Wave number reconstruction for the acoustic problem

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We formulate a three dimensional inverse medium problem in which the inhomogeneity is bounded, has a cylindrical shape and only depends on base's variables. We prove that the wave number can be uniquely reconstructed as soon as we know the scattered field on the cylinder's boundary. The reconstruction algorithm is explicitly given and proved to be stable.

1. INTRODUCTION

This paper considers the inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium, which is usually referred to as the acoustic inverse medium problem (see [3] and references therein).

As is very well known, solving such an inverse problem would mean to determine the wave number of a (compactly supported) scatterer immersed in an homogeneous host medium. The typical experiments consist in scattering some particularly chosen incident fields, and then their corresponding scattered fields are measured somewhere outside the scatterer (or equivalently, we measure their far field patterns).

There are at least two very interesting questions regarding the full three dimensional inverse medium problem. First, the uniqueness of its solution under some a priori assumptions on the scatterer's properties and second, the availability of a stable, as explicit and simple as possible reconstruction procedure.

The uniqueness aspects are well understood by now and many elegant ways of proving such results may be found in the literature. Among these, probably the most complete results were obtained employing the powerful method of Lippmann-Schwinger-Fadeev scattering solutions. Let us cite here only the celebrated works of Nachman [5], Novikov [6] and Ramm [7], [8] .

Even though the above mentioned method also allows one to reconstruct the wave number (actually, in this case the uniqueness is a simple consequence of the unique reconstruction), it is rather difficult to follow up all the steps from which the reconstruction algorithm is made of. It is even more difficult to formulate stability results, especially when reconstruction procedures are not available. For the conductivity problem this goal was achieved by Alessandrini [1] who inspired Stefanov [10] in the inverse problem of potential scattering; Stefanov's proof was subsequently carried over to the electromagnetic case by Hähner [4].

In order to be able to formulate inverse medium problems with more easily solvable uniqueness, reconstruction and stability issues, one has to impose some more restrictive conditions on the scatterer's assumed properties. This is what we actually do in our paper: we formulate a three dimensional inverse medium problem in which the scatterer is bounded, has a cylindrical shape and the wave number only depends on base's variables.

We then show that the wave number can be uniquely reconstructed as soon as we know the scattered field on the cylinder's boundary. The reconstruction algorithm is explicitly given and proved to be stable.

Let us now briefly describe the structure of the paper:

Section 2 contains the rigorous description of our setting and the statements of our results. We start by describing the a-priori assumptions we make on the scatterer, and after a few definitions concerning various direct problems we arrive at our inverse problem (see Definition 2.4). The next step we make is to define a space of "admissible data"; a function belongs to this space if it fulfills a certain list of conditions. These conditions build in fact the algorithm one should follow for the reconstruction of the wave number.

Clearly, the difficult problem resides in proving that these conditions are also necessary for scattering fields coming from "almost all" forward problems. For more precision, see Theorem 2.1.

Theorem 2.2 and Corollary 2.1 reformulate the "almost" equivalence between the inverse problem and the space of admissible data. Corollary 2.2 states a uniqueness result from the knowledge of partial data.

Finally, Theorem 2.3 employs the reconstruction procedure in order to conclude that small changes in the measured boundary field lead to small changes in its corresponding wave number.

Section 3 contains the main technical core of our paper, being entirely dedicated to the proof of Theorem 2.1. For reader's convenience, we added a concluding overview intended to "put all the things together".

Section 4 gives the proofs for Theorem 2.2 and Corollaries 2.1 and 2.2.

Section 5 contains the proof of Theorem 2.3.

2. PRELIMINARIES AND THE RESULTS

2.1. General notations

The acoustic scatterer will be modeled by a cylindrical domain $\Lambda \subset \mathbf{R}^3$. If $\Omega \subset \mathbf{R}^2$ is an open, bounded and connected C^∞ domain, then for $a > 0$ we define

$$\Lambda = \Omega \times [0, a] \ . \tag{1}$$

Throughout the paper, three dimensional vectors $\vec{x}$ will sometimes be represented as $\vec{x} = (\underline{x}, x_3)$, where $\underline{x} \in \mathbf{R}^2$ and $x_3 \in \mathbf{R}$ (or $\mathbf{C}$). The characteristic function of a set M will be denoted by $\mathbf{1}_M$.

From technical reasons, we introduce a smoother domain $\Lambda' \subset \{\vec{x} \in \mathbf{R}^3,\ x_3 \geq 0\}$, obeying

$$\Lambda \subset \Lambda', \quad (\Omega, 0), (\Omega, a) \subset \partial\Lambda',\ \partial\Lambda' \in C^\infty \ . \tag{2}$$

Let us now enumerate our a-priori assumptions on k^2; the set of all such wave numbers will be generically denoted by W:

Assumptions 2.2.1.

1.Outside the scatterer, the square of the wave number is constant and equal to $k_0^2 = \frac{\omega^2}{c_0^2} + i\omega\sigma_0,\ \sigma_0 > 0,\ \omega > 0$*, while inside it only depends on two variables:* $k^2(\vec{x}) = \kappa_\omega(\underline{x})\mathbf{1}_{[0,a]}(x_3) + k_0^2, \mathrm{supp}(\kappa) \subseteq \Omega$*;*

2.There exist two real valued functions $\kappa_1, \kappa_2 \in C_0^1(\Omega)$ *such that for* $\omega > 0$*:*

(i)

$$\kappa_\omega(\underline{x}) = \omega^2\kappa_1(\underline{x}) + i\ \omega\kappa_2(\underline{x}); \tag{3}$$

(ii)

$$\begin{aligned} k^2(\vec{x}) &= \kappa_\omega(\underline{x})\mathbf{1}_{[0,a]}(x_3) + k_0^2 \\ &= \omega^2[\kappa_1(\underline{x})\mathbf{1}_{[0,a]}(x_3) + 1/c_0^2] + i\ \omega\ [\kappa_2(\underline{x})\mathbf{1}_{[0,a]}(x_3) + \sigma_0] \ ; \end{aligned} \tag{4}$$

(iii)

$$\inf_{\vec{x}\in\mathbf{R}^3}[\kappa_2(\underline{x})\mathbf{1}_{[0,a]}(x_3) + \sigma_0] > 0. \tag{5}$$

Throughout the paper, by $\sqrt{z}$ we mean the principal branch of the complex square root, holomorphic on $\mathbf{C} \setminus (-\infty, 0]$. Define

$$\vec{p} = (\underline{p}, p_3) = (\underline{p}, \sqrt{k_0^2 - |\underline{p}|^2}) \in \mathbf{C}^3,\ \underline{p} \in \mathbf{R}^2,\ p_3 = p_3(\underline{p}) \in \mathbf{C}. \tag{6}$$

Notice since $\Im(p_3) > 0$ that $e^{ip_3 a} - 1 \neq 0$ for all $\underline{p}$ and moreover,

$$\lim_{|\underline{p}| \to \infty} \frac{1}{|e^{ip_3 a} - 1|} = 1.$$

The incidents fields we work with are:

$$u^{in}(\vec{x}; \vec{p}) = u^{in}(\underline{x}, x_3; \underline{p}) = e^{i\underline{x}\cdot\underline{p} + ip_3 x_3}\ , \tag{7}$$

for all possible values of $\underline{p} \in \mathbf{R}^2$ at some frequency $\omega > 0$.

DEFINITION 2.1. We say that $u(\vec{x})$ solves the forward acoustic scattering problem if:

1. $(\Delta + k^2(\vec{x}))u = 0$, $u \in H^2_{\mathrm{loc}}(\mathbf{R}^3)$;
2. $u(\vec{x}) = u^{in}(\vec{x}) + u^{sc}(\vec{x})$;
3. $(\Delta + k_0^2)u^{in} = 0$ in $\mathbf{R}^3$, $k_0^2 = \frac{\omega^2}{c_0^2} + i\omega\sigma_0,\ \sigma_0 > 0,\ \omega > 0$;
4. $\lim_{r\to\infty} r\left(\frac{\partial u^{sc}}{\partial r} - ik_0 u^{sc}\right) = 0.$

Under our a-priori assumptions on k^2, the above forward problem has a unique solution (see [3, theorem 8.7]). We also know that the solution to the forward problem also solves the Lippmann-Schwinger equation in $H^2_{loc}(\mathbf{R}^3)$:

$$u(x) = u^{in}(\vec{x}) - \int_\Lambda G_0(\vec{x}, \vec{y}, \omega)(k^2(\vec{y}) - k_0^2)u(\vec{y})d\vec{y}, \tag{8}$$

where

$$G_0(\vec{x}, \vec{y}; \omega) = (4\pi\ |\vec{x} - \vec{y}|)^{-1} \exp\left[i\ k_0(\omega)|\vec{x} - \vec{y}|\right]. \tag{9}$$

Denote by $\mathcal{G}_\omega(k^2)$ the integral operator having the kernel $G_0(\vec{x}, \vec{y}; \omega)(k^2(\vec{y}) - k_0^2)$. Then $\mathcal{G}_\omega(k^2)$ is a compact operator on $C^0(\Lambda)$. Due to the unique solvability of (8), it can be easily argued that the operator $\mathbf{1} + \mathcal{G}_\omega(k^2)$ is one-to-one, therefore invertible (the Fredholm alternative). Denote by $\xi(\vec{x}, \underline{p}, \omega) =$

$\{\mathcal{G}_\omega(k^2)u^{in}(\cdot\ ;\underline{p},\omega)\}(\vec{x}) \in C^0(\Lambda)$; the same notation will be employed for its natural extension to $H^2(\mathbf{R}^3)$:

$$\xi(\vec{x},\underline{p},\omega) = \int_\Lambda G_0(\vec{x},\vec{y};\omega)(k^2(\vec{y}) - k_0^2)u^{in}(\vec{y},\underline{p})d\vec{y}. \tag{10}$$

Then the restriction to Λ of the scattered field u^{sc} is represented as:

$$u^{sc}(\vec{x};\underline{p},\omega) = -\{[\mathbf{1} + \mathcal{G}_\omega(k^2)]^{-1}\xi(\cdot\ ,\underline{p},\omega)\}(\vec{x}) \tag{11}$$

The solution to (8) is:

$$\begin{aligned} u(\vec{x};\underline{p},\omega) \ &= \ u^{in}(\vec{x},\underline{p},\omega) - \xi(\vec{x},\underline{p},\omega) + \\ &\int_\Lambda G_0(\vec{x},\vec{y};\omega)(k^2(\vec{y}) - k_0^2)\{[\mathbf{1} + \mathcal{G}_\omega(k^2)]^{-1}\xi(\cdot\ ,\underline{p},\omega)\}(\vec{y}). \end{aligned} \tag{12}$$

DEFINITION 2.2. We say that $\tilde{u}(f)$ solves the exterior Dirichlet problem if:

1. $(\Delta + k_0^2)\tilde{u} = 0$ in $\mathbf{R}^3 \setminus \overline{\Lambda'}$, $\tilde{u} \in H^2_{\mathrm{loc}}(\mathbf{R}^3 \setminus \overline{\Lambda'})$;
2. $\tilde{u}|_{\partial\Lambda'} = f \in H^{3/2}(\partial\Lambda') \subset C^0(\partial\Lambda')$;
3. $\tilde{u}$ satisfies the Sommerfeld radiation condition at infinity.

