Perturbation of eigenvalues embedded at a threshold

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Abstract

Results are obtained on perturbation of eigenvalues and half-bound states (zero-resonances) embedded at a threshold. The results are obtained in a two-channel framework for small off-diagonal perturbations. The results are based on given asymptotic expansions of the component Hamiltonians.

1 Introduction

We consider Hamiltonians which can be represented in a two channel framework. Assume that the diagonal part has an eigenvalue embedded at a threshold. Then it is shown that under small off-diagonal perturbations this eigenvalue never moves into the continuous spectrum. The results require a kind of effective interaction assumption on the off-diagonal part.

We first give the results in an abstract form, and then we show how to apply them to Schrödinger operators.

Let us now describe some of the results in detail. We consider Hamiltonians which are decomposed as

$$H(g) = \begin{pmatrix} H_a & 0 \\ 0 & H_b \end{pmatrix} + g \begin{pmatrix} 0 & V_{ab} \\ V_{ba} & 0 \end{pmatrix} \tag{1.1}$$

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on a Hilbert space $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_b$. Here H_a and H_b are self-adjoint operators on $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively. For the sake of simplicity we assume $V_{ab} \in \mathcal{B}(\mathcal{H}_b, \mathcal{H}_a)$, the bounded operators, and require $V_{ba} = V_{ab}^*$.

Due to the diagonal structure of $H(0)$ its spectral nature can be an arbitrary combination of those of H_a and H_b. There are many possible cases, and we have only treated some of them in detail. A particular case of interest is the following. We assume $\sigma(H_a) = \sigma_{\rm ac}(H_a) = [\lambda, \infty)$ and $\sigma_{\rm ac}(H_b) = [\lambda_1, \infty)$ for some $\lambda_1 > \lambda$. Furthermore, we assume that λ is an isolated eigenvalue of H_b with eigenprojection P_b. Thus $H(0)$ has an eigenvalue embedded at the threshold λ. We would like to know what happens when the interaction is turned on. To obtain results we require some information on the threshold of H_a. We assume some type of asymptotic expansion of the resolvent $R_a(\zeta)$ near λ. More precisely, let $\mathcal{K}_a$ be a Hilbert (or Banach) space, which is densely and continuously embedded in $\mathcal{H}_a$. We assume the existence of an expansion, valid in the norm topology of $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$,

$$R_a(\zeta) = G_0 + i(\zeta - \lambda)^{1/2} G_1 - (\zeta - \lambda) G_2 + o(|\zeta - \lambda|) \qquad (1.2)$$

as $\zeta \to \lambda$, $\zeta \in \mathbf{C} \setminus [\lambda, \infty)$. Thus in particular the resolvent has a well-defined limit G_0 at the threshold point in the norm topology of $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. This type of asymptotic expansion is know to hold generically for a Schrödinger operator $-\Delta + V(x)$ on $L^2(\mathbf{R}^d)$ for d odd, provided $V(x)$ decays sufficiently rapidly.

Some assumptions on the interaction are needed. Assume that $V_{ab} \in \mathcal{B}(\mathcal{H}_b, \mathcal{K}_a)$ and that the operator $P_b V_{ba} G_0 V_{ab} P_b$ is strictly positive and invertible in $\mathcal{B}(P_b \mathcal{H}_b)$.

Under these assumptions the following result holds, see Theorem 3.9. There exist $\eta_0 > 0$, $\delta_0 > 0$, and a function $\delta_l(g)$, which satisfies $cg^2 \leq \delta_l(g) \leq Cg^2$, such that

$$(\lambda - \delta_l(g), \lambda + \delta_0) \cap \sigma_{\rm pp}(H(g)) = \emptyset \qquad (1.3)$$

for all g with $0 < |g| < \eta_0$.

Thus this result shows that the eigenvalues at λ may move left and become isolated eigenvalues of $H(g)$. But the eigenvalues cannot move into the continuous spectrum. If one assumes that the limiting absorption principle holds for H_a on (λ, λ_1), and that this interval is in the resolvent set of H_b, then the result (1.3) holds for any $\delta_0 < \lambda_1 - \lambda$, see Remark 3.3. In this case we may need to have a smaller η_0. See Section 5.2 for the case of a Schrödinger operator with confined channels, where all the above conditions are verified explicitly. The power g^2 behavior of $\delta_l(g)$ is optimal as a general result, as shown by an example in Section 5.1. This particular behavior is a consequence of the form of the asymptotic expansion in (1.2).

In some cases one would expect that embedded eigenvalues become resonances under perturbation. In this paper we are not imposing assumptions on the Hamiltonians, which make is possible to give a reasonable definition of a resonance, hence we have no results in this direction here.

The result (1.3) is obtained by using the asymptotic expansion (1.2) in combination with the Feshbach formula and a technique based on factoring out the identity plus a finite rank operator.

We give a number of other results of the same type in Section 3, and in Section 5 we give a few applications. Further applications will be given elsewhere.

Concerning the literature, then there seems to be few results on this problem. A general result on absorption of eigenvalues in the continuum is given in [17]. In the paper [2] a result is obtained, concerning the survival of the ground state of a Pauli-Fierz Hamiltonian. In the paper [3] the possibility of having a modified Fermi Golden Rule at a threshold is considered. In a time-dependent framework the perturbation of threshold eigenvalues has been discussed in the paper [18].

The paper [6] contains a number of related results, obtained using the Feshbach formula, for eigenvalues embedded in the continuum.

Our results rely on having asymptotic expansions of resolvents around thresholds. Such results have been obtained for Schrödinger type operators by a number of authors. We mention the papers [10, 8, 14, 19, 9, 13], the survey paper [4], and the references therein.

2 Preliminaries

Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T)$. The spectrum and resolvent set are denoted by $\sigma(T)$ and $\rho(T)$, respectively. We use standard terminology for the various parts of the spectrum, see for example [16]. The resolvent is $R(\zeta) = (T - \zeta)^{-1}$.

If λ is an isolated eigenvalue of T with associated eigenprojection P, then the reduced resolvent is given as

$$C = \lim_{\zeta \to \lambda} (I - P)R(\zeta), \tag{2.1}$$

and we have the norm convergent expansion

$$R(\zeta) = -\frac{P}{\zeta - \lambda} + \sum_{n=0}^{\infty} (\zeta - \lambda)^n C^{n+1}. \tag{2.2}$$

The expansion is valid for $0 < |\zeta - \lambda| < \delta$ for some small $\delta > 0$. See for example [12, 15].

