# Components of the Fundamental Category

by

Lisbeth Fajstrup, Éric Goubault, Emmanuel Haucourt, and Martin Raussen

January 2003 R-2003-02

Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 96 35 80 80 ▪ Telefax: +45 98 15 81 29
URL: www.math.auc.dk/research/reports/reports.htm

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

# COMPONENTS OF THE FUNDAMENTAL CATEGORY

L. FAJSTRUP, M. RAUSSEN, E. GOUBAULT, AND E. HAUCOURT

Abstract. In this article we study the fundamental category [10, 9] of a partially ordered topological space [15, 12], as arising in e.g. concurrency theory [5]. The "algebra" of dipaths modulo dihomotopy (the fundamental category) of such a po-space is essentially finite in a number of situations: We define a component category of a category of fractions with respect to a suitable system, which contains all relevant information. Furthermore, some of these simpler invariants are conjectured to also satisfy some form of a van Kampen theorem, as the fundamental category does [9, 11]. We end up by giving some hints about how to carry out some computations in simple cases.

## 1. Introduction

The aim of this paper is to show how to compute some algebraic topological invariants relevant to questions about concurrent and distributed systems.

A class of examples, which will be used throughout this text, generating geometrical invariants, arises from a toy langage manipulating semaphores. Using Dijkstra's notation [3], we consider processes to be sequences of locking operations $Pa$ on semaphores $a$ and unlocking operations $Va$. In this introduction, we consider only binary semaphores, ensuring mutual exclusion of accesses, but in further examples, we will also model and use counting semaphores, or $k$-semaphores ($k > 1$) which can be accessed concurrently by up to $k$ processes.

In the example where two processes share two resources $a$ and $b$:

$$T1 = Pa.Pb.Vb.Va$$

$$T2 = Pb.Pa.Va.Vb$$

the geometric model is the "Swiss flag", Fig. 1, regarded as a subset of $\mathbb{R}^2$ with the componentwise partial order $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq y_1$ and $x_2 \leq y_2$. The (interior of the) horizontal dashed rectangle comprises global states that are such that $T_1$ and $T_2$ both

*Key words and phrases.* po-space, dihomotopy, fundamental category, category of fractions, component, weakly invertible morphism, pure system.

A preliminary version of this work was presented and discussed at the Seminar *Structures for Computable Topology and Geometry* at Schloss Dagstuhl. Part of this work was accomplished while the first two authors were visiting Ecole Polytechnique, France. This work was completed during an academic visit of the third author to the computer science department of Macquarie University, Sydney. Acknowledgments are due to the members of Sydney Category Theory Seminar for numerous discussions on the subject. Part of this work was done while the fourth author was visiting Aalborg University, with support from BRICS and Paris 7.