It is well known (see [2, Theorem 3.21] or[3, Theorem 3.9]) that the exterior Dirichlet problem has a unique solution.

2.2. Stating the inverse problem

As we have already outlined in the introduction, our main interest resides in reconstructing the wave number from the measured boundary data. The experiments we consider consist in the scattering of the particularly chosen incident fields $u^{in}(\cdot\ ;\underline{p},\omega)$ (see (7)) at a fixed frequency $\omega > 0$ and all possible values of $\underline{p} \in \mathbf{R}^2$.

A particularly interesting set is composed from the values of the scattered field on the boundary of Λ', generated by all possible $k \in W$; this set will be denoted by F_ω:

$$F_\omega := \{u^{sc}(\vec{x};\underline{p},\omega) \mid \vec{x} \in \partial\Lambda',\ \underline{p} \in \mathbf{R}^2,\ k \in W\}\ . \tag{13}$$

Another important definition comes next:

DEFINITION 2.3. Fix $N > 0$. We denote by $W_N \subseteq W$ the set of wave numbers for which $\max\{||\kappa_1||_{C^1}, ||\kappa_2||_{C^1}\} \leq N$. Then by $F_{N,\omega} \subset F_\omega$ we understand the subset of only those Φ_ω's which correspond (via the forward problem) to wave numbers in W_N.

The inverse problem we study can be stated as follows:

DEFINITION 2.4. Fix a frequency $\omega > 0$ and perform the above experiments, for every possible $\underline{p} \in \mathbf{R}^2$. Denote by $\Phi_\omega(\cdot, \underline{p})$ the scattered field restricted to $\partial\Lambda'$. Then
i. Existence: find sufficient conditions for a function $\Phi_\omega(\cdot, \underline{p}) \in H^{3/2}(\partial\Lambda')$ to be an element of F_ω.
ii. Unique reconstruction: if $\Phi_\omega \in F_\omega$, then construct a unique $k^2(\Phi_\omega) \in W$ such that by introducing it into the forward problem (see Definition 2.1), the scattered field such obtained coincides with Φ_ω on $\partial\Lambda'$.

Remark. In this paper we only answer the unique reconstruction question *ii*. The more difficult problem of giving sufficient conditions for a Φ_ω in order to be an element of F_ω, remains open.

In what follows, we are mainly interested in three things:

1. First, to formulate a list of sufficient conditions on the measured data $\Phi_\omega \in F_\omega$ which should hold in order to permit the unique reconstruction of the wave number (or κ_ω); these conditions will also provide the reconstruction algorithm for κ_ω. The set of measured scattered fields having those properties will be called *the space of admissible data* and generically denoted by $\mathcal{A}_\omega$; clearly, $\mathcal{A}_\omega \subseteq F_\omega$.
2. Second, to prove that $\mathcal{A}_\omega$ "is not empty" and sometimes equals F_ω (for more precision see Theorem 2.1).
3. Third, to study some stability properties of the mapping $F_{\omega,N} \ni \Phi_\omega \mapsto k^2(\Phi_\omega) \in W$ (see definition 2.3).

Let us now start the rigorous description of $\mathcal{A}_\omega$:

DEFINITION 2.5. We say that $\Phi \in \mathcal{A}_\omega$ if

1. $\Phi(\cdot, \underline{p}) \in H^{3/2}(\partial\Lambda')$, for all $\underline{p} \in \mathbf{R}^2$;
2. Denote by $[\tilde{u}(\Phi)](\cdot, \underline{p})$ the solution to the exterior Dirichlet problem corresponding to the boundary value $\Phi(\cdot, \underline{p})$ and denote by

$$[u(\Phi)](\cdot; \underline{p}) = u^{in}(\cdot; \vec{p}) + [\tilde{u}(\Phi)](\cdot; \underline{p}) \; . \tag{14}$$

Take $\rho > 0$ so large that the ball centered at the origin with radius ρ (i.e. $B(0, \rho)$) includes $\overline{\Lambda'}$. Define

$$[\eta(\Phi)](\underline{p}) = \frac{1}{e^{ip_3 a} - 1} \int_{|\vec{y}|=\rho} \{-[\partial_3 u]\,[\partial_\nu u] + u[\partial_\nu \partial_3 u]\}\; d\sigma(\vec{y}), \tag{15}$$

where ν is the exterior normal; our second condition is

$$[\eta(\Phi)] \in L^2(\mathbf{R}^2). \tag{16}$$

3. If $\underline{x} \in \Omega$, define

$$\mathcal{L}(\underline{x},\underline{p}) := \frac{1}{e^{ip_3 a}-1}\{[u(\Phi)]^2(\underline{x},a;\underline{p}) - [u(\Phi)]^2(\underline{x},0;\underline{p})\} = \\ = e^{i2\underline{p}\cdot\underline{x}} + \mathcal{L}_1(\underline{x},\underline{p}), \tag{17}$$

where

$$\mathcal{L}_1(\underline{x},\underline{p}) := \frac{1}{e^{ip_3 la}-1}\{2[u^{in}\Phi](\underline{x},a;\underline{p}) \\ -2[u^{in}\Phi](\underline{x},0;\underline{p}) + \Phi^2(\underline{x},a;\underline{p}) - \Phi^2(\underline{x},0;\underline{p})\}. \tag{18}$$

Our third condition is:

$$\mathcal{L}_1 \in L^2(\Omega\times\mathbf{R}^2),\ \text{i.e.}\ \int_{\Omega\times\mathbf{R}^2} |\mathcal{L}_1|^2(\underline{x},\underline{p})d\underline{x}\ d\underline{p} < \infty\ ; \tag{19}$$

4. If $\mathcal{F}_{\underline{p}}$ denotes the partial Fourier transform with respect to the “$\underline{p}$” variable, then define

$$\mathcal{K}(\underline{x},\underline{y}) = \left[\mathcal{F}_{\underline{p}}\left(\mathcal{L}_1(\underline{y},\underline{p}/2)\right)\right](\underline{x}). \tag{20}$$

Denote by K the Hilbert-Schmidt operator corresponding to $\mathcal{K}$ acting on $L^2(\mathbf{R}^2\times\Omega)$ and with the same letter its natural restriction to $L^2(\Omega\times\Omega)$. Then our fourth condition is that the operator $\mathbf{1}+K$ is invertible in $B(L^2(\Omega\times\Omega))$;

5. Denote by

$$[\tilde{\eta}(\Phi)](\underline{y}) = \left[\mathcal{F}_{\underline{p}}\left(\eta(\Phi)(\underline{p}/2)\right)\right](\underline{y}), \tag{21}$$

and with the same letter its natural restriction to $L^2(\Omega)$.

Denote by

$$\kappa(\Phi) = (\mathbf{1}+K)^{-1}\tilde{\eta}(\Phi) \in L^2(\Omega), \tag{22}$$

extended by zero outside Ω. Our fifth condition is that

$$k^2(\Phi) := \kappa(\Phi)\mathbf{1}_{[0,a]} + k_0^2, \tag{23}$$

obeys the Assumptions 2.2.1;

6. Introduce $k^2(\Phi)$ in (11) and denote the scattered field such obtained by $v^{sc}(\Phi) \in H^2_{\text{loc}}(\mathbf{R}^3)$. The last condition is

$$v^{sc}(\Phi)|_{\partial\Lambda'} = \Phi. \tag{24}$$

Remarks. 1. It may seem that the last two conditions are awkward and superfluous as soon as we assume that Φ_ω comes from a forward problem (i.e. belongs to F_ω). But they are justified if we reason from the point of view of a practical application. Indeed, the scattering experiments provide us with a Φ_ω which could come from a wave number which is not in W (this would mean that our a-priori assumptions for the scatterer are wrong). Hence even if the computation of $k^2(\Phi)$ in (23) is possible, we still have to check that (24) holds in order to conclude that our a-priori assumptions about the scatterer are correct.

2. Although equation (24) is highly nonlinear, it only involves Φ; a characterization of its solutions would automatically lead us to an affirmative answer to the "existence" part of our inverse problem.

2.3. The results

We now are prepared to give our first result. It essentially says that the space of admissible data $\mathcal{A}_\omega$ is not empty:

THEOREM 2.1. *Let $\omega_0 > 0$ and $N > 0$ be arbitrarily large, but fixed.*
i. Assume that $k \in W$ and ω is allowed to take values in $(0, \omega_0)$. Denote by $u \in H^2_{\rm loc}(\mathbf{R}^3)$ the solution to the forward problem corresponding to k^2 (see Definition 2.1 and formula (12)). Then there exists a finite set $M \subset (0, \omega_0)$ such that for any frequency $\omega \in (0, \omega_0) \setminus M$ one has $u^{sc}|_{\partial\Lambda'} \in \mathcal{A}_\omega$.
ii. There exists $\omega_N > 0$ such that for any $\omega \in (0, \omega_N)$ and any $k \in W_N$ (see Definition 2.3) one has $u^{sc}|_{\partial\Lambda'} \in \mathcal{A}_\omega$. Therefore, in this case $\mathcal{A}_\omega = F_{N,\omega}$.

