dimensions, and balls in two and three dimensions. We refer to [**KS00**] and [**Ike00**] for this problem. For the inverse problem for the Schrödinger operator a similar structure of the potential is considered in [**KKY01**], where uniqueness for balls in two and three dimensions is shown.

We note also that in the recent paper [**Ber02**], Berntsen considered the related inverse scattering problem for the Schrödinger equation having a potential of the form $q(x) = 1 + \sum_{j=1}^{N} k_j \chi_{D_j}(x)$, where the location of the N disjoint scatterers D_j is known and the constants k_j are unknown. The result is then that the constants k_j are uniquely determined from one scattering experiment.

Appendix A

Notation

In this appendix we will review the notation used in the thesis. The definitions are standard but included here for completeness and reference.

Spaces. We will denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space and by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. Furthermore for any measurable set $\Omega \subset \mathbb{R}^n$ we denote by $\mathcal{D}(\Omega)$ the space consisting of smooth functions supported inside Ω and by $\mathcal{D}'(\Omega)$ the space of distributions. The spaces of distributions are in particular relevant when taken derivatives, and we will always understand derivatives as defined in the weak sense on $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{D}'(\Omega)$.

Let $\Omega \subset \mathbb{R}^n$ be a measurable set and let the Lebesgue space $L^p(\Omega), 1 \leq p < \infty$ be the space of (equivalence classes of) measurable functions in Ω with

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{1/p} < \infty,$$

and let $L^\infty(\Omega)$ be the space of essentially bounded measurable functions equipped with

$$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}(u) < \infty.$$

With the norm $\| \cdot \|_{L^p(\Omega)}$, $L^p(\Omega)$, $1 \leq p \leq \infty$ is a Banach space.

For $m \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$ we denote by $W^{m,p}(\Omega)$ the L^p-based Sobolev space of order m consisting of functions $u \in L^p(\Omega)$ with

$$D^\alpha u \in L^p(\Omega)$$

for any multi-index α with $|\alpha| < m$. $W^{m,p}(\Omega)$ is a Banach space with norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p} < \infty.$$

We write $H^m(\Omega) = W^{m,2}(\Omega)$. Note that $W^{m,p}(\mathbb{R}^n)$ coincides with the Sobolev space $H^m_p(\mathbb{R}^n)$ defined by the Fourier transform. The completion of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ is denoted $H^1_0(\Omega)$ and consists of the elements in $H^1(\Omega)$ that vanishes on the boundary $\partial\Omega$.

Denote by $\mathcal{M}_2$ the vector space of complex 2×2 matrices. Then we denote by $L^p(\mathbb{R}^n; \mathcal{M}_2)$ the L^p-space consisting of functions defined on $\mathbb{R}^n$ with values in $\mathcal{M}_2$ having each matrix-element in $L^p(\mathbb{R}^n)$. The natural norm on this space is

$$\|u\|_{L^p(\mathbb{R}^n;\mathcal{M}_2)} = \sum_{i,j=1}^{2} \|u_{ij}\|_{L^p(\mathbb{R}^n)}.$$

Let $L^p_\delta(\mathbb{R}^n)$ be the weighted L^p-space

$$L^p_\delta(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) \mid \langle x \rangle^\delta u \in L^p(\mathbb{R}^n)\}$$

with norm

$$\|u\|_{L^p_\delta(\mathbb{R}^n)} = \|\langle x \rangle^\delta u\|_{L^p(\mathbb{R}^n)}.$$

Denote by $W^{s,p}_\delta(\mathbb{R}^n)$ the natural weighted Sobolev space. When $p = 2$ we write $H^s_\delta(\mathbb{R}^n) = W^{s,2}_\delta(\mathbb{R}^n)$.

In the context of L^p-spaces we use the notation p' for the conjugate exponent of p defined by

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

and the notation $\tilde{p}$ defined for $1 < p < 2$ by

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}.$$

For $m \in \mathbb{Z}_+$ we denote by $C^m(\overline{\Omega})$ the space of bounded continuous functions on $\overline{\Omega}$ with continuous and bounded derivatives up to order m. This space we equip with the norm

$$\|u\|_{C^m(\overline{\Omega})} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

We write $C(\overline{\Omega}) = C^0(\overline{\Omega})$. Furthermore, for $0 < \alpha < 1$ we denote by $C^\alpha(\overline{\Omega})$ the Hölder space with exponent α. This space consists of functions $u \in$

$C(\overline{\Omega})$ for which there is a constant C such that

$$|u(x) - u(y)| \leq C|x - y|^{\alpha}$$

for all $x, y \in \overline{\Omega}$. This space is normed by

$$\|u\|_{C^{\alpha}(\overline{\Omega})} = \inf_{x,y \in \overline{\Omega}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}.$$