The Feshbach formula gives a convenient explicit representation of the resolvent $R(g;\zeta)$ of $H(g)$. There are two variants. We give only one of them. The other version is just an interchange of indices. Define

$$R_a(\zeta) = (H_a - \zeta)^{-1}, \tag{2.3}$$

$$T_b(\zeta) = H_b - \zeta - g^2 V_{ba} R_a(\zeta) V_{ab}. \tag{2.4}$$

Then for $\operatorname{Im}\zeta \neq 0$ we have

$$R(g;\zeta) = \begin{pmatrix} R_a(\zeta) + g^2 R_a(\zeta) V_{ab} T_b(\zeta)^{-1} V_{ba} R_a(\zeta) & -g R_a(\zeta) V_{ab} T_b(\zeta)^{-1} \\ -g T_b(\zeta)^{-1} V_{ba} R_a(\zeta) & T_b(\zeta)^{-1} \end{pmatrix} \tag{2.5}$$

To state the abstract results we will need an abstract formulation of the limiting absorption principle. We have settled on the following simplified version.

Definition 2.1. Let T be a self-adjoint operator on $\mathcal{H}$ and $I \subset \sigma(T)$ an open interval. The limiting absorption principle is said to hold on I, if there exists a Hilbert (or Banach) space $\mathcal{K}$, densely and continuously embedded in $\mathcal{H}$, such that $\zeta \mapsto R(\zeta)$ has a norm-continuous extension from the upper half plane $\mathbf{C}_+$ to $I \cup \mathbf{C}_+$, with values in $\mathcal{B}(\mathcal{K}, \mathcal{K}^*)$.

Thus our definition introduces the triple of spaces $\mathcal{K} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{K}^*$, as is common in abstract formulations of the limiting absorption principle. The boundary values are denoted by $R(\lambda + i0) \in \mathcal{B}(\mathcal{K}, \mathcal{K}^*)$ for $\lambda \in I$.

The notation $J \Subset I$ means that the open interval J is relatively compact in the open interval I.

We sometimes use the following remark to avoid repetition of proofs.

Remark 2.2. Note that the extension property in Definition 2.1 holds trivially in the case $I \subset \rho(T)$, with $\mathcal{K} = \mathcal{H}$.

Let $\delta(g)$ and $f(g)$ be two positive functions, defined for $g \in F$, a parameter set. We write $\delta(g) \asymp f(g)$, if there exist constants $c > 0$, $C > 0$, such that $cf(g) \leq \delta(g) \leq Cf(g)$ for all $g \in F$.

For a complex number $z \in \mathbf{C} \setminus [0, \infty)$ we denote by $z^{1/2}$ the branch of the square root with positive imaginary part.

Finally we recall some elementary results from operator theory. Let $X, Y \in \mathcal{B}(\mathcal{H})$, and assume that X is invertible. If $\|Y\| < \|X^{-1}\|^{-1}$, then $X + Y$ is invertible, and we have the estimate

$$\|(X+Y)^{-1}\| \leq \frac{\|X^{-1}\|}{1 - \|Y\| \cdot \|X^{-1}\|}.$$

Let P be a projection in $\mathcal{B}(\mathcal{H})$ and let $X \in \mathcal{B}(\mathcal{H})$. Assume that the operator $P + PXP$ is invertible in $\mathcal{B}(P\mathcal{H})$. Then the operators $I + XP$ and $I + PX$ are invertible, and we have, with an obvious notation,

$$(I + XP)^{-1} = I - XP(P + PXP)^{-1}P, \tag{2.6}$$

and

$$(I + PX)^{-1} = I - P(P + PXP)^{-1}PX. \tag{2.7}$$

These results are verified by straightforward computations. We also note that (2.6) implies

$$P(I + XP)^{-1} = P(P + PXP)^{-1}P. \tag{2.8}$$

3 Absence of embedded eigenvalues

In several different settings it is shown that embedded eigenvalues for $H(0)$ leave the continuous spectrum of $H(g)$ for $0 < |g| < \eta_0$ for some positive η_0. These eigenvalues may show up as resonances or discrete eigenvalues. We do not discuss resonances in this paper, since we are not making assumptions that enable us to give a meaningful definition of a resonance.

We start with an easy auxiliary result. It will be combined with other results later.

Assumption 3.1. Let $I \subset \sigma_{\mathrm{ac}}(H_a) \cap \sigma_{\mathrm{ac}}(H_b)$. Assume that the limiting absorption principle holds on I for H_a and H_b in spaces $\mathcal{K}_a$ and $\mathcal{K}_b$, respectively. Assume that V_{ab} extends to $\mathcal{K}_b^*$, such that $V_{ab} \in \mathcal{B}(\mathcal{K}_b^*, \mathcal{K}_a)$.

Proposition 3.2. *Let Assumption* 3.1 *hold on I. Let $J \Subset I$. Then there exists an $\eta_0 > 0$ such that for $|g| < \eta_0$ we have $J \subset \sigma_{\mathrm{ac}}(H(g))$ and $\sigma_{\mathrm{pp}}(H(g)) \cap J = \emptyset$.*

Proof. We use (2.5). Write

$$T_b(\zeta) = (H_b - \zeta)\left(I_b - g^2 R_b(\zeta) V_{ba} R_a(\zeta) V_{ab}\right).$$

Let

$$\mu = \sup_{\lambda \in J} \|R_b(\lambda + i0) V_{ba} R_a(\lambda + i0) V_{ab}\|_{\mathcal{B}(\mathcal{K}_b^*)}.$$

Then the claims in the proposition are true for $\eta_0 < 1/\sqrt{\mu}$. We omit the details. ∎

Remark 3.3. Using Remark 2.2 we get that this result also holds in the two cases $I \subset \sigma_{\mathrm{ac}}(H_a) \cap \rho(H_b)$ and $I \subset \rho(H_a) \cap \sigma_{\mathrm{ac}}(H_b)$. In the case $I \subset \rho(H_a) \cap \rho(H_b)$ we get $J \subset \rho(H(g))$ and $\sigma_{\mathrm{pp}}(H(g)) \cap J = \emptyset$.

Assumption 3.4. Let $\lambda \in \sigma(H_a)$.
(i) Assume that there exists a Hilbert space $\mathcal{K}_a$, densely and continuously embedded in $\mathcal{H}_a$, such that for some $\delta > 0$ we have an asymptotic expansion in the norm of $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$, viz.

$$R_a(\zeta) = G_0 + i(\zeta - \lambda)^{1/2} G_1 - (\zeta - \lambda) G_2 + o(|\zeta - \lambda|) \tag{3.1}$$

for $|\zeta - \lambda| < \delta$, $\operatorname{Im} \zeta > 0$. Assume furthermore that $G_j = G_j^*$, $j = 0, 1, 2$, as operators in $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$.
(ii) Assume that $V_{ab} \in \mathcal{B}(\mathcal{H}_b, \mathcal{K}_a)$.
(iii) Assume that λ is a simple isolated eigenvalue of H_b, with normalized eigenfunction ψ. Its reduced resolvent is denoted by C_b.

We use the notation $P_b = \langle \cdot, \psi \rangle \psi$ for the eigenprojection. The following real numbers are needed to state the results.

$$\alpha_0 = \langle V_{ba} G_0 V_{ab} \psi, \psi \rangle, \tag{3.2}$$
$$\beta_0 = \langle V_{ba} G_1 V_{ab} \psi, \psi \rangle, \tag{3.3}$$
$$\gamma_0 = \langle V_{ba} G_0 V_{ab} C_b V_{ba} G_0 V_{ab} \psi, \psi \rangle. \tag{3.4}$$

Before treating the case where λ is an eigenvalue of H_b we briefly consider two other cases. We collect the results in the following proposition.