Remark. During the proof of Theorem 2.1, we will show that $u^{sc}|_{\partial\Lambda'}$ always obeys the first three conditions in Definition 2.5. The only problem which could appear in the reconstruction process is that the operator $\mathbf{1} + K$ introduced in the fourth condition might not be invertible. That is why the following definition is justified:

DEFINITION 2.6. Fix $\omega > 0$ and take $k^2 = k^2(\omega)$ as in (3). Construct the integral kernel $\mathcal{K}(\underline{x}, \underline{y}; \omega)$ as in (20). We say that ω is regular with respect to κ_1 and κ_2 if the operator $\mathbf{1} + K(\omega)$ is invertible.

The next theorem couples Theorem 2.1, Definition 2.5 and Definition 2.6, stating the conditions we need such that the reconstruction part of our inverse problem to have a unique solution:

THEOREM 2.2. *Fix $\omega_0 > 0$ and choose some ω in $(0, \omega_0)$. Then the following two statements are equivalent:*
i. The scattered field restricted to the boundary belongs to the space of admissible data;

ii. The inverse problem (see Definition 2.4) has a unique solution $k \in W$ given by some $\kappa_1, \kappa_2 \in C_0^1(\Omega)$ (see (3)), and the frequency is regular with respect to κ_1 and κ_2.

The next corollary is a natural consequence of the above theorems, saying that if the wave numbers are restricted to some W_N, we can uniquely reconstruct them from the knowledge of the scattered field on the boundary, for some sufficiently small frequency:

COROLLARY 2.1. *Fix $N > 0$ and assume that the wave numbers are only allowed to belong to W_N (see Definition 2.3). Then there exists $\omega_N > 0$ such that for any frequency $\omega \in (0, \omega_N)$, the inverse problem has a unique solution in W_N, given by (23) and (22).*

We can also formulate a uniqueness result claiming that if we have two wave numbers in W for which we know that their corresponding scattered fields are equal at the boundary for some frequencies and some (not all) $\underline{p}$'s, then those wave numbers must be equal:

COROLLARY 2.2. *Let $\omega > 0$ and let $k_1, k_2 \in W$. Denote by $u_1(\vec{x}; \underline{p}, \omega)$ ($u_2(\vec{x}; \underline{p}, \omega)$) the total field obtained by introducing k_1 (k_2) in (12), i.e. the solution to the forward problem.*

Assume that $u_1|_{\partial\Lambda'}(\cdot\ ; \underline{p}, 1/n) = u_2|_{\partial\Lambda'}(\cdot\ ; \underline{p}, 1/n)$ for all $\underline{p}$ in an open subset of $\mathbf{R}^2$ and $n > N_0$. Then $k_1 = k_2$.

Finally, let us state our stability result for the inverse problem. We will prove (see (34)) that if $\Phi \in F_\omega$, then $\sup_{\vec{y} \in \partial\Lambda'} |\Phi(\vec{y}; \cdot)| \in L^2(\mathbf{R}^2)$. Then introduce the following norm on F_ω:

$$|||\Phi|||^2 := \int_{\mathbf{R}^2} \left(\sup_{\vec{y} \in \partial\Lambda'} |\Phi(\vec{y}; \underline{p})| \right)^2 d\underline{p}. \tag{25}$$

It is not difficult to see that

$$||\cdot||^2_{L^2(\partial\Lambda' \times \mathbf{R}^2)} \leq |\partial\Lambda'| \quad |||\cdot|||^2. \tag{26}$$

We thus justified the use of $|||\cdot|||$-norm for measuring the distance between two boundary data:

THEOREM 2.3. *Fix $N > 0$ and find ω_N as in Theorem 2.1.ii. Choose an $\omega \in (0, \omega_N)$ and fix an arbitrary $\Phi \in F_{N,\omega}$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that $||k^2(\Psi) - k^2(\Phi)||_{L^2(\Omega)} < \epsilon$ whenever $\Psi \in F_{N,\omega}$ and $|||\Psi - \Phi||| < \delta$.*

3. PROOF OF THEOREM 2.1

The strategy will consists in showing that if $k \in W$, all the conditions in Definition 2.5 can be verified except for the case in which the frequency ω belongs to a finite subset M of $(0, \omega_0)$.

3.1. An equation for κ

Consider $\rho > 0$ large enough such that the ball $B(0,\rho)$ includes $\overline{\Lambda'}$. Then:

LEMMA 3.1. *Let $k \in W$ (see Assumptions 2.2.1) and let $u(\cdot, \underline{p}) \in H^2_{loc}(\mathbf{R}^3)$ be the corresponding solution to the forward problem (see Definition 2.1). Denote by $\Phi(\cdot\,;\underline{p}) \in H^{3/2}(\partial\Lambda')$ the restriction of $u^{sc}(\cdot\,;\underline{p})$ to $\partial\Lambda'$. Then κ satisfies the following integral equation (see (15) and (16)):*

$$\int_\Omega \mathcal{L}(\underline{x},\underline{p})\kappa(\underline{x})d\underline{x} = [\mathcal{F}^{-1}(\kappa)](2\underline{p}) + \int_\Omega \mathcal{L}_1(\underline{x},\underline{p})\kappa(\underline{x})d\underline{x} = [\eta(\Phi)](\underline{p}) \;. \tag{27}$$

Proof. Due to the uniqueness properties for both exterior and forward problems (see Definitions 2.2 and 2.1), one can easily see that (27) is equivalent to

$$\int_\Omega [u^2(\underline{y},0) - u^2(\underline{y},a)]\kappa(\underline{y})d\underline{y} = \int_{|\vec{y}|=\rho} \{(\partial_3 u)(\partial_\nu u) - u(\partial_\nu\partial_3 u)\}d\sigma(\vec{y}), \tag{28}$$

where ν is the exterior normal.

We will first give the formal derivation of (28) and then we will argue why the formal computations are justified. First, write the Helmholtz equation $(\Delta + k^2)u = 0$ and differentiate it with respect to x_3:

$$\begin{aligned} 0 = \partial_{x_3}(\nabla^2 u + k^2 u) &= \nabla^2(\partial_{x_3}u) + k^2(\partial_{x_3}u) + u\kappa\{\delta(x_3) - \delta(x_3 - a)\} \\ &= (\nabla^2 + k^2)(\partial_{x_3}u) + u[\partial_{x_3}(k^2 - k_0^2)], \end{aligned} \tag{29}$$

where the Dirac distribution acts on $\mathcal{S}(\mathbf{R})$.

Then

$$\int_{|\vec{x}|<\rho} \{(\partial_{x_3}u)[(\nabla^2 + k^2)u] - u[(\nabla^2 + k^2)(\partial_{x_3}u)]\}d\vec{x} = \int_{|\vec{x}|<\rho} u^2[\partial_{x_3}(k^2 - k_0^2)]d\vec{x}. \tag{30}$$

Apply Green's formula in the left hand side, taking into account the fact that the distribution $\partial_{x_3}(k^2 - k_0^2)$ is supported on $(\Omega, 0)$ and (Ω, a):

$$\int_{|\vec{x}|=\rho} \{-(\partial_{x_3}u)(\partial_\nu u) + u(\partial_{x_3}\partial_\nu u)\}d\sigma(\vec{x}) = \int_\Omega \kappa(\underline{y})[u^2(\underline{y},a) - u^2(\underline{y},0)]d\underline{y}. \tag{31}$$

Notice that the above integrals are well defined, since u is a C^∞ function outside Λ and continuous on $\partial\Lambda'$ (in fact, the restriction of u to $\partial\Lambda'$ belongs to $H^{3/2}(\partial\Lambda')$).

In order to justify these formal computations, consider $\{\alpha_n\}_{n\geq 1} \subset \mathcal{S}(\mathbf{R}^3)$ an approximation of the Dirac distribution, where

$$\operatorname{supp}(\alpha_n) \subset B(0,1/n) \text{ and } \alpha_n(\vec{x}) = \alpha_n(|\vec{x}|). \tag{32}$$

Define $u_n = \alpha_n * u$; they are C^∞ functions and for any $r > 0$:

$$\lim_{n\to\infty}\{||u_n - u||_{H^2(B(0,r))} + \\ ||[\partial_{x_3} u_n]\alpha_n * [(k^2 - k_0^2)u] - u_n\alpha_n * [(k^2 - k_0^2)\partial_{x_3} u]||_{L^1(\mathbf{R}^3)}\} = 0.$$

Then due to (32) the following identity holds (n sufficiently large):

$$\int_{|\vec{x}|<\rho} \big\{[\partial_{x_3} u_n][(\Delta + k^2)u_n + \alpha_n * ((k^2 - k_0^2)u)] - \\ - u_n[(\Delta + k^2)\partial_{x_3} u_n + \alpha_n * ((k^2 - k_0^2)\partial_{x_3} u)]\big\}\, d\vec{x} = \\ = \int_\Omega \kappa(\underline{y}) \int_{|\vec{x}|<\rho} \big[u_n(\vec{x})\alpha_n(\vec{x} - (\underline{y},0))u(\underline{y},0) \\ -u_n(\vec{x})\alpha_n(\vec{x} - (\underline{y},a))u(\underline{y},a)\big]\, d\vec{x} = \\ = \int_\Omega \kappa(\underline{y})\{u(\underline{y},0)[\alpha_n * u_n](\underline{y},0) - u(\underline{y},a)[\alpha_n * u_n](\underline{y},a)\}d\underline{y}\,. \tag{33}$$

Notice that $\alpha_n * u_n = \beta_n * u$, where $\beta_n = \alpha_n * \alpha_n$ is another approximation of the Dirac distribution and

$$\lim_{n\to\infty} ||\beta_n * u - u||_{H^2(B(0,\rho))} = 0.$$

Taking n to the limit, employing Green's formula and the continuity of the trace operator between $H^2(B(0,\rho))$ and $L^2(\partial\Lambda')$, (28) follows. ▮

3.2. Checking that $\mathcal{L}_1 \in L^2(\Omega \times \mathbf{R}^2)$

This subsection will prove that the scattered field $u^{sc}(\vec{x};\underline{p})$ corresponding to a wave number $k \in W$ is sufficiently well localized in $\underline{p}$ in order to insure (19). Looking at the definition of $\mathcal{L}_1$, one sees that it would be enough proving for $u^{sc}(\vec{x};\underline{p})$ an estimate of the form

$$\sup_{\vec{x}\in\Lambda} |u^{sc}(\vec{x};\underline{p})| \leq \frac{C}{(1+\underline{p}^2)^{\frac{1+\delta}{2}}}, \tag{34}$$

where C is some constant and $\delta > 0$. The next lemmas will make this precise.