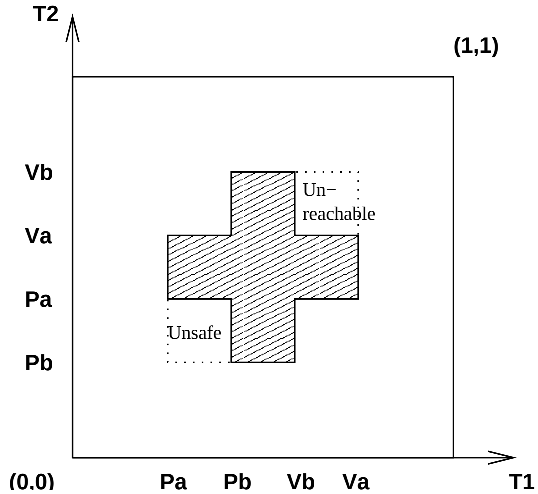



FIGURE 1. The Swiss Flag example - two processes sharing two resources

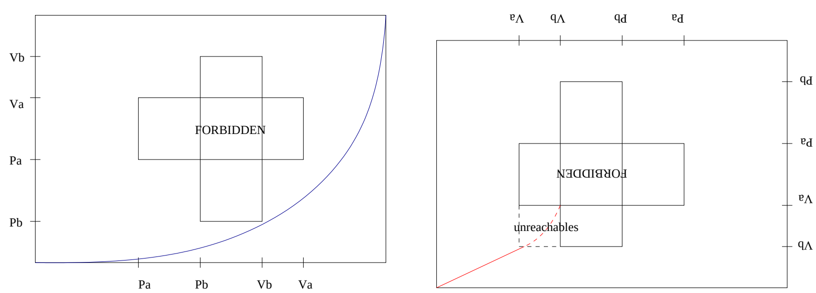



FIGURE 2. Deadlocks and unreachables

hold a lock on $a$: this is impossible by the very definition of a binary semaphore. Similarly, the (interior of the) vertical rectangle consists of states violating the mutual exclusion property on $b$. Therefore both dashed rectangles form the *forbidden region*, which is the complement of the space $X$ of (legal) states. This space with the inherited partial order provides us with a particular po-space $X$ [15, 12], as defined in Sect. 2.

Moreover, legal execution paths, called *dipaths*, are increasing maps from the po-space $\vec{I}$ (the unit segment with its natural order) to $X$. The partial order on $X$ thus reflects (at least) the time ordering on all possible execution paths.

Many different execution paths have the same global effect: In the "Swiss Flag" example, for any execution path shaped like the one at the left of Figure 3, $T_1$ gets hold of locks $a$ and $b$ before $T_2$ does. This implies that for the actual assignments on variable $b$ that we have chosen in this example: $T_1$ does $b := b+1$ and $T_2$ does $b = b*2$, starting with an initial value of 2, all execution paths below the hole will end up with the value $b = 6$. In fact, there are only two essentially different execution paths from the initial point $(0, 0)$ to the final point $(1, 1)$, that fully determine the computer-scientific behaviour of the system: one is the type of dipaths just discussed, the other one runs to the left and above the central

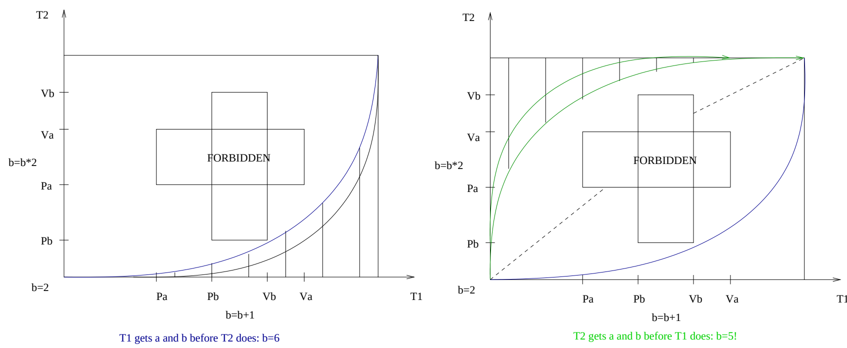



FIGURE 3. Essential schedules for the swiss flag.

hole (see picture at the right hand side of Figure 3). In terms of schedules of executions, the latter corresponds to executions in which $T_2$ is the first to read and write (after having got the corresponding locks) on $a$ and $b$, before $T_1$ does (ending up with result $b = 5$). These are in fact the only two classes of dipaths from $(0, 0)$ to $(1, 1)$ modulo "continuous deformations" that do not reverse time, i.e., up to *dihomotopy* as defined in [5] and in Section 2.

Other interesting dipaths in our example space start in the initial point $(0, 0)$ and end in a deadlock, cf. the first picture of Figure 2 or start in an unreachable state and end in the final point $(1, 1)$, cf. the right hand side of Figure 2).

In general, one of the important invariants of a concurrent system is its *fundamental category* [9, 10], classifiying dipaths between any pair of points up to dihomotopy, i.e, a directed version of the fundamental groupoid [2] of a topological space. A drawback of the fundamental category is that it is less easy to compute than the fundamental groupoid or the fundamental group. There are similarities though, for instance there is a van Kampen theorem in the directed case [9, 11].

Our aim is to go further in the study of the *algebraic properties* of the fundamental category in order to manipulate and compute it for a variety of systems. In nice cases, the relevant information in the fundamental category is essentially finite. This is shown using a construction based on categories of fractions [6], which are briefly explained in Section 3. The principle is to formally "invert" systems of "inessential" morphisms in the fundamental category.