Proposition 3.5. *Let Assumption* 3.4(i) *hold for H_a at $\lambda \in \mathbf{R}$.*
(i) *Assume that $\lambda \in \sigma(H_b)$ and that the limiting absorption principle holds for H_b on $I \ni \lambda$ in $\mathcal{K}_b$. Assume that $V_{ab} \in \mathcal{B}(\mathcal{K}_b^*, \mathcal{K}_a)$. Then there exist an $\eta_0 > 0$ and an interval $J \Subset I$, $\lambda \in J$, such that for $|g| < \eta_0$ we have $J \subset \sigma_{\mathrm{ac}}(H(g))$ and $J \cap \sigma_{\mathrm{pp}}(H(g)) = \emptyset$.*
(ii) *Assume that $\lambda \in \rho(H_b)$ and that $V_{ab} \in \mathcal{B}(\mathcal{H}_b, \mathcal{K}_a)$. Then there exist $\eta_0 > 0$ and $\delta_0 > 0$ such that for $|g| < \eta_0$ we have $(\lambda, \lambda + \delta_0) \subset \sigma_{\mathrm{ac}}(H(g))$ and $(\lambda - \delta_0, \lambda + \delta_0) \cap \sigma_{\mathrm{pp}}(H(g)) = \emptyset$.*

The proof is similar to the proof of Proposition 3.2 and is omitted.

We now consider the case where λ is an eigenvalue of H_b. Let us first consider the case $\alpha_0 \neq 0$.

Theorem 3.6. *Let Assumption* 3.4 *hold at $\lambda \in \mathbf{R}$.*
(i) *Assume that $\alpha_0 > 0$. Then there exist $\eta_0 > 0$, $\delta_0 > 0$ and $\delta_l(g) \asymp g^2$ such that for all g with $0 < |g| < \eta_0$ we have $(\lambda - \delta_l(g), \lambda + \delta_0) \cap \sigma_{\mathrm{pp}}(H(g)) = \emptyset$.*

(ii) *Assume that* $\alpha_0 < 0$ *and* $\beta_0 \neq 0$. *Then there exist* $\eta_0 > 0$ *and* $\delta_0 > 0$ *such that for all* g *with* $0 < |g| < \eta_0$ *we have* $(\lambda - \delta_0, \lambda + \delta_0) \cap \sigma_{\mathrm{pp}}(H(g)) = \emptyset$.
(iii) *Assume that* $\alpha_0 < 0$ *and* $\beta_0 = 0$. *Then there exist* $\eta_0 > 0$, $\delta_0 > 0$, *and* $\delta_r(g) \asymp g^2$ *such that for all* g *with* $0 < |g| < \eta_0$ *we have* $(\lambda - \delta_0, \lambda + \delta_r(g)) \cap \sigma_{\mathrm{pp}}(H(g)) = \emptyset$.

Proof. The strategy of the proof is to factor the operator $T_b(\zeta)$ in order to show that the resolvent of $H(g)$ remains bounded in the norm topology of $\mathcal{B}(\mathcal{K}_a \oplus \mathcal{H}_b, \mathcal{K}_a^* \oplus \mathcal{H}_b)$ near a certain interval around λ. In some cases this interval depends on g.

In the sequel we always assume at least $|\zeta - \lambda| < \delta$ (with the δ from Assumption 3.4) and $\operatorname{Im} \zeta > 0$. We use the factorization

$$T_b(\zeta) = \bigl(I_b - g^2 V_{ba} R_a(\zeta) V_{ab} R_b(\zeta)\bigr)(H_b - \zeta). \tag{3.5}$$

The assumption gives the following asymptotic expansion in $\mathcal{B}(\mathcal{H}_b)$.

$$\begin{aligned} I_b - &g^2 V_{ba} R_a(\zeta) V_{ab} R_b(\zeta) \\ &= I_b + \frac{1}{\zeta - \lambda} g^2 V_{ba} G_0 V_{ab} P_b + \frac{i}{(\zeta - \lambda)^{1/2}} g^2 V_{ba} G_1 V_{ab} P_b \\ &\quad - g^2 \left(V_{ba} G_0 V_{ab} C_b + V_{ba} G_2 V_{ab} P_b + o(1)\right). \end{aligned} \tag{3.6}$$

We first consider the identity plus the terms containing singularities. Let

$$S(\zeta) = I_b + \frac{1}{\zeta - \lambda} g^2 V_{ba} G_0 V_{ab} P_b + \frac{i}{(\zeta - \lambda)^{1/2}} g^2 V_{ba} G_1 V_{ab} P_b, \tag{3.7}$$

and define

$$\kappa(g; \zeta) = \frac{g^2}{g^2 \alpha_0 + i(\zeta - \lambda)^{1/2} g^2 \beta_0 + \zeta - \lambda}. \tag{3.8}$$

For all ζ, g such that $\kappa(g; \zeta) \neq 0$ the operator $S(\zeta)$ is invertible in $\mathcal{B}(\mathcal{H}_b)$. The inverse is given by (see (2.6))

$$S(\zeta)^{-1} = I_b - \kappa(g; \zeta) \langle \cdot, \psi \rangle \left(V_{ba} G_0 V_{ab} \psi + i(\zeta - \lambda)^{1/2} V_{ba} G_1 V_{ab} \psi\right). \tag{3.9}$$

Since we are assuming $\alpha_0 \neq 0$, the inverse $S(\zeta)^{-1}$ is regular at $\zeta = \lambda$. We factor as follows

$$\begin{aligned} &I_b - g^2 V_{ba} R_a(\zeta) V_{ab} R_b(\zeta) = \\ &\quad \left[I_b - g^2 \left(V_{ba} G_0 V_{ab} C_b + V_{ba} G_2 V_{ab} P_b + o(1)\right) S(\zeta)^{-1}\right] S(\zeta). \end{aligned} \tag{3.10}$$

The terms in $[\cdots]$ are denoted by $U(\zeta)$. We compute the leading terms

$$\begin{aligned}U(\zeta) &= I_b - g^2 V_{ba} G_2 V_{ab} P_b - g^2 V_{ba} G_0 V_{ab} C_b \\ &\quad + \alpha_0^{-1} \langle \cdot, \psi \rangle g^2 \left(V_{ba} G_2 V_{ab} P_b V_{ba} G_0 V_{ab} \psi + V_{ba} G_0 V_{ab} P_b V_{ba} G_0 V_{ab} \psi \right) \\ &\quad + g^2 o(1).\end{aligned}$$

From the definition of α_0 follows

$$\alpha_0^{-1} \langle \cdot, \psi \rangle V_{ba} G_2 V_{ab} P_b V_{ba} G_0 V_{ab} \psi = V_{ba} G_2 V_{ab} P_b.$$