Remember that $k \in W$ is determined by $\kappa_\omega(\underline{x}) = \omega^2\kappa_1(\underline{x}) + i\omega\kappa_2(\underline{x})$, where $\omega > 0$ and $\kappa_{1,2} \in C_0^1(\Omega)$. In particular, this implies:

$$||\kappa_{1,2}||_{C^1} := \max_{|\alpha|\leq 1} \sup_{\underline{x}\in\overline{\Omega}} |D^\alpha \kappa_{1,2}(\underline{x})| < \infty. \tag{35}$$

Recall first that $k_0^2 = \omega^2/c_0^2 + i\sigma_0\omega$, $\sigma_0 > 0$, $\omega > 0$, $p_3 = \sqrt{k_0^2 - \underline{p}^2}$ and $\Im(p_3) > 0$. For further purposes, we introduce $0 < \omega_0$ and

$$\mathcal{S}_0 := \{z \in \mathbf{C};\ 0 < \Re(z) < \omega_0,\ |\Im(z)| < \sigma_0 c_0/4\}. \tag{36}$$

If $\omega \in \mathcal{S}_0$, then $\Im(p_3) > 0$. Clearly, there exists a constant $A > 1$, only depending on $\mathcal{S}_0$ such that if $|\underline{p}| \geq A$ then $p_3 \sim i\ |\underline{p}|$ i.e.

$$\Im(p_3) \geq \frac{|\underline{p}|}{2}, \quad |\underline{p}| \geq A > 1. \tag{37}$$

LEMMA 3.2. *Fix $\vec{x} \in \overline{\Lambda}$, $\underline{p} \in \mathbf{R}^2$ and $\omega \in \mathcal{S}_0$. Consider (see also (10) and (7))*

$$\xi(\vec{x}, \underline{p}, \omega) = \int_\Lambda G_0(\vec{x}, \vec{y}; \omega)\kappa_\omega(\underline{y})\mathbf{1}_{[0,a]}(y_3)e^{i\underline{p}\cdot\underline{y}+ip_3y_3}d\vec{y}\,. \tag{38}$$

Fix $0 < \delta < 1$. Then there exists a positive constant $C(\delta, \Lambda, \mathcal{S}_0)$ such that for any $\omega \in \mathcal{S}_0$ and $\underline{p} \in \mathbf{R}^2$:

$$\sup_{\vec{x}\in\overline{\Lambda}} |\xi(\vec{x}, \underline{p}, \omega)| \leq \frac{C(\delta, \Lambda, \mathcal{S}_0)}{(1 + |\underline{p}|)^{1+\delta}} ||\kappa_\omega||_{C^1} \left[1 + \frac{1}{\Im(k_0)^{2\delta}} + \frac{|k_0|}{\Im(k_0)}\right]. \tag{39}$$

Proof. Fix $0 < \delta < 1$. Define $\gamma(\underline{p}) = (1 + |\underline{p}|)^{-\delta}$. One of the key ingredients used in the proof of (39) is the following estimate, uniform in $\omega \in \mathcal{S}_0$ (see also (37)):

$$\exp\left[-a\gamma(\underline{p})\Im(p_3)\right] \leq \begin{cases} 1 & \text{if} \quad |\underline{p}| \leq A(\delta, \mathcal{S}_0) \\ \exp\left[-a|\underline{p}|^{1-\delta}/2\right] & \text{if} \quad |\underline{p}| \geq A(\delta, \mathcal{S}_0) \end{cases} \tag{40}$$

First, rewrite ξ as

$$\begin{aligned} \xi(\vec{x}, \underline{p}, \omega) &= \int_\Omega \int_0^a G_0(\vec{x}, \vec{y}; \omega)\kappa_\omega(\underline{y})e^{i\underline{p}\cdot\underline{y}+ip_3y_3}d\underline{y}\, dy_3 \\ &:= \xi_1(\vec{x}, \underline{p}, \omega) + \xi_2(\vec{x}, \underline{p}, \omega), \end{aligned} \tag{41}$$

where

$$\xi_1(\vec{x},\underline{p},\omega) = \int_\Omega \int_{a\gamma(\underline{p})}^{a} G_0(\vec{x},\vec{y};\omega)\kappa_\omega(\underline{y}) e^{i\underline{p}\cdot\underline{y}+ip_3y_3} d\underline{y}\, dy_3 \tag{42}$$

and

$$\xi_2(\vec{x},\underline{p},\omega) = \int_\Omega \int_0^{a\gamma(\underline{p})} G_0(\vec{x},\vec{y};\omega)\kappa_\omega(\underline{y}) e^{i\underline{p}\cdot\underline{y}+ip_3y_3} d\underline{y}\, dy_3. \tag{43}$$

Let us first treat ξ_1. Since $a\gamma(\underline{p}) < y_3 < a$,

$$|e^{ip_3y_3}| \leq e^{-\Im(p_3)y_3} \leq \exp\left[-a\gamma(\underline{p})\Im(p_3)\right],$$

it follows from (40) that

$$\begin{aligned}(1+|\underline{p}|)^{1+\delta}|\xi_1(\vec{x},\underline{p},\omega)| \;&\leq\; C(\delta,\mathcal{S}_0)\;||\kappa_\omega||_{C^1}\; \int_\Lambda \frac{1}{4\pi|\vec{x}-\vec{y}|}d\vec{y} \leq \\ &\leq\; C(\delta,\mathcal{S}_0)||\kappa_\omega||_{C^1}\;\mathrm{const}(\Lambda).\end{aligned} \tag{44}$$

Therefore, ξ_1 obeys (39).

Secondly, let us study ξ_2. If $|\underline{p}| \leq A(\delta,\mathcal{S}_0)$, then ξ_2 obeys an estimate similar to (43), therefore the “nontrivial” region is $|\underline{p}| \geq A(\delta,\mathcal{S}_0)$ (where we also have (37)).

For technical reasons, we introduce $\xi_{2,\varepsilon}$ by

$$\xi_{2,\varepsilon}(\vec{x},\underline{p},\omega) = \int_0^{a\gamma(\underline{p})} e^{ip_3y_3}\psi_\varepsilon(\vec{x},y_3;\underline{p},\omega)dy_3 \tag{45}$$

where

$$\psi_\varepsilon(\vec{x},y_3;\underline{p},\omega) = \int_{\Omega\cap\{|\underline{x}-\underline{y}|\geq\varepsilon\}} G_0(\vec{x},\vec{y};\omega)\kappa_\omega(\underline{y})e^{i\underline{p}\cdot\underline{y}}d\underline{y}. \tag{46}$$

Since (see (9))

$$|G_0(\vec{x},\vec{y},\omega)| \leq \frac{1}{4\pi|\underline{x}-\underline{y}|}, \tag{47}$$

we have $\lim_{\varepsilon\to 0}\xi_{2,\varepsilon}(\vec{x},\underline{p},\omega) = \xi_2(\vec{x},\underline{p},\omega)$. Our goal now consists in proving (39) for $\xi_{2,\varepsilon}$ uniformly in ε, which would end the proof.

Let us first remark a useful identity:

$$\begin{aligned}&p_3\xi_{2,\varepsilon}(\vec{x},\underline{p},\omega) = \\ &-i\left[\psi_\varepsilon(\vec{x},y_3;\underline{p},\omega)e^{ip_3y_3}\right]_0^{a\gamma(\underline{p})} + i\int_0^{a\gamma(\underline{p})} e^{ip_3y_3}\frac{\partial\psi_\varepsilon}{\partial y_3}(\vec{x},y_3;\underline{p},\omega)dy_3\ .\end{aligned} \tag{48}$$

Taking the limit:

$$\begin{aligned} p_3\xi_2(\vec{x},\underline{p},\omega) &= -i\psi_0(\vec{x},a\gamma(\underline{p});\underline{p},\omega)e^{ip_3a\gamma(\underline{p})} + i\psi_0(\vec{x},0;\underline{p},\omega) + \\ &+ \ i\lim_{\varepsilon\to 0}\int_0^{a\gamma(\underline{p})} e^{ip_3y_3}\frac{\partial\psi_\varepsilon}{\partial y_3}(\vec{x},y_3;\underline{p},\omega)dy_3 \ . \end{aligned} \tag{49}$$

The first term in (48) is exponentially small, due to (40) and to the estimate (see (46))

$$|\psi_0(\vec{x},y_3;\underline{p},\omega)| \leq \text{const}(\Lambda)\ ||\kappa_\omega||_{C^1}. \tag{50}$$

We decompose ψ_0 into two terms $\psi_0^{(1)}+\psi_0^{(2)}$:

$$\psi_0^{(1)}(\vec{x},0;\underline{p},\omega) = \kappa_\omega(\underline{x})e^{i\underline{p}\cdot\underline{x}}\int_{\mathbf{R}^2} e^{i\underline{p}\cdot(\underline{y}-\underline{x})}G_0(\vec{x}-(\underline{y},0);\omega)d\underline{y} \tag{51}$$

$$\psi_0^{(2)}(\vec{x},0;\underline{p},\omega) = \int_{\mathbf{R}^2} e^{i\underline{p}\cdot\underline{y}}G_0(\vec{x}-(\underline{y},0);\omega)[\kappa_\omega(\underline{y})-\kappa_\omega(\underline{x})]d\underline{y} \tag{52}$$

In order to finish the proof of (39) for $|\underline{p}|\geq A>1$, it would be enough having three more estimates:

$$\sup_{\vec{x}\in\Lambda}|\psi_0^{(1)}(\vec{x},0;\underline{p},\omega)| \leq \text{const}(\Lambda)\ ||\kappa_\omega||_{C^1}|\underline{p}|^{-1}, \tag{53}$$

$$\sup_{\vec{x}\in\Lambda}|\psi_0^{(2)}(\vec{x},0;\underline{p},\omega)| \leq \text{const}(\Lambda,\mathcal{S}_0)\ (1+|\underline{p}|)^{-\delta}\frac{||\kappa_\omega||_{C^1}}{\Im(k_0)^{2\delta}} \tag{54}$$

and

$$\sup_{\varepsilon>0}\left|\frac{\partial\psi_\varepsilon}{\partial y_3}(\vec{x},y_3;\underline{p},\omega)\right| \leq \text{const}(\Lambda,\mathcal{S}_0)\ ||\kappa_\omega||_{C^1}\left[1+\frac{|k_0|}{\Im(k_0)}\right]. \tag{55}$$

Indeed, if we replace them in (48), then (39) follows.