By $C^{0,1}(\overline{\Omega})$ we denote the space of Lipschitz continuous functions consisting of functions $u \in C(\overline{\Omega})$ satisfying

$$|u(x) - u(y)| \leq C|x - y|.$$

The natural norm is

$$\|u\|_{C^{0,1}(\overline{\Omega})} = \inf_{x,y \in \overline{\Omega}} \frac{|u(x) - u(y)|}{|x - y|}.$$

Boundaries. Concerning regularity of the boundary let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a bounded and open subset. Then we say that $\partial\Omega$ is globally C^k provided that for any $x \in \partial\Omega$ there is an open neighborhood U, which has a local coordinate patch such that

$$\begin{aligned} \Omega \cap U &= \{x = (x', x_n) \in U \mid x_n > \sigma(x')\}, \\ \partial\Omega \cap U &= \{x = (x', x_n) \in U \mid x_n = \sigma(x')\}, \end{aligned} \tag{A.1}$$

where the real function $\sigma \in C^k(\mathbb{R}^{n-1})$. When σ is only Lipschitz continuous we shall say that Ω has Lipschitz boundary. We note that when $\partial\Omega$ is C^1 then a normal vector field to $\partial\Omega$ is well-defined. We will by $\nu = \nu(x)$ denote the out pointing unit normal at $x \in \partial\Omega$.

To define Sobolev spaces on the boundary, let $\{U_j\}_{j=1}^N$ be an open cover of $\partial\Omega$ consisting of such neighborhoods in (A.1). Due to the compactness of $\partial\Omega$ we can assume $N < \infty$. Let $\{\phi_j\}_{j=1}^N$ be a smooth partition of unity subordinate to this covering and define the map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi_j(x_1, \cdots, x_n) = (x_1, \cdots, x_n - \sigma_j(x')).$$

This map straightens out the boundary, i.e. if $y = \Phi_j(x)$ for some point $x \in \partial\Omega \cap U_j$, then $y_n = 0$. Then we will say that $f \in W^{s,p}(\partial\Omega),\ s > 0, 1 \leq p \leq \infty$ if and only if

$$\sum_{j=1}^{N} f(\phi_j \Phi_j^{-1}(y', 0)) \in W^{s,p}(\mathbb{R}^{n-1}).$$

The norm on $H^s(\partial\Omega)$ is defined as

$$\|f\|_{W^{s,p}(\partial\Omega)} = \|\sum_{j=1}^{N} f(\phi_j \Phi_j^{-1}(y', 0))\|_{W^{s,p}(\mathbb{R}^{n-1})}.$$

Again we write $H^s(\partial\Omega) = W^{2,p}(\partial\Omega)$. A fundamental result states that the trace operator initially defined on $C(\overline{\Omega})$ extends to a bounded operator on $W^{s,p}(\Omega), s > 1/p$ with range $W^{s-1/p,p}(\partial\Omega)$. We note that the definitions of the spaces on the boundary are independent of the choice of the cover $\{U_j\}_{j=1}^N$. For $s < 0$, we define $H^s(\partial\Omega) = (H^{-s}(\partial\Omega))^*$, the dual space of $H^{-s}(\partial\Omega)$ taken with respect to the norm topology.

A useful space when dealing with boundary value problems is the subspace $H_0^s(\partial\Omega)$ of $H^s(\partial\Omega)$ consisting of functions, which integrate to zero along the boundary, i.e.

$$H_0^s(\partial\Omega) = \{g \in H^s(\partial\Omega) \mid \langle g, 1\rangle = 0\}. \tag{A.2}$$

Here we use the fact that $1 \in H^{-s}(\partial\Omega)$ for any $s \in \mathbb{R}$. We will in the same way denote by $C_0^\alpha(\partial\Omega) \subset C^\alpha(\partial\Omega)$ the subset of functions, which integrate to zero along the boundary. The space $H_0^1(\partial\Omega)$ should not be confused with the space $H_0^1(\Omega)$ defined above.

Concerning integration along C^1 curves in the plane we will use the notation

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

where $\gamma : [a,b] \mapsto \Gamma$ is a parameterization of Γ. We note that if $d\sigma$ is the usual surface measure on Γ, then

$$\int_\Gamma f(z)dz = \int_\Gamma f(x)i(\nu_1 + i\nu_2)(x)d\sigma(x),$$

where $x_1 + ix_2 = z$ and (ν_1, ν_2) is the unit normal vector to Γ in x pointing to the right relative to the direction of γ.

Fourier transform. The n-dimensional Fourier transform is defined for $\phi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{\phi}(\xi) = \mathcal{F}\phi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\phi(x)dx$$

and extended in the usual way to $L^1(\mathbb{R}^n), L^2(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$. The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$.

Miscellaneous. All constants are generically denoted by C. Occasionally we elaborate the dependency of a constant on certain parameters. Furthermore we will use the notation $f \sim g$ to indicate that $f - g$ is asymptotically zero. In the concrete applications we are more exact about in which sense this holds.

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