Thus we get

$$\begin{aligned}U(\zeta) &= I_b - g^2 V_{ba} G_0 V_{ab} C_b \\ &\quad + \alpha_0^{-1} \langle \cdot, \psi \rangle g^2 V_{ba} G_0 V_{ab} P_b V_{ba} G_0 V_{ab} \psi + g^2 o(1).\end{aligned} \tag{3.11}$$

We see that for small $|g|$ and small $|\zeta - \lambda|$ this operator is invertible. Putting the terms together we have

$$T_b(\zeta)^{-1} = R_b(\zeta) S(\zeta)^{-1} U(\zeta)^{-1}.$$

Since $R_b(\zeta)$ has a pole at λ, we need to compute the products to check whether the singular terms cancel. We have

$$\begin{aligned}R_b(\zeta) S(\zeta)^{-1} &= \left(-\frac{P_b}{\zeta - \lambda} + C_b + O(|\zeta - \lambda|) \right) \cdot \\ &\quad \cdot \left(I_b - \kappa(g;\zeta) \left(V_{ba} G_0 V_{ab} P_b + i(\zeta - \lambda)^{1/2} V_{ba} G_1 V_{ab} P_b \right) \right) \\ &= -\frac{P_b}{\zeta - \lambda} + \frac{\kappa(g;\zeta)}{\zeta - \lambda} \left(P_b V_{ba} G_0 V_{ab} P_b + i(\zeta - \lambda)^{1/2} P_b V_{ba} G_1 V_{ab} P_b \right) \\ &\quad + C_b - \kappa(g;\zeta) C_b \left(V_{ba} G_0 V_{ab} P_b + i(\zeta - \lambda)^{1/2} V_{ba} G_1 V_{ab} P_b \right) \\ &\quad + g^2 O(|\zeta - \lambda|) \\ &= -\kappa(g;\zeta) P_b + C_b \\ &\quad - \kappa(g;\zeta) \left(C_b V_{ba} G_0 V_{ab} P_b + i(\zeta - \lambda)^{1/2} C_b V_{ba} G_1 V_{ab} P_b \right) \\ &\quad + g^2 O(|\zeta - \lambda|).\end{aligned}$$

Here we have used $\alpha_0 P_b = P_b V_{ba} G_0 V_{ab} P_b$, $\beta_0 P_b = P_b V_{ba} G_1 V_{ab} P_b$, and the definition of $\kappa(g;\zeta)$ to simplify the expressions. Thus under the condition $\alpha_0 \neq 0$ there is no singularity in the above expression at $\zeta = \lambda$.

The singularities in $\kappa(g;\zeta)$ occur at the zeroes of the polynomial $g^{-2} z^2 + i\beta_0 z + \alpha_0$. Thus they occur at

$$\zeta - \lambda = z^2 = -\alpha_0 g^2 - \tfrac{1}{2} \beta_0^2 g^4 \pm \tfrac{1}{2} \beta_0 g^2 \sqrt{\beta_0^2 g^4 + 4\alpha_0 g^2}.$$

The results in cases (i), (ii), and (iii) follow from a straightforward analysis of this expression. $\square$

We now consider the case $\alpha_0 = 0$.

Theorem 3.7. *Let Assumption* 3.4 *hold at* $\lambda \in \mathbf{R}$. *Assume* $\alpha_0 = 0$ *and* $\gamma_0 \neq 0$.
(i) *Assume* $\beta_0 \neq 0$. *Then there exist* $\eta_0 > 0$, $\delta_0 > 0$ *and* $\delta_l(g) \asymp g^4$ *such that for all* g *with* $0 < |g| < \eta_0$ *we have* $(\lambda - \delta_l(g), \lambda + \delta_0) \cap \sigma_{\mathrm{pp}}(H(g)) = \emptyset$.
(ii) *Assume* $\beta_0 = 0$ *and* $\gamma_0 > 0$. *Then there exist* $\eta_0 > 0$, $\delta_0 > 0$ *and* $\delta_l(g) \asymp g^4$ *such that for all* g *with* $0 < |g| < \eta_0$ *we have* $(\lambda - \delta_l(g), \lambda + \delta_0) \cap \sigma_{\mathrm{pp}}(H(g)) = \emptyset$.
(iii) *Assume* $\beta_0 = 0$ *and* $\gamma_0 < 0$. *Then there exist* $\eta_0 > 0$, $\delta_0 > 0$ *and* $\delta_r(g) \asymp g^4$ *such that for all* g *with* $0 < |g| < \eta_0$ *we have* $(\lambda - \delta_0, \lambda + \delta_r(g)) \cap \sigma_{\mathrm{pp}}(H(g)) = \emptyset$.

Proof. The proof is similar to the proof of Theorem 3.6. We use $\kappa(g;\zeta)$ defined in (3.8). Under the assumption that $\alpha_0 = 0$ this expression has a singularity at $\zeta = \lambda$. Thus this time $S(\zeta)^{-1}$ given in (3.9) has a singularity here. This means that we have to look carefully at the term $U(\zeta)$. The computation in (3.11) is not valid in the present case.

We introduce the shorthand notation

$$\phi = V_{ba} G_0 V_{ab} C_b V_{ba} G_0 V_{ab} \psi.$$

Then a computation shows that the singularities in $U(\zeta)$ are contained in the term

$$S_1(\zeta) = I_b + \kappa(g;\zeta) g^2 \langle \cdot, \psi \rangle \phi. \tag{3.12}$$

We then introduce

$$\mu(g;\zeta) = \frac{g^4}{g^4 \gamma_0 + i(\zeta - \lambda)^{1/2} g^2 \beta_0 + \zeta - \lambda}. \tag{3.13}$$

For all g, ζ such that $\mu(g,\zeta) \neq 0$ we have that $S_1(\zeta)$ is invertible, and the inverse is given by

$$S_1(\zeta)^{-1} = I_b - \mu(g;\zeta) \langle \cdot, \psi \rangle \phi. \tag{3.14}$$

Since we are assuming that $\gamma_0 \neq 0$, we find that this inverse has no singularity at $\zeta = \lambda$. We then compute as follows.

$$\begin{aligned} U(\zeta) &= \left(I_b + g^2 \left(K + o(1) \right) S_1(\zeta)^{-1} \right) S_1(\zeta) \\ &= W(\zeta) S_1(\zeta). \end{aligned}$$

We have collected the constant terms in the term K in $W(\zeta)$. It follows that $W(\zeta)$ is invertible for g and $|\zeta - \lambda|$ sufficiently small. This means that we have a factorization

$$T_b(\zeta) = W(\zeta) S_1(\zeta) S(\zeta) (H_b - \zeta),$$

and therefore

$$T_b(\zeta)^{-1} = R_b(\zeta)S(\zeta)^{-1}S_1(\zeta)^{-1}W(\zeta)^{-1}.$$

The first two terms contain singularities. We therefore have to multiply out to check for cancellations. The computation of $R_b(\zeta)S(\zeta)^{-1}$ from the proof of Theorem 3.6 is valid under the current assumptions. Thus we have

$$\begin{aligned} R_b(\zeta)S(\zeta)^{-1} = &-\kappa(g;\zeta)P_b + C_b \\ &- \kappa(g;\zeta)\big(C_bV_{ba}G_0V_{ab}P_b + i(\zeta-\lambda)^{1/2}C_bV_{ba}G_1V_{ab}P_b\big) \\ &+ g^2O(|\zeta-\lambda|). \end{aligned}$$