Let us now prove (53). We transform $\psi_0^{(1)}$ using a partial Fourier transform identity for G_0:

$$\frac{1}{2\pi}\int_{\mathbf{R}} dq\frac{e^{iqx_3}}{k_0^2-\underline{p}^2-q^2} = \int e^{-i\underline{p}\cdot(\underline{x}-\underline{y})}G_0((\underline{x}-\underline{y},x_3);\omega)d\underline{y}, \tag{56}$$

Using that $\Im(k_0^2) > 0$, we split $k_0^2 - \underline{p}^2 - q^2 = -(q - \sqrt{k_0^2 - \underline{p}^2})(q + \sqrt{k_0^2 - \underline{p}^2})$ and apply the residue theorem to the left hand side of (56), yielding:

$$-i\frac{e^{i\sqrt{k_0^2 - p^2}|x_3|}}{2\sqrt{k_0^2 - p^2}},$$

hence (53) holds for $\psi_0^{(1)}$.

For $\psi_0^{(2)}$ we will finally need the C^1 regularity on $\kappa_{1,2}$:

$$\begin{aligned} p_{1,2}\psi_0^{(2)} &= -i\int_{\mathbf{R}^2}\left[\frac{\partial}{\partial y_{1,2}} e^{i\underline{p}\cdot\underline{y}}\right] G_0(\vec{x} - (\underline{y}, 0); \omega)(\kappa_\omega(\underline{y}) - \kappa_\omega(\underline{x}))d\underline{y} \\ &= i\int_{\mathbf{R}^2} e^{i\underline{p}\cdot\underline{y}}\left\{\left[\frac{\partial}{\partial y_{1,2}} G_0(\vec{x} - (\underline{y}, 0); \omega)\right](\kappa_\omega(\underline{y}) - \kappa_\omega(\underline{x})) \right. \\ &\qquad \left. + \frac{\partial \kappa_\omega}{\partial y_{1,2}}(\underline{y}) G_0(\vec{x} - (\underline{y}, 0); \omega)\right\}. \end{aligned} \tag{57}$$

Since $\kappa_{1,2}(\underline{y}) \in C^1$, $|\kappa_{1,2}(\underline{y}) - \kappa_{1,2}(\underline{x})| \leq \text{const}|\underline{y} - \underline{x}|$. Also,

$$\left|\frac{\partial}{\partial y_{1,2}} G_0(\vec{x} - (\underline{y}, 0); \omega)\right| \leq \text{const}\ \left(1 + \frac{1}{|\underline{x} - \underline{y}|^2}\right) e^{-\Im(k_0)\ |\underline{x} - \underline{y}|}\ . \tag{58}$$

Inserting the above estimate into (57) we get:

$$|\psi_0^{(2)}(\vec{x}, 0; \underline{p}, \omega)| \leq \text{const}\ (1 + |\underline{p}|)^{-1}||\kappa_\omega||_{C^1}\frac{1}{\Im(k_0)^2}. \tag{59}$$

Unfortunately, (59) does not have a sufficiently good behavior when ω is real and tends to zero. To improve it, we will use a small trick: combine (53) and (50) to obtain

$$|\psi_0^{(2)}(\vec{x}, 0; \underline{p}, \omega)| \leq \text{const}(\Lambda, \mathcal{S}_0)||\kappa_\omega||_{C^1}. \tag{60}$$

Then write

$$|\psi_0^{(2)}| = |\psi_0^{(2)}|^\delta\ |\psi_0^{(2)}|^{1-\delta} \leq \text{const}(\Lambda, \mathcal{S}_0)\ (1 + |\underline{p}|)^{-\delta}\frac{||\kappa_\omega||_{C^1}}{\Im(k_0)^{2\delta}}$$

which proves (54).

Finally, let us prove (55). Performing the derivative with respect to y_3 in (46) we get:

$$\frac{\partial\psi_\varepsilon}{\partial y_3}(\vec{x}, y_3; \underline{p}, \omega) = \int_{\Omega\cap\{|\underline{x}-\underline{y}|\geq\varepsilon\}} \left\{ -\frac{e^{ik_0|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|^3} + ik_0\frac{e^{ik_0|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|^2} \right\} (y_3 - x_3)\kappa_\omega(\underline{y})e^{i\underline{p}\cdot\underline{y}}d\underline{y}. \tag{61}$$

Assume $x_3 - y_3 \neq 0$ (otherwise there is nothing to prove). Using polar coordinates in (61) (i.e. $|\underline{y} - \underline{x}| = r > \varepsilon$) we get

$$\left|\frac{\partial\psi_\varepsilon}{\partial y_3}\right| \leq 2\pi||\kappa_\omega||_{C^1}|x_3 - y_3| \int_\varepsilon^\infty \left\{ \frac{e^{-\Im(k_0)r}}{(r^2 + |x_3 - y_3|^2)^{3/2}} + |k_0|\frac{e^{-\Im(k_0)r}}{(r^2 + |x_3 - y_3|^2)} \right\} r\, dr. \tag{62}$$

Changing the variable in $r' = r/|x_3 - y_3|$ we get:

$$\left|\frac{\partial\psi_\varepsilon}{\partial y_3}\right| \leq 2\pi||\kappa_\omega||_{C^1} \int_0^\infty \left\{ \frac{r'}{(r'^2+1)^{3/2}} + |k_0|\ |x_3 - y_3|e^{-\Im(k_0)|x_3-y_3|r'} \right\} dr'. \tag{63}$$

While the first term in the above integrand is well behaved in terms of ω, the second one will generate a $k_0/\Im(k_0)$; hence, (55) is proved. ∎

LEMMA 3.3. *The mapping* $\mathcal{S}_0 \ni z \mapsto \xi(\cdot\ , \underline{p}, z) \in C^0(\Lambda)$ *is analytic.*

Proof. Take $z_0 \in \mathcal{S}_0$. It is easy to see that for fixed $\vec{x} \in \overline{\Lambda}$ and $\underline{p} \in \mathbf{R}^2$, the function $\xi(\vec{x}, \underline{p}, \cdot) : \mathcal{S}_0 \mapsto \mathbf{C}$ is holomorphic (see (38)). Take a disk $B(z_0, r) \subset \mathcal{S}_0$ and apply the Cauchy integral formula ($z \in B(z_0, r)$):

$$\xi(\vec{x}, \underline{p}, z) = \frac{1}{2\pi i}\int_{|\zeta-z_0|=r} \frac{\xi(\vec{x}, \underline{p}, \zeta)}{\zeta - z} = \sum_{n\geq 0}(z - z_0)^n a_n(\vec{x}, \underline{p}), \tag{64}$$

where

$$a_n(\vec{x}, \underline{p}) = \frac{1}{2\pi i}\int_{|\zeta-z_0|=r} \frac{\xi(\vec{x}, \underline{p}, \zeta)}{(\zeta - z_0)^{n+1}}.$$

Finally, employ (39) and obtain

$$||a_n(\cdot\ , \underline{p})||_\infty \leq \frac{\mathrm{const(r, \delta, \Lambda, z_0)}}{r^n}(1 + |\underline{p}|)^{-1-\delta} \tag{65}$$

which finishes the proof. ▮

LEMMA 3.4. *Consider the natural extension of* $k^2 = k_\omega^2 \in W$ *for complex values of* ω *(see Assumptions 2.2.1).*
i. There exists an open strip $\mathcal{S} \subseteq \mathcal{S}_0$ *containing* $(0, \omega_0)$ *such that whenever* $\omega \in \mathcal{S}$, $\inf_{\vec{x}\in\mathbf{R}^3} \Im(k_\omega^2(\vec{x})) > 0$*;*
ii. The forward problem corresponding to k_ω^2 *has a unique solution, whose scattered field* $u^{sc}(\vec{x}; \underline{p}, \omega)$ *is given by the natural extension of (11) to* $\mathcal{S}$*.*
Moreover,the mapping $\mathcal{S} \ni z \mapsto u^{sc}(\cdot\,; \underline{p}, z) \in C^0(\Lambda)$ *is analytic.*

Proof i. Relation (3) yields

$$\begin{aligned}\Im(k_\omega^2(\vec{x})) &= \Re(\omega)\,\{2\Im(\omega)[\kappa_1(\underline{x})\mathbf{1}_{[0,a]}(x_3) + 1/c_0^2] + [\kappa_2(\underline{x})\mathbf{1}_{[0,a]}(x_3) + \sigma_0]\} \\ &\geq \Re(\omega)\,\left\{\inf_{\vec{y}\in\mathbf{R}^3}[\kappa_2(\underline{y})\mathbf{1}_{[0,a]}(y_3) + \sigma_0] - |\Im(\omega)|\,[||\kappa_1||_\infty + 1/c_0^2]\right\} \qquad (66)\end{aligned}$$

Now use the condition (5); then $\mathcal{S}$ may be defined as the intersection (see also (36)):

$$\mathcal{S}_0 \cap \left\{z \in \mathbf{C},\ |\Im(z)| < \frac{\inf_{\vec{y}\in\mathbf{R}^3}[\kappa_2(\underline{y})\mathbf{1}_{[0,a]}(y_3) + \sigma_0]}{2[||\kappa_1||_\infty + 1/c_0^2]}\right\}. \qquad (67)$$

ii. Since *i* holds, the argument which led to (11) still works. It is now easy to establish that the mapping

$$\mathcal{S} \ni \omega \mapsto \mathcal{G}_\omega(k_\omega^2) \in B(C^0(\Lambda))$$

is analytic, and this also remains true for

$$\mathcal{S} \ni \omega \mapsto [\mathbf{1} + \mathcal{G}_\omega(k_\omega^2)]^{-1} \in B(C^0(\Lambda)).$$