Our aim is to decompose the fundamental category into big chunks as the regions 1 to 10 in Figure 4. Basically, inside these regions, or components, nothing important happens: first of all, there is at most one dihomotopy class of dipath between any two points in the same component. Moreover, composing with morphisms (= dihomotopy classes of dipaths) within these regions does not affect the "shape" of the future nor of the past. We will consider the category of fractions with these morphisms formally inverted. A certain quotient of the fundamental category with respect to this system of " inessential" morphisms forms then the category of components, which, in our example is the following

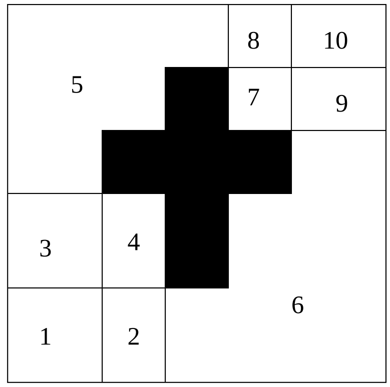
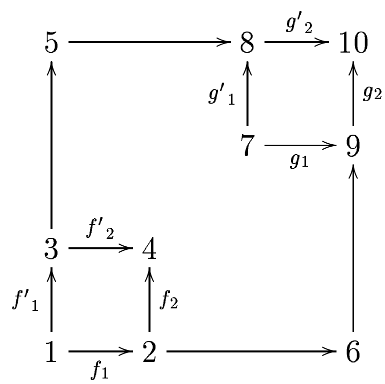



FIGURE 4. The components of the Swiss flag

finite category:




together with relations $g'_2 \circ g'_1 = g_2 \circ g_1$ and $f'_2 \circ f'_1 = f_2 \circ f_1$ (compare with e.g. [8]). In some sense, this category of components finitely presents the fundamental category. In particular, we can infer from this component category, all dihomotopy information because of a lifting property, see Propositions 3 and 7.

In general it is not obvious how to characterize the inessential morphisms, i.e., the morphisms which should be inverted formally. This leads to a more specific calculus of fractions, in particular left and right categories of fractions as defined in Section 4. Moreover, as shown in [6], finite limits and finite colimits are preserved when taking left and right categories of fractions. We can view equalizers, sums and products (when they exist, at least locally in some subcategories) as expressing particular equations between dipaths modulo dihomotopy which the category of fractions we construct has to preserve.

We then apply this "abstract nonsense" to various topological situations, arising from questions regarding dihomotopies. Last but not least, we give some hints about how to compute these invariants for simple spaces like some compact subsets of $\mathbb{R}^n$ with the componentwise ordering. This is done in Section 5.

There are two importants points that need to be outlined, particularly with respect to earlier work [17, 10]:

First, the "inessential" morphisms used to be defined with respect to sets of initial and final points. Dipaths were considered as pieces of dipaths between a set of initial points and a set of final points. The new definitions allow us to be more natural, without any reference to specific sets of points. This also tackles some of the problems with the "homotopy history equivalence" relation as defined in [5] which also needed to be bipointed by sets of initial and final points. In some sense, Proposition 6 shows that the new definition of inessential morphisms allows us to encompass all possible choices of initial and final sets.

Another modification arises from the fact that the fundamental category does not satisfy cancellation properties, in general. This is the reason for introducing the additional concept of a *pure* system (cf. Definition 4.3): For instance, the fundamental category of a cube minus an inner cube (see Section 6) is *not* trivial (as its fundamental group). Close to the inner deleted cube, there are local obstructions to directed homotopy of directed paths. But these are cancelled out under any long enough extensions, in the future as well as in the past. In general it is not clear whether composites of essential morphisms can become inessential. To avoid this, we ask a system of morphims to be inverted to satisfy the pureness property from Def. 4.3. And furthermore, this property has some very nice consequences as we show in section 5. Unfortunately, it is not clear in general how to construct significant systems of inessential morphisms satisfying this pureness property.