However, under the current assumptions this expression is not regular at $\zeta = \lambda$. We continue the computation, using $P_b\langle\cdot,\,\psi\rangle\phi = \gamma_0P_b$.

$$\begin{aligned} R_b(\zeta)S(\zeta)^{-1}S_1(\zeta)^{-1} = &\, C_b - \mu(g;\zeta)\langle\cdot,\,\psi\rangle C_b\phi \\ &+ (\gamma_0\kappa(g;\zeta)\mu(g;\zeta) - \kappa(g;\zeta))\,P_b \\ &+ (\gamma_0\kappa(g;\zeta)\mu(g;\zeta) - \kappa(g;\zeta))\,\cdot \\ &\cdot\big(C_bV_{ba}G_0V_{ab}P_b + i(\zeta-\lambda)^{1/2}C_bV_{ba}G_1V_{ab}P_b\big) \\ &+ g^2O(|\zeta-\lambda|^{1/2}). \end{aligned}$$

Now we use that

$$\gamma_0\kappa(g;\zeta)\mu(g;\zeta) - \kappa(g;\zeta) = -g^{-2}\mu(g,\zeta)$$

to conclude that the singularities have cancelled out. It remains to analyze the zeroes of the polynomial $z^2 + ig^2\beta_0 z + g^4\gamma_0$. This analysis is similar to the one given in the proof of Theorem 3.6, and leads to the three cases stated in the theorem. □

We now consider the case when λ is an isolated eigenvalue of H_b of arbitrary multiplicity. We limit ourselves to discussing the simplest case.

Assumption 3.8. Let parts (i) and (ii) of Assumption 3.4 hold at $\lambda \in \mathbf{R}$. Assume that λ is an isolated eigenvalue of H_b with eigenprojection P_b such that the operator $P_bV_{ba}G_0V_{ab}P_b$ is strictly positive and invertible in $\mathcal{B}(P_b\mathcal{H}_b)$.

Theorem 3.9. *Let Assumption* 3.8 *hold at* $\lambda \in \mathbf{R}$. *Then there exist* $\eta_0 > 0$, $\delta_0 > 0$, *and* $\delta_l(g) \asymp g^2$ *such that for* g *with* $0 < |g| < \eta_0$ *we have* $(\lambda-\delta_l(g), \lambda+\delta_0) \cap \sigma_{\mathrm{pp}}(H(g)) = \emptyset$.

Proof. We follow the strategy of the proof of Theorem 3.6. To facilitate comparison with this proof we use analogous notation for various operators. The expansion in (3.6) is still valid. We define

$$Y(g;\zeta) = g^2 V_{ba} G_0 V_{ab} + i(\zeta-\lambda)^{1/2} g^2 V_{ba} G_1 V_{ab}. \tag{3.15}$$

Then we consider

$$S(\zeta) = I_b + \frac{1}{\zeta-\lambda} Y(g;\zeta) P_b. \tag{3.16}$$

Suppose that for some g, ζ the operator

$$Z(g;\zeta) = (\zeta-\lambda)P_b + P_b Y(g;\zeta) P_b \tag{3.17}$$

is invertible in $\mathcal{B}(P_b\mathcal{H}_b)$. Then $S(\zeta)$ is invertible, and the inverse is given by

$$S(\zeta)^{-1} = I_b - Y(g;\zeta) P_b Z(g;\zeta)^{-1} P_b, \tag{3.18}$$

see (2.6). It is clear from (3.10) that if we can control singularities in this inverse, then we can obtain invertibility of $U(\zeta)$ for suitably restricted values of g and ζ.

Let us analyze in detail the restrictions on ζ and the dependence on g in order for $Z(g;\zeta)$ to be invertible. Let

$$c_0 = \|(P_b V_{ba} G_0 V_{ab} P_b)^{-1}\|_{\mathcal{B}(P_b\mathcal{H}_b)}.$$

Due to the positivity assumption the operator

$$g^2 P_b V_{ba} G_0 V_{ab} P_b + \mathrm{Re}(\zeta-\lambda) P_b$$

is invertible for ζ satisfying $\mathrm{Re}(\zeta-\lambda) > -g^2/2c_0$, and the norm of the inverse is bounded by $2c_0/g^2$. Let

$$c_1 = \|P_b V_{ba} G_1 V_{ab} P_b\|_{\mathcal{B}(P_b\mathcal{H}_b)},$$

and let $\delta = (4c_0c_1)^{-2}$. Then for $|\zeta-\lambda| < \delta$ and $\mathrm{Re}(\zeta-\lambda) > -g^2/2c_0$ we have existence of the inverse and the estimate

$$\|(g^2 P_b V_{ba} G_0 V_{ab} P_b + i(\zeta-\lambda)^{1/2} g^2 P_b V_{ba} G_1 V_{ab} P_b + \mathrm{Re}(\zeta-\lambda)P_b)^{-1}\|_{\mathcal{B}(P_b\mathcal{H}_b)} \leq \frac{4c_0}{g^2}.$$

Now finally we assume $0 \leq \mathrm{Im}(\zeta-\lambda) < g^2/8c_0$. Then $Z(g;\zeta)$ is invertible, and we have

$$\|Z(g;\zeta)^{-1}\|_{\mathcal{B}(P_b\mathcal{H}_b)} \leq \frac{8c_0}{g^2}$$

for all ζ satisfying the three conditions. For these values of ζ we then have

$$\|S(\zeta)^{-1}\|_{\mathcal{B}(\mathcal{H}_b)} \leq 1 + 8c_0(\|V_{ba}G_0V_{ab}\| + \sqrt{\delta}\|V_{ba}G_1V_{ab}\|).$$

We note that this estimate is independent of g. It follows that we can determine an $\eta_0 > 0$ such that for $0 < |g| < \eta_0$ and ζ satisfying the above restrictions the operator $U(\zeta)$ is invertible with a uniformly bounded inverse.