Employing the analyticity of ξ (see the previous lemma), and by a similar argument with that one used for establishing (64) one gets

$$u^{sc}(\vec{x}\,; \underline{p}, z) = \sum_{n\geq 0}(z - z_0)^n u_n(\vec{x}, \underline{p}) \qquad (68)$$

where

$$u_n(\vec{x}, \underline{p}) = -\frac{1}{2\pi i}\int_{|\zeta - z_0| = r}\frac{1}{(\zeta - z_0)^{n+1}}\left\{[\mathbf{1} + \mathcal{G}_\zeta(k_\zeta^2)]^{-1}\xi(\cdot\,, \underline{p}, \zeta)\right\}(\vec{x}) \qquad (69)$$

Since

$$\sup_{\{|\zeta - z_0| = r\}} \left\| [\mathbf{1} + \mathcal{G}_\zeta(k_\zeta^2)]^{-1} \right\| < \infty,$$

(69) and (39) imply

$$||u_n(\cdot\ , \underline{p})||_\infty \leq \frac{\mathrm{const}(r, \delta, \Lambda, z_0)}{r^n}(1 + |\underline{p}|)^{-1-\delta}, \tag{70}$$

therefore the sequence in (68) converges in $C^0(\Lambda)$ and the proof is completed. ∎

Denote by $\Phi(\cdot\ , \underline{p}, \omega)$ the restriction to $\partial\Lambda'$ of $u^{sc}(\cdot\ ; \underline{p}, \omega)$, where $\omega \in \mathcal{S}$ and $\underline{p} \in \mathbf{R}^2$. Define (see also (18)):

$$\begin{aligned}\mathcal{L}_1(\underline{x}, \underline{p}; \omega) \ &:= \ \frac{1}{e^{ip_3(\omega)a} - 1} \left\{ 2[u^{in}\Phi](\underline{x}, a; \underline{p}, \omega) - 2[u^{in}\Phi](\underline{x}, 0; \underline{p}, \omega) \ + \right. \\ &+ \ \left. \Phi^2(\underline{x}, a; \underline{p}, \omega) - \Phi^2(\underline{x}, 0; \underline{p}, \omega) \right\}. \end{aligned} \tag{71}$$

Among other things, the next corollary proves (19).

COROLLARY 3.1. *Denote by $L(\omega)$ the integral operator corresponding to $\mathcal{L}_1(\underline{x}, \underline{p}; \omega)$ acting on $L^2(\Omega \times \mathbf{R}^2)$. Then:*
i. $L(\omega)$ is a Hilbert-Schmidt operator (i.e. (19) holds) and the following mapping is analytic:

$$\mathcal{S} \ni \omega \mapsto L(\omega) \in \mathcal{B}(L^2(\Omega \times \mathbf{R}^2)) \ .$$

ii. For real values of $\omega \in \mathcal{S}$,

$$\lim_{\omega \searrow 0} ||L(\omega)|| = 0 \ . \tag{72}$$

Proof i. First notice that (see (7)) $|u^{in}(\vec{x}, \underline{p}, \omega)| \leq 1$ if $\vec{x} \in \Lambda$, $\underline{p} \in \mathbf{R}^2$ and $\omega \in \mathcal{S}$. Introduce (with $n = 0$) (70) in (70) and (19) follows. Furthermore, reasoning as in the previous two lemmas, one can write a power series expansion for $\mathcal{L}_1(\underline{x}, \underline{p}; \omega)$ similar to that one in (68), where the coefficients obey an estimate as in (70). Therefore, the power expansion holds in $L^2(\Omega \times \mathbf{R}^2)$, too, and the mapping $L(\omega)$ is analytic even in the Hilbert-Schmidt norm.

ii. Let us begin by noticing that (see (11)):

$$\lim_{\omega \searrow 0} ||\mathcal{G}_\omega(k_\omega^2)||_{B(C^0(\Lambda))} = 0 \ .$$

Using (39) in estimating the sup-norm in the right side of (11) we obtain that for $0 < \omega < \epsilon(c_0, \sigma_0, \omega_0, \delta, \Lambda)$

$$||u^{sc}(\cdot\ , \underline{p}, \omega)||_{C^0(\Lambda)} \leq 2||\xi(\cdot\ , \underline{p}, \omega)||_{C^0(\Lambda)} \leq \frac{\mathrm{const}}{(1 + |\underline{p}|)^{1+\delta}}\omega^{1-\delta}, \tag{73}$$

where the above constant depends on everything but ω and $\underline{p}$.

We have already seen that there exists a constant $A > 0$ such that if $|\underline{p}| > A$, one has $\Im(p_3(\omega)) > |\underline{p}|/2$ uniformly in $\omega \in \mathcal{S}_0$.

If $|\underline{p}| > A$, then

$$|\mathcal{L}_1(\underline{x}, \underline{p}; \omega)| \leq \frac{\text{const}}{1 - e^{-Aa/2}} (1 + |\underline{p}|)^{-1-\delta} \omega^{1-\delta}.$$

If $|\underline{p}| < A$ and $\omega \in \mathcal{S}_0$, define the (bounded and continuous) function

$$f(\underline{p}, \omega) = \frac{ip_3(\omega)a}{e^{ip_3(\omega)a} - 1}.$$

Since $|p_3(\omega)| \geq \sigma_0^{1/2} \omega^{1/2}$ for any $\underline{p}$, we have

$$|\mathcal{L}_1(\underline{x}, \underline{p}; \omega)| \leq \text{const } (\sup |f|) \; (1 + |\underline{p}|)^{-1-\delta} \omega^{1/2-\delta}.$$

Now choose some $0 < \delta < 1/2$; then the L^2 norm of $\mathcal{L}_1(\cdot\;, \cdot\;; \omega)$ yet the Hilbert-Schmidt norm of $L(\omega)$ tends to zero with ω; therefore (72) holds. ▮

3.3. Completing the proof of Theorem 2.1

The next lemma verifies the remaining needed conditions for u^{sc} in order to be an element of $\mathcal{A}_\omega$:

LEMMA 3.5. *i. The function* $[\eta(\Phi)](\cdot)$ *defined in (15) belongs to* $L^2(\mathbf{R}^2)$ *(i.e. (16) holds);*

ii. Having in mind (20), re-denote by $\mathcal{K}(\underline{x}, \underline{y}; \omega)$ *the partial Fourier transform of* $\mathcal{L}_1$*, and by* $K(\omega)$*,* $\omega \in \mathcal{S}$*, the Hilbert-Schmidt operator corresponding to* $\mathcal{K}$ *acting on* $L^2(\Omega \times \Omega)$*. Then the operator* $\mathbf{1} + K(\omega)$ *is invertible in* $B(L^2(\Omega \times \Omega))$ *for all frequencies except maybe for a discrete set* $\mathcal{M} \subset \mathcal{S}$*;*

iii. The intersection $M := \mathcal{M} \cap (0, \omega_0)$ *is finite.*

Proof i. Since both terms in the left hand side of (27) are L^2 vectors, $\eta(\Phi)$ is modulus square integrable, too. Notice that this result is not at all obvious from the definition of $\eta(\Phi)$ in (15), since the incident field has an exponential increase in $|\underline{p}|$ when $y_3 < 0$.

ii. We intend to use the analytic Fredholm theorem ([9, Theorem VI.14]). Since the partial Fourier transform is a unitary operator on $L^2(\Omega \times \mathbf{R}^2)$, we can restate Corollary 3.1 with $L(\omega)$ replaced by $K(\omega) \in B(L^2(\Omega \times \Omega))$. Next, notice that $\mathbf{1} + K(\omega)$ is invertible for sufficiently small and positive frequencies (see (72)), therefore the "right" Fredholm alternative holds.

iii. Firstly, we have just seen that 0 is not an accumulation point for the eventual positive singularities. Secondly, let us show that ω_0 cannot be an accumulation point for $\mathcal{M}$. Indeed, one can reconsider the first two statements of this lemma for strips containing the interval $(0, 2\omega_0)$, hence if ω_0 is not regular, it must be isolated from the other singularities. ▮

3.3.1. Putting all the things together

Since the proof of Theorem 2.1 required a lot of intermediary technical results, let us now give an overview of the argument. We started by choosing $k \in W$ of the form given in (3) and (3). We then considered the scattering field appearing in the forward problem (see (11), (12) and the argument leading to them).

Denote by $\Phi_\omega(\cdot, \underline{p})$ the restriction of the scattered field to $\partial\Lambda'$ and let us see that all six conditions listed in Definition 2.5 are fulfilled, i.e. k can be reconstructed from the knowledge of $\Phi_\omega(\cdot, \underline{p})$ except maybe for a finite number of frequencies in the interval $(0, \omega_0)$:

1. $\Phi_\omega(\cdot, \underline{p}) \in H^{3/2}(\partial\Lambda')$ by the trace theorem;

2. Since the scattered field $u^{sc}(\cdot; \underline{p}, \omega)$ coincides outside Λ' with the solution to the exterior problem $\tilde{u}(\Phi_\omega)$ (see Definition 2.2), we conclude that $u(\Phi_\omega)$ which enters in the definition of $[\eta(\Phi_\omega)](\underline{p})$ (see (15)) is in fact the total field which solves the forward problem. We then established the key equation (27) which allow us to conclude that $\eta(\Phi_\omega)(\cdot) \in L^2(\mathbf{R}^2)$ as soon as (19) holds;

3. As we have already mentioned, equation (27) lies at the very foundation of our arguments. The whole reconstruction process depends on its solvability. Since we see $\mathcal{L}_1$ as an integral kernel of a certain operator $L(\omega)$ (see Corollary 3.1), we would very much like to have a property as (19) since it would automatically imply that $L(\omega)$ is compact, and moreover, even after the partial Fourier transform with respect to the "$\underline{p}$" variable, the newly obtained operator $K(\omega)$ (see Lemma 3.5) remains compact.

As one can see in (18) or (70), the proof of (19) can be reduced to the proof of an estimate as in (34). The main idea consists in using (11), where one can see that the "$\underline{p}$" dependence is only contained by the "free" term ξ. Hence we concentrated our efforts in proving (39), and then we used (11) in passing on the decay in $\underline{p}$ to the scattered field.

4. We also payed a lot of attention to the frequency dependence. Since we eventually want to invert the operator $\mathbf{1} + K(\omega)$ in the Hilbert space $L^2(\Omega \times \Omega)$, we focused on verifying the conditions of the analytic Fredholm alternative. This has been ultimately achieved in Corollary 3.1 and Lemma 3.5. The main idea consisted in proving that the compact operators $L(\omega)$ and $K(\omega)$ admit analytic extensions to a strip $\mathcal{S}$ containing the interval $(0, \omega_0)$, and secondly, proving that there are points in this strip for which the inverse exist. This was the main reason for the careful study of the frequency behavior of the constant appearing in (39)

(since k_0^2 is ω dependent). In fact, we should emphasize that (see Lemma 3.5 *ii.* and *iii.*) for sufficiently small frequency, we can always invert $\mathbf{1}+K(\omega)$ and hence to solve (27) and reconstruct k.

5 and 6. These points are now automatically satisfied.

4. PROOF OF THEOREM 2.2 AND COROLLARIES 2.1 AND 2.2

At this point, having explained and motivated the structure of the space of admissible data $\mathcal{A}_\omega$ at a given positive frequency ω, the proofs which follow are more or less straightforward.

4.1. Arguing for Theorem 2.2

We prove the theorem by double implication.

Assume *i* holds. The existence in W of a solution for the inverse problem is guaranteed by (23) and (24). That ω is regular (see Definition 2.6) with respect to any such solution follows from the fourth condition in Definition 2.5.

As for the uniqueness of the solution to the inverse problem, assume that we have two solutions in W corresponding to the same $\Phi_\omega \in \mathcal{A}_\omega$. Then they both solve equation (27), and since ω is regular with respect to both solutions (the operator $K(\omega)$ is the same), the uniqueness follows since $\mathbf{1}+K(\omega)$ is one to one.

Assume *ii* holds. Then since $k \in W$, by following the proof of Theorem 2.1 we see that the first three conditions in Definition 2.5 are satisfied. Since ω is regular with respect to k, we conclude that the fourth condition holds, too. Therefore the measured data belongs to $\mathcal{A}_\omega$. ▮

4.2. Arguing for Corollary 2.1

We know from Theorem 2.1 *ii* that there exists an $\omega_N > 0$ such that for any frequency $0 < \omega < \omega_N$, the restriction to $\partial\Lambda'$ of the scattered field corresponding to any wave number in W_N belongs to the space of admissible data $\mathcal{A}_\omega$. Therefore, the reconstruction process can take place (according to Theorem 2.2), yielding a unique solution to the inverse problem. ▮

4.3. Arguing for Corollary 2.2

The proof will have two steps:

First, let us argue why the fields are in fact equal for all $\underline{p}$'s, not only for an open set as stated in the hypothesis. Without loss, assume that the fields coincide for all $|\underline{p}| < 1$ at some frequency $\omega > 0$. We know that both u_1 and u_2 can be expressed as in (12) and (10) where k is replaced by k_1 and k_2 respectively. Then we employ the next lemma:

LEMMA 4.1. *Let $R > 1$ be arbitrary large. Fix $\omega > 0$, $k \in W$ and $\vec{x} \in \partial\Lambda'$. Then there exists an open strip $S_R \subset \mathbf{C}^2$ containing the "real" ball $B(0,R)$ such that the mapping (see (12) and (10))*

$$B(0,R) \ni \underline{p} \mapsto u(\vec{x}; \underline{p}, \omega) \in \mathbf{C}$$

admits an analytic extension to S_R.

Proof. The strip is chosen such that extending the function (see (6))

$$B(0,R) \ni \underline{p} \mapsto p_3(\underline{p}) = \sqrt{k_0^2 - \underline{p}^2} \in \mathbf{C}$$

to it, becomes analytic in $\underline{p}$. This construction is always possible since the imaginary part of k_0^2 is strictly positive. Then we "propagate" this extension from (7) to (10) and finally to (12) employing a similar approach to that one used in proving Lemma 3.3. ∎

An immediate consequence of this lemma is that the fields must now coincide on $B(0,R)$ and since R was arbitrary, they must coincide for all $\underline{p} \in \mathbf{R}^2$.

Second, since we now know that the fields are equal for all $\underline{p}$'s and for a sequence of decreasing frequencies ($\omega = 1/n,\ n > N_0$), there must exist $m > N_0$ such that $1/m$ is a regular frequency for both k_1 and k_2 (since the set M in Theorem 2.1 *i* is finite). Therefore the fields are in the space of admissible data and the reconstruction procedure assures the equality $k_1 = k_2$. ∎

5. PROOF OF THEOREM 2.3

We know from Theorem 2.1 *ii*. that $F_{N,\omega} = \mathcal{A}_\omega$ if $\omega \in (0, \omega_N)$. Therefore, one can associate to any $\Psi \in F_{N,\omega}$ (via the reconstruction procedure) a unique $k(\Psi) \in W_N$. In fact (see (23), what we really need to compute is $\kappa(\Psi)$ given by (22). The rest of this section will prove that small changes in Ψ lead to small changes in $\kappa(\Psi)$.

Let us rewrite (22) as:

$$\begin{aligned} &\kappa(\Psi) - \kappa(\Phi) \qquad (1)\\ &= \{[\mathbf{1} + K(\Psi)]^{-1} - [\mathbf{1} + K(\Phi)]^{-1}\}\tilde{\eta}(\Phi) + [\mathbf{1} + K(\Psi)]^{-1}\{\tilde{\eta}(\Psi) - \tilde{\eta}(\Phi)\}\\ &= [\mathbf{1} + K(\Psi)]^{-1}\left\{[K(\Phi) - K(\Psi)][\mathbf{1} + K(\Phi)]^{-1}\tilde{\eta}(\Phi) + \tilde{\eta}(\Psi) - \tilde{\eta}(\Phi)\right\}. \end{aligned}$$

The theorem would be concluded if we can prove the following three technical lemmas (see also (25)):

LEMMA 5.1. *Under the same conditions as in Theorem 2.3, one can find* $\delta_1 > 0$ *such that* $||K(\Psi) - K(\Phi)||_{B(L^2(\Omega))} < \epsilon$ *whenever* $\Psi \in F_{N,\omega}$ *and* $||\Psi - \Phi||_{L^2(\partial\Lambda' \times \mathbf{R}^2)} < \delta_1$.

LEMMA 5.2. *Under the same assumptions, given* $\epsilon > 0$ *there exist* $\delta_2 > 0$ *and a constant* $C > 0$ *such that* $||[\mathbf{1} + K(\Psi)]^{-1}||_{B(L^2(\Omega))} < C$ *whenever* $\Psi \in F_{N,\omega}$ *and* $||\Psi - \Phi||_{L^2(\partial\Lambda' \times \mathbf{R}^2)} < \delta_2$.

LEMMA 5.3. *Under the same assumptions, given* $\epsilon > 0$ *there exists* $\delta_3 > 0$ *such that (see (25))* $||\tilde{\eta}(\Psi) - \tilde{\eta}(\Phi)||_{L^2(\Omega)} < \epsilon$ *whenever* $\Psi \in F_{N,\omega}$ *and* $|||\Psi - \Phi||| < \delta_3$.

5.1. Proof of Lemmas 5.1 and 5.2

Instead of a "direct" study of the $B(L^2(\Omega))$-norm for the operator $K(\Phi) - K(\Psi)$, we will estimate its Hilbert-Schmidt norm. We know that each operator K corresponds to an integral kernel $\mathcal{K}(\underline{x}, \underline{y})$ (see (20) and (18)), obtained from $\mathcal{L}_1(\underline{x}, \underline{p})$ via a partial Fourier transform over the "$\underline{p}$"-variable. Hence,

$$||K(\Phi) - K(\Psi)||_{B(L^2(\Omega))} \leq ||\mathcal{L}_1(\Phi) - \mathcal{L}_1(\Psi)||_{L^2(\Omega \times \mathbf{R}^2)},$$

where (see (18)) the kernel $\mathcal{L}_1(\Phi) - \mathcal{L}_1(\Psi)$ looks like

$$\frac{1}{e^{ip_3(\omega)a} - 1} \left\{ 2[u^{in}(\Phi - \Psi)](\underline{x}, a; \underline{p}, \omega) - 2[u^{in}(\Phi - \Psi)](\underline{x}, 0; \underline{p}, \omega) + \right.$$
$$\left. + \ \Phi^2(\underline{x}, a; \underline{p}, \omega) - \Psi^2(\underline{x}, a; \underline{p}, \omega) - \Phi^2(\underline{x}, 0; \underline{p}, \omega) + \Psi^2(\underline{x}, 0; \underline{p}, \omega) \right\}. \qquad (2)$$

Remember that $\omega > 0$ is fixed; then (see (6) and (7))

$$\sup_{\underline{p} \in \mathbf{R}^2} \left| \frac{1}{e^{ip_3(\underline{p}, \omega)a} - 1} \right| \leq \text{const}, \quad \sup_{\vec{x} \in \Lambda} |u^{in}(\vec{x}; \underline{p})| \leq 1.$$

Another important thing is to apply the estimate (73) for both Φ and Ψ. Before that, let us mention that the constant appearing in (73) also depends on the C^1-norm of its corresponding wave number (i.e. κ_1 and κ_2) which by assumption is bounded from above by N. Therefore,

$$\sup_{\vec{x}\in\Lambda}\sup_{\underline{p}\in\mathbf{R}^2}|\Psi|(\vec{x};\underline{p},\omega)\leq \mathrm{const}(N,\Lambda) \tag{3}$$

and a similar estimate can be written for Φ, too.

It follows that

$$||\mathcal{L}_1(\Phi)-\mathcal{L}_1(\Psi)||_{L^2(\Omega\times\mathbf{R}^2)}\leq \mathrm{const}(N,\Lambda)||\Phi-\Psi||_{L^2(\partial\Lambda'\times\mathbf{R}^2)}$$

which ends the proof of Lemma 5.1.