An important point in the proof of Theorem 3.6 is the cancellation of singularities in $R_b(\zeta)S(\zeta)^{-1}$. We now verify that also in the present case the singularities cancel (see also (2.8)).

$$\begin{aligned} R_b(\zeta)S(\zeta)^{-1} &= \Bigl(-\frac{P_b}{\zeta-\lambda} + C_b + O(|\zeta-\lambda|)\Bigr)\cdot \\ &\quad \cdot \bigl(I_b - Y(g;\zeta)P_bZ(g;\zeta)^{-1}P_b\bigr) \\ &= -\frac{1}{\zeta-\lambda}\bigl(P_b - P_bY(g;\zeta)P_bZ(g;\zeta)^{-1}P_b\bigr) \\ &\quad + C_b - C_bY(g;\zeta)P_bZ(g;\zeta)^{-1}P_b + O(|\zeta-\lambda|) \\ &= -P_bZ(g;\zeta)^{-1}P_b + C_b \\ &\quad - C_bY(g;\zeta)P_bZ(g;\zeta)^{-1}P_b + O(|\zeta-\lambda|). \end{aligned}$$

Let us now put all the estimates together. Fix g, $0 < |g| < \eta_0$. Then for a suitable $\delta_0 \leq \delta$ (determined above) we have for all ζ satisfying $|\zeta-\lambda| < \delta_0$, $\mathrm{Re}(\zeta-\lambda) < -g^2/2c_0$, and $0 \leq \mathrm{Im}(\zeta-\lambda) < g^2/8c_0$ that $\|T_b(\zeta)^{-1}\|$ is uniformly bounded. This estimate proves the theorem. □

We now turn to the case where $\lambda \in \rho(H_b)$ and λ is a threshold eigenvalue of H_a. We assume that the asymptotic expansion of $R_a(\zeta)$ around λ has a particular structure which we know occurs for Schrödinger-type operators, see [10, 8, 14].

Assumption 3.10. Let λ be an eigenvalue of H_a with associated eigenprojection P_a.
(i) Assume that there exists a Hilbert space $\mathcal{K}_a$, densely and continuously embedded in $\mathcal{H}_a$, such that for some $\delta > 0$ we have an asymptotic expansion

$$R_a(\zeta) = -\frac{1}{\zeta-\lambda}P_a - \frac{i}{(\zeta-\lambda)^{1/2}}G_{-1} + G_0 + o(1) \tag{3.19}$$

for $|\zeta-\lambda| < \delta$, $\mathrm{Im}\,\zeta > 0$, in norm in $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. Assume that $G_j = G_j^*$ for $j = -1, 0$, in $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. Assume $P_a \in \mathcal{B}(\mathcal{K}_a)$ and furthermore $G_{-1}P_a = G_{-1}$.
(ii) Assume that $V_{ab} \in \mathcal{B}(\mathcal{H}_b, \mathcal{K}_a)$.
(iii) Assume that $\lambda \in \rho(H_b)$.

We recall that for $\lambda \in \rho(H_b)$ we have

$$R_b(\zeta) = \sum_{n=0}^{\infty} (\zeta - \lambda)^n C_n \tag{3.20}$$

for $|\zeta - \lambda|$ sufficiently small. The series converges in $\mathcal{B}(\mathcal{H}_b)$, and we have $C_n = R_b(\lambda)^{n+1}$.

Assumption 3.11. Let Assumption 3.10 hold. Assume that the operator $P_a V_{ab} C_0 V_{ba} P_a$ is strictly positive and invertible in $\mathcal{B}(\mathcal{K}_a)$.

Theorem 3.12. *Let Assumption* 3.11 *hold at* $\lambda \in \mathbf{R}$. *Then there exist* $\eta_0 > 0$, $\delta_0 > 0$, *and* $\delta_l(g) \asymp g^2$ *such that for all* g *with* $0 < |g| < \eta_0$ *we have* $(\lambda - \delta_l(g), \lambda + \delta_0) \cap \sigma_{\mathrm{pp}}(H(g)) = \emptyset$.

Proof. We proceed as in the proofs of the other results. However, this time we interchange the roles of a and b in the Feshbach formula, since now $R_b(\zeta)$ is regular at $\zeta = \lambda$.

The first step is again to factor $T_a(\zeta)$.

$$T_a(\zeta) = \left(I_a - g^2 V_{ab} R_b(\zeta) V_{ba} R_a(\zeta)\right)(H_a - \zeta).$$

Inserting the two expansions we find the following asymptotic expansion in $\mathcal{B}(\mathcal{K}_a)$.

$$\begin{aligned} I_a - {} & g^2 V_{ab} R_b(\zeta) V_{ba} R_a(\zeta) \\ &= I_a + \frac{1}{\zeta - \lambda} g^2 V_{ab} C_0 V_{ba} P_a + \frac{i}{(\zeta - \lambda)^{1/2}} g^2 V_{ab} C_0 V_{ba} G_{-1} \\ &\quad - g^2 \left(V_{ab} C_0 V_{ba} G_0 + V_{ab} C_1 V_{ba} P_a + o(1)\right) \end{aligned} \tag{3.21}$$

We see that the singular part is contained in

$$S(\zeta) = I_a + \frac{1}{\zeta - \lambda} Y(g;\zeta) P_a,$$

where we have introduced

$$Y(g;\zeta) = g^2 V_{ab} C_0 V_{ba} + i(\zeta - \lambda)^{1/2} g^2 V_{ab} C_0 V_{ba} G_{-1}.$$

The operator $S(\zeta)$ is invertible, if

$$Z(g;\zeta) = (\zeta - \lambda) P_a + i(\zeta - \lambda)^{1/2} g^2 P_a V_{ab} C_0 V_{ba} G_{-1} P_a + g^2 P_a V_{ab} C_0 V_{ba} P_a$$

is invertible in the space $\mathcal{B}(\mathcal{K}_a)$, and we have again

$$S(\zeta)^{-1} = I_a - Y(g;\zeta) P_a Z(g;\zeta)^{-1} P_a,$$

see (2.6). Thus the remainder of the proof is analogous to the proof of Theorem 3.9, and is omitted. □

The case where λ is an eigenvalue of H_a and $\lambda \in \sigma_{\rm ac}(H_b)$ can be treated with essentially the same arguments. We state the result and outline the proof.

Assumption 3.13. Let λ be an eigenvalue of H_a of finite multiplicity.
(i) Let Assumption 3.10(i) hold.
(ii) Assume that $\lambda \in \sigma_{\rm ac}(H_b)$ and that there exist an open interval $J \ni \lambda$ and a Hilbert space $\mathcal{K}_b$, densely and continuously embedded in $\mathcal{H}_b$, such that we have

$$R_b(\zeta) = C_0 + (\zeta - \lambda)C_1 + o(|\zeta - \lambda|) \tag{3.22}$$

for $\zeta \to 0$ with $\operatorname{Re}\zeta \in J$, $\operatorname{Im}\zeta \geq 0$, in the norm topology of $\mathcal{B}(\mathcal{K}_b, \mathcal{K}_b^*)$. In particular, the limiting absorption principle holds on J. Here $C_0, C_1 \in \mathcal{B}(\mathcal{K}_b, \mathcal{K}_b^*)$, and furthermore $C_0 = R_b(\lambda + i0)$.
(iii) Assume $V_{ab} \in \mathcal{B}(\mathcal{K}_b^*, \mathcal{K}_a)$.
(iv) Assume that $\operatorname{Im} P_a V_{ab} C_0 V_{ba} P_a$ is a strictly positive operator in $\mathcal{B}(P_a\mathcal{K}_a)$.