As for Lemma 5.2, we only remark that it is a straightforward consequence of Lemma 5.2 and of a well known identity:

$$[\mathbf{1}+K(\Psi)]^{-1}=[\mathbf{1}+K(\Phi)]^{-1}+[\mathbf{1}+K(\Psi)]^{-1}[K(\Phi)-K(\Psi)][\mathbf{1}+K(\Phi)]^{-1}.$$

∎

5.2. Proof of Lemma 5.3

First, from the definition of $\tilde{\eta}$ (see (21)) we conclude that it would be enough proving a similar statement with η instead of $\tilde{\eta}$. More precisely, we prove two technical results:

PROPOSITION 5.1. *Consider the same conditions as in Theorem 2.3. Then for any $\epsilon>0$ there exists $P_\epsilon>0$ such that for any $\Psi\in F_{N,\omega}$ we have*

$$\int_{|\underline{p}|\geq P_\epsilon}|\eta(\Psi)|^2(\underline{p})d\underline{p}\leq\epsilon^2/6.$$

Remark. The above proposition states that the L^2-"tail" of $\eta(\Psi)$ is small, uniformly in Ψ.

The second result states the following:

PROPOSITION 5.2. *Under the same assumptions as above, given $\epsilon>0$ there exists $\delta_4>0$ such that (see (25)) $\int_{|\underline{p}|\leq P_\epsilon}|\eta(\Psi)-\eta(\Phi)|^2(\vec{p})<\epsilon^2/3$ whenever $\Psi\in F_{N,\omega}$ and $|||\Psi-\Phi|||<\delta_4$.*

Before actually proving these two propositions, let us see why they are implying Lemma 5.3. Indeed, choose a positive ϵ and apply Proposition 5.1 for getting P_ϵ; then

$$||\eta(\Phi)-\eta(\Psi)||^2_{L^2(\mathbf{R}^2)} \leq 2\epsilon^2/3 + \int_{|\underline{p}|\leq P_\epsilon} |\eta(\Psi)-\eta(\Phi)|^2(\vec{p}).$$

Identify δ_3 with δ_4 and we are done. ▮

5.2.1. Proof of Proposition 5.1

The main ingredient we employ is equation (27), where Φ should be replaced with Ψ. In other words, we will show that its left hand side has the property stated in the proposition.

Let us start with the first term, i.e. the Fourier transform of κ. Since κ was assumed to belong to $C^1_0(\Omega)$, we employ integration by parts in deriving the next formula:

$$\underline{p}^2 \left| \mathcal{F}^{-1}\kappa \right|^2(\underline{p}) = \left| \mathcal{F}^{-1}(\partial_1\kappa) \right|^2(\underline{p}) + \left| \mathcal{F}^{-1}(\partial_2\kappa) \right|^2(\underline{p}). \qquad (4)$$

Hence, for any $P>1$ we get

$$\begin{aligned} \int_{|\underline{p}|\geq P} \left| \mathcal{F}^{-1}\kappa \right|^2 d\underline{p} &= \frac{1}{P^2}\int_{|\underline{p}|\geq P} [\left| \mathcal{F}^{-1}(\partial_1\kappa) \right|^2 + \left| \mathcal{F}^{-1}(\partial_2\kappa) \right|^2] d\underline{p} \\ &\leq \frac{1}{P^2}\left[||\partial_1\kappa||^2_{L^2(\mathbf{R}^2)} + ||\partial_2\kappa||^2_{L^2(\mathbf{R}^2)} \right] \\ &\leq \frac{\mathrm{const}(N,\Lambda)}{P^2}, \qquad (5) \end{aligned}$$

where in the second line we employed Plancherel's identity and that $k \in W_N$. Clearly, the right hand side of (5) can be made arbitrarily small as soon as P is increased, uniformly in κ and therefore in Ψ.

Let us now say a few words about the second term in the left hand side of equation (27). One can combine the estimates (34) and (3) with (70) getting

$$\left| \int_\Omega \mathcal{L}_1(\underline{x},\underline{p})\kappa(\underline{x})d\underline{x} \right| \leq \frac{\mathrm{const}}{(1+|\underline{p}|)^{1+\delta}},$$

where the above constant only depends on N and Λ. Now its L^2-tail can be made arbitrarily small, and we are done. ▮

5.2.2. Proof of Proposition 5.2

We will show that a stronger estimate holds, uniformly in $|\underline{p}| \leq P_\epsilon$

$$|[\eta(\Phi)](\underline{p}) - [\eta(\Psi)](\underline{p})| \leq \mathrm{const}_\epsilon \cdot \sup_{\vec{y}\in\partial\Lambda'} |\Psi(\vec{y};\underline{p}) - \Phi(\vec{y};\underline{p})|. \qquad (6)$$

After we introduce the expression of $u(\Psi)$ from (14) in (15), we obtain several integrals; one of them will only contain the incident field hence being independent of Ψ, two of them will contain products between (derivatives of) the incident field and (derivatives of) the radiating field $\tilde{u}(\Psi)$. The last integral will only contain the radiating field.

Therefore, the difference $[\eta(\Phi)](\underline{p}) - [\eta(\Psi)](\underline{p})$ will only consists from integrals whose integrands contain at least one term of the form $[\tilde{u}(\Psi)](\vec{y};\underline{p}) - [\tilde{u}(\Phi)](\vec{y};\underline{p})$ or $[\tilde{u}(\Psi)]^2(\vec{y};\underline{p}) - [\tilde{u}(\Phi)]^2(\vec{y};\underline{p})$, eventually with some $\vec{y}$-derivatives acting on them.

Let us investigate a typical term:

$$\frac{1}{e^{ip_3 a}-1}\int_{|\vec{y}|=\rho}\{u^{in}(\vec{y};\underline{p})[[\partial_\nu\partial_3\tilde{u}(\Phi)](\vec{y};\underline{p}) - [\partial_\nu\partial_3\tilde{u}(\Psi)](\vec{y};\underline{p})]\}d\sigma(\vec{y}). \quad (7)$$

First, we want to get rid of the $\vec{y}$-derivatives acting on the radiating solutions in the above formula. We intend to apply Theorem 3.9 in [3]; in order to do that, we introduce a few notations.

Denote by S and D the single- and double-layer operators acting on $C^0(\partial\Lambda')$ and given by (see 9):

$$(S\phi)(\vec{x}) := 2\int_{\partial\Lambda'} G_0(\vec{x},\vec{x}')\phi(\vec{x}')d\sigma(\vec{x}') \quad (8)$$

and

$$(T\phi)(\vec{x}) := 2\int_{\partial\Lambda'} \frac{\partial G_0}{\partial\nu(\vec{x}')}(\vec{x},\vec{x}')\phi(\vec{x}')d\sigma(\vec{x}'), \quad (9)$$

where $\vec{x}$ is restricted to the boundary.

Following [3], we express the radiating solution as

$$[\tilde{u}(\Psi)](\vec{x};\vec{p}) = \int_{\partial\Lambda'}\left\{\frac{\partial G_0}{\partial\nu(\vec{x}')}(\vec{x},\vec{x}') - i\alpha G_0(\vec{x},\vec{x}')\right\}\psi(\vec{x}';\underline{p})d\sigma(\vec{x}'), \quad (10)$$

where $\vec{x}\in\mathbf{R}^3\setminus\Lambda'$, $\alpha>0$ is a positive coupling parameter and $\psi(\cdot;\underline{p})\in C^0(\partial\Lambda')$ is an yet unknown continuous function, which is to be found. Indeed, one can prove that the operator $\mathbf{1}+T-i\alpha S$ has a bounded inverse in $B(C^0(\partial\Lambda'))$ and

$$\psi(\cdot;\underline{p}) = 2(\mathbf{1}+T-i\alpha S)^{-1}\Psi(\cdot;\underline{p}). \quad (11)$$

Introducing (11) in (10), we get that for $|\vec{y}|=\rho$ i.e. away from the boundary $\partial\Lambda'$ we have

$$\sup_{\vec{y}\in\partial\Lambda'}\left|[\partial_\nu\partial_3\tilde{u}(\Phi)](\vec{y};\underline{p}) - [\partial_\nu\partial_3\tilde{u}(\Psi)](\vec{y};\underline{p})]\right| \leq \text{const}||\Psi(\cdot;\underline{p}) - \Phi(\cdot;\underline{p})||_\infty. \quad (12)$$

Using the above estimate in (7) and remembering that (see (7))

$$\sup_{|\vec{y}|=\rho} \sup_{|\underline{p}|\leq P_\epsilon} \left|u^{in}\right| (\vec{y};\underline{p}) = \mathrm{const}(\epsilon,\Lambda),$$

we obtain an estimate as in (6).

Besides terms like that one in (7), we also have typical "quadratic" terms as the next one:

$$\frac{1}{e^{ip_3 a}-1}\int_{|\vec{y}|=\rho} \left\{[\tilde{u}(\Phi)]\,[\partial_\nu\partial_3\tilde{u}(\Phi)](\vec{y};\underline{p}) - [\tilde{u}(\Psi)]\,[\partial_\nu\partial_3\tilde{u}(\Psi)](\vec{y};\underline{p})\right\} d\sigma(\vec{y}). \tag{13}$$

We split the above term in two, trying to "linearize" it:

$$\frac{1}{e^{ip_3 a}-1}\int_{|\vec{y}|=\rho} [\tilde{u}(\Phi)](\vec{y};\underline{p}) \left\{[\partial_\nu\partial_3\tilde{u}(\Phi)](\vec{y};\underline{p}) - [\partial_\nu\partial_3\tilde{u}(\Psi)](\vec{y};\underline{p})\right\} d\sigma(\vec{y}) \tag{14}$$

and

$$\frac{1}{e^{ip_3 a}-1}\int_{|\vec{y}|=\rho} \left\{[\tilde{u}(\Phi)](\vec{y};\underline{p}) - [\tilde{u}(\Psi)](\vec{y};\underline{p})\right\} [\partial_\nu\partial_3\tilde{u}(\Psi)](\vec{y};\underline{p}) d\sigma(\vec{y}). \tag{15}$$

While (14) brings nothing new compared to (7), equation (15) still requires an estimate more; that is, uniformly in Ψ and $|\underline{p}| \leq P_\epsilon$ we have (see also (3))

$$\sup_{|\vec{y}|=\rho} |\partial_\nu\partial_3\tilde{u}(\Psi)|\,(\vec{y};\underline{p}) \leq \mathrm{const}(\Lambda)||\Psi(\cdot;\underline{p})||_\infty \leq \mathrm{const}(N,\Lambda).$$

We then conclude that (6) holds and so does Proposition 5.2. ▮

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