Remark 3.14. Let us note that a result of the type (3.22) can be obtained from the extended Mourre theory, see [11], and for more recent results [1].

Theorem 3.15. *Let Assumption* 3.13 *hold at* $\lambda \in \mathbf{R}$. *Then there exist* $\eta_0 > 0$ *and* $\delta_0 > 0$ *such that for all* g *with* $0 < |g| < \eta_0$ *we have* $(\lambda - \delta_0, \lambda + \delta_0) \cap \sigma_{\rm pp}(H(g)) = \emptyset$, *and* $(\lambda - \delta_0, \lambda + \delta_0) \subseteq \sigma_{\rm ac}(H(g))$.

Proof. The computations in the proof of Theorem 3.12 hold with minor modifications under the present assumptions. The problem is to show that the operator

$$Z(g;\zeta) = (\zeta - \lambda)P_a + i(\zeta - \lambda)^{1/2} g^2 P_a V_{ab} C_0 V_{ba} G_{-1} P_a + g^2 P_a V_{ab} C_0 V_{ba} P_a$$

is invertible in the space $\mathcal{B}(\mathcal{K}_a)$. Since $\operatorname{rank} P_a$ is finite and $\operatorname{Im} P_a V_{ab} C_0 V_{ba} P_a$ is assumed strictly positive in $\mathcal{B}(P_a\mathcal{K}_a)$, it follows that there exists $c < 0$ such that the numerical range of $(\zeta - \lambda)P_a + g^2 P_a V_{ab} C_0 V_{ba} P_a$ is contained in the set $\{z \in \mathbf{C} \mid \operatorname{Im} z \leq g^2 c\}$. Hence we have invertibility of this operator and a norm estimate for the inverse of the form cg^{-2}, such that the middle term in $Z(g;\zeta)$ can be treated as a perturbation (see also [6] for related arguments). It now follows from the Feshbach formula and the assumptions that the limiting absorption principle holds on a small g-independent neighborhood of λ. This proves the result. $\square$

Finally we consider the case when H_a has a so-called half-bound state (or zero-resonance) at λ. Motivated by the known results for Schrödinger operators in dimensions one and three, see e. g. [4, 10, 13], we assume a particular form of the singularity of the resolvent.

Assumption 3.16. Let λ be a resonance of H_a.
(i) Assume that there exists a Hilbert space $\mathcal{K}_a$, densely and continuously embedded in $\mathcal{H}_a$, such that for some $\delta > 0$ we have an asymptotic expansion

$$R_a(\zeta) = \frac{i}{(\zeta-\lambda)^{1/2}} Q_a + G_0 + o(1) \tag{3.23}$$

for $|\zeta - \lambda| < \delta$, $\operatorname{Im} \zeta > 0$, in norm in $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. Assume that $G_0 = G_0^*$ for in $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. Assume $Q_a = \langle \cdot, \varphi \rangle \varphi$ for some $\varphi \in \mathcal{K}_a^*$.
(ii) Assume that $V_{ab} \in \mathcal{B}(\mathcal{H}_b, \mathcal{K}_a)$.
(iii) Assume that $\lambda \in \rho(H_b)$.

Under this assumption (3.20) holds. We introduce the real constant

$$\theta_0 = \langle V_{ab} C_0 V_{ba} \varphi, \varphi \rangle.$$

Theorem 3.17. *Let Assumption* 3.16 *hold at* $\lambda \in \mathbf{R}$. *Assume that* $\theta_0 \neq 0$. *Then there exist* $\eta_0 > 0$, $\delta_0 > 0$, *and* $\delta_l(g) \asymp g^4$ *such that for all* g *with* $0 < |g| < \eta_0$ *we have* $(\lambda - \delta_l(g), \lambda + \delta_0) \cap \sigma_{\mathrm{pp}}(H(g)) = \emptyset$.

Proof. Under the present assumptions the expansion in (3.21) becomes

$$\begin{aligned} I_a - g^2 V_{ab} R_b(\zeta) V_{ba} R_a(\zeta) = I_a &- \frac{ig^2}{(\zeta-\lambda)^{1/2}} \langle \cdot, \varphi \rangle V_{ab} C_0 V_{ba} \varphi \\ &- g^2 \left(V_{ab} C_0 V_{ba} G_0 + O(|\zeta - \lambda|) \right). \end{aligned} \tag{3.24}$$

Thus this time the singular part is

$$S(\zeta) = I_a - \frac{ig^2}{(\zeta-\lambda)^{1/2}} \langle \cdot, \varphi \rangle V_{ab} C_0 V_{ba} \varphi.$$

We introduce

$$\kappa(g; \zeta) = \frac{g^2}{i(\zeta-\lambda)^{1/2} + g^2 \theta_0}.$$

Then the inverse of $S(\zeta)$ is given by

$$S(\zeta) = I_a - \kappa(g; \zeta) \langle \cdot, \varphi \rangle V_{ab} C_0 V_{ba} \varphi,$$

provided $\zeta - \lambda \neq -g^4 \theta_0^2$.

The remainder of the proof is similar to the proof of Theorem 3.12 and is omitted. □

4 Further results

There are several other cases which could be considered. It is possible to have both an eigenvalue and a resonance at a threshold for H_a, and furthermore, an eigenvalue of H_b could also occur at λ. It seems that the present technique is difficult to adapt to these problems. One will have to go through several stages of decomposition.

We should also mention that the results obtained here, except Theorem 3.15, depend crucially on the self-adjointness of the coefficients (as we have defined them, see for example (3.1)) in the asymptotic expansions around a threshold, which means that the threshold is at the bottom of the essential spectrum. Theorem 3.15 is an example showing how to adapt the arguments to a case where the threshold is not at the bottom of the essential spectrum of $H(0)$.

We have modelled our assumptions concerning asymptotic expansion on a Schrödinger operator $-\Delta + V(x)$ on $L^2(\mathbf{R}^d)$ for d odd. It is easy to extend the results cover to the even-dimensional case. One has for such Schrödinger operators in dimensions $d \geq 6$ an expansion of the form

$$R(\zeta) = -\frac{1}{\zeta}P_0 - \ln\zeta B^0_{-1} + B^0_0 + \zeta(\ln\zeta)^2 B^2_1 + \zeta\ln\zeta B^1_1 + \zeta B^0_1 + o(\zeta)$$

as $\zeta \to 0$. We always have $B^0_{-1}P_0 = B^0_{-1}$, and generically $P_0 = 0$, i. e. zero is not an eigenvalue, see [8]. In dimension $d \geq 5$ there is no zero resonance (half-bound state). Similar expansions hold in dimensions $d = 2, 4$, but here additionally the zero resonance may occur. The above abstract arguments can clearly be adapted to cover this type of expansion.

We note that the method of proof used above can be extended to give asymptotic expansions of the resolvent of a two-channel Hamiltonian around a threshold. Some results in this direction have been obtained in [13] in the case when both H_a and H_b are one-dimensional Schrödinger operators.

5 Applications

We now give some applications of the results in Section 3.

5.1 The Friedrichs model

We start by an application to the Friedrichs model [7]. Let $\mathcal{H}_a = L^2([0,1])$ and H_a multiplication by x, which is self-adjoint on its maximal domain. Fix

$s > 1/2$ and let

$$\mathcal{K}_a = \{f \in C^1([0,1]) \,|\, f(0) = f'(0) = 0, \\ \text{and } f' \text{ is Hölder continuous of order } s\}$$

with the norm

$$\|f\|_{\mathcal{K}_a} = \sup_x (|f(x)| + |f'(x)|) + \sup_{x \neq y} \frac{|f'(x) - f'(y)|}{|x-y|^s}.$$

We have in $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$ the expansion

$$R_a(\zeta) = G_0 - \zeta G_2 + o(\zeta),$$

where G_0 denotes multiplication by x^{-1}, and G_2 multiplication by x^{-2}. Let $\mathcal{H}_b = \mathbf{C}$ and $H_b = 0$. Then 0 is a simple eigenvalue of H_b. Furthermore, let $v \in \mathcal{K}_a$, $v \neq 0$, be a real-valued function. We define $V_{ab}z = v(x)z$ for $z \in \mathcal{H}_b$ and $V_{ba}f = \int_0^1 \overline{v(x)} f(x)dx$ for $f \in \mathcal{H}_a$. Then it is easy to verify that all conditions in Assumption 3.4 are satisfied. Since

$$\alpha_0 = \int_0^1 \frac{1}{x} |v(x)|^2 dx > 0,$$

and since the limiting absorption principle holds for H_a in $\mathcal{K}_a$, we can combine Theorem 3.6 with Remark 3.3 and a covering argument to conclude that for a given $\varepsilon > 0$ there exist an $\eta_0 > 0$ and a function $\delta_l(g) \asymp g^2$ such that $(-\delta_l(g), 1-\varepsilon) \cap \sigma_{\rm pp}(H(g)) = \emptyset$ for all g, $0 < |g| < \eta_0$.

In this model we can show directly that there is a simple negative eigenvalue for small g. Thus the embedded eigenvalue becomes a discrete eigenvalue in this case. We are looking for solutions to

$$H(g) \begin{pmatrix} f \\ 1 \end{pmatrix} = \mu \begin{pmatrix} f \\ 1 \end{pmatrix}.$$

Analyzing the conditions we get

$$f(x) = g \frac{v(x)}{x - \mu}$$

and

$$\mu = -g^2 \int_0^1 \frac{|v(x)|^2}{x - \mu} dx. \tag{5.1}$$

A contraction argument shows that (5.1) has a unique solution $\mu(g) < 0$ for g sufficiently small. Furthermore, this solution satisfies

$$\mu(g) = -g^2 \int_0^1 \frac{|v(x)|^2}{x} dx + O(g^4).$$

Thus this example also shows that the result in Theorem 3.6(i) is optimal.

5.2 Schrödinger operators

As an example we take the problem of confined channels discussed in [5]. The model considered in that paper is of the form (1.1). We introduce the following set-up, which agrees with the one considered in [5], up to minor changes in constants and notation.

$$\mathcal{H}_a = \mathcal{H}_b = L^2(\mathbf{R}^3),$$
$$H_a = -\Delta, \quad H_b = -\Delta + U(x),$$
$$V_{ab}(x) = V_{ba}(x) \equiv V(x) \in L^\infty(\mathbf{R}^3), \quad \text{real-valued.}$$

We further assume that $V(x) = O(|x|^{-\beta})$ as $|x| \to \infty$ for some $\beta > 0$. Here $U(x)$ is a confining potential, meaning that H_b has a purely discrete spectrum, bounded below. As an example one can take $U(x) = x^2 + \omega_0$. By a suitable choice of ω_0 one of the eigenvalues of H_b coincides with zero.

The resolvent $R_a(\zeta)$ has the integral kernel

$$\frac{e^{i\zeta^{1/2}|x-y|}}{4\pi|x-y|}.$$

Let $L^{2,s}(\mathbf{R}^3) = L^2(\mathbf{R}^3; (1+|x|)^{2s}dx)$ denote the weighted space. We assume that the decay rate satisfies $\beta > 5/2$ and fix $5/2 < s < \beta$. Then we choose $\mathcal{K}_a = L^{2,s}(\mathbf{R}^3)$ and use the identification $\mathcal{K}_a^* = L^{2,-s}(\mathbf{R}^3)$.

It follows from Taylor's formula (see [10]) that we have the asymptotic expansion

$$R_a(\zeta) = G_0 + i\zeta^{1/2}G_1 - \zeta G_2 + o(|\zeta|)$$

as $\zeta \to 0$, $\operatorname{Im}\zeta > 0$, in the norm topology of $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. The operators G_j are given by their integral kernels as follows.

$$G_0\colon \frac{1}{4\pi|x-y|}, \quad G_1\colon \frac{1}{4\pi}, \quad G_2\colon \frac{|x-y|}{8\pi}.$$

We have now verified the conditions in parts (i) and (ii) of Assumption 3.4. In order to verify the last part of this assumption, or Assumption 3.8, let P_b denote the projection onto the eigenspace of eigenvalue zero for H_b. We are assuming $0 < \operatorname{rank} P_b < \infty$. Given our decay assumption on the potential, the operator $V_{ba}G_0V_{ab}$ is bounded and strictly positive on $\mathcal{H}_b$, as can be seen using the explicit integral representation and the Fourier transform. It follows that the operator $P_bV_{ba}G_0V_{ab}P_b$ is strictly positive and invertible in $\mathcal{B}(P_b\mathcal{H}_b)$.

In the case of a simple eigenvalue we have $\alpha_0 > 0$ (see (3.2)), and in the general case all conditions in Assumption 3.8 are satisfied. Thus the results in Theorem 3.6(i) or Theorem 3.9 hold for the example under consideration. Combining these results with the assumed discreteness of the spectrum of H_b and Remark 3.3, we find that if λ_1 denotes the smallest eigenvalue of H_b larger than zero, then given $\varepsilon > 0$ we can find $\eta_0 > 0$ such that for $0 < |g| < \eta_0$ we have $(-\delta_l(g), \lambda_1 - \varepsilon) \cap \sigma_{\rm pp}(H(g)) = \emptyset$. As above $\delta_l(g) \asymp g^2$.

The above results can be extended to cover the case $H_a = -\Delta + W(x)$, using the results on asymptotic expansions in [10], provided W has sufficient decay, and we are in the generic case. Since the results are fairly obvious, we omit further details.

5.3 Magnetic Schrödinger operators

The results obtained in Section 3 can be applied to a Schrödinger operator in $L^2(\mathbf{R}^3)$ with a constant magnetic field and an axisymmetrical electric potential. Under these assumptions the operator can be represented in a multi-channel framework. For the lowest Landau level we can fit the problem into the two-channel framework considered here. It requires considerable preparation to apply our results. Preliminary results on this case are contained in [13]. Complete results will be published elsewhere.

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