## 2. Basic definitions

The framework for the applications we have in mind is mostly based on the simple notion of a po-space:

**Definition 1.** (1) *A po-space is a topological space $X$ with a (global) closed partial order $\leq$ (i.e. $\leq$ is a closed subset of $X \times X$).*

(2) *A dimap $f : X \to Y$ between po-spaces $X$ and $Y$ is a continuous map that respects the partial orders (is non-decreasing).*

(3) *A dipath $f : \vec{I} \to X$ is a dimap whose source is the interval $\vec{I}$ with the usual order.*

Po-spaces and dimaps form a category. To a certain degree, our methods apply to the more general categories of lpo-spaces [5] (with a local partial order), of flows [7] and of $d$-spaces [11], but for the sake of simplicity, we stick to po-spaces in the present paper. Dihomotopies between dipaths $f$ and $g$ (with fixed extremities $\alpha$ and $\beta$ in $X$) are dimaps $H : \vec{I} \times I \to X$ such that for all $x \in \vec{I}$, $t \in I$,

$$H(x, 0) = f(x),\ H(x, 1) = g(x),\ H(0, t) = \alpha,\ H(1, t) = \beta.$$

A dihomotopy is to be understood as a 1-parameter family of dimaps without order requirements in the second $I$-coordinate[1]. Now, we can define the main object of study of this paper:

**Definition 2.** *The fundamental category is the category $\vec{\pi}_1(X)$ with:*

- *as objects: the points of $X$,*

[1]This is slightly different for $d$-spaces, but coincides in important cases.

- *as morphisms, the dihomotopy classes of dipaths: a morphism from $x$ to $y$ is a dihomotopy class $[f]$ of a dipath $f$ from $x$ to $y$.*

Concatenation of dipaths factors over dihomotopy and yields the composition of morphisms in the fundamental category. A dimap $f : X \to Y$ between po-spaces induces a functor $f_\# : \vec{\pi}_1(X) \to \vec{\pi}_1(Y)$, and we obtain thus a functor $\vec{\pi}_1$ from the category of po-spaces to the category of categories.

The fundamental category of a po-space generalizes the fundamental group $\pi_1(X)$ of a topological space $X$ (a single object=base point; morphisms=homotopy classes of loops). It is often an enormous gadget (with uncountably many objects and morphisms) and possesses less structure than a group. It is the aim of this paper to "shrink" the essential information in the fundamental category to an associated component category, that in many cases is finite and possesses a comprehensible structure.

## 3. Categories of fractions and component categories

Many of the tools we need for the study of the fundamental category can in fact be applied to at least all small categories. These are the notions of categories of fractions, of left and right calculi of fractions and of pure systems. The first two notions are well-known in the category theory literature [6, 1] and were already applied to the analysis of fundamental categories in [17, 10]. The new notion in this paper is that of *pure* systems yielding far more satisfactory applications.

3.1. **Categories of fractions.** In the sequel, we will only consider *small* categories (most of the results would still hold with locally small categories [14], but we do not need these in the applications to the fundamental category).

**Definition and lemma 1.** [1] *Let $\mathcal{C}$ be a category.*

(1) *A subset $\Sigma \subset Mor(\mathcal{C})$ is called a* system of morphisms *of $\mathcal{C}$ if*
  (i) $\forall x$ *object of $\mathcal{C}$, $Id_x \in \Sigma$*
  (ii) $\forall \sigma_1 : x \longrightarrow x', \sigma_2 : x' \longrightarrow x'' \in \Sigma, \sigma_2 \circ \sigma_1 \in \Sigma$.
  *(In other words, the objects of $\mathcal{C}$ together with $\Sigma$ form a wide subcategory of $\mathcal{C}$.)*
(2) *Given a system $\Sigma$ of morphisms[2] in $\mathcal{C}$, there is, up to isomorphism of categories, a unique category (denoted $\mathcal{C}[\Sigma^{-1}]$) and a functor $P_\Sigma : \mathcal{C} \longrightarrow \mathcal{C}[\Sigma^{-1}]$, such that:*
  - *$\forall \sigma \in \Sigma, P_\Sigma(\sigma)$ is an isomorphism of $\mathcal{C}[\Sigma^{-1}]$.*
  - *For any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ such that if $\sigma$ is an isomorphism of $\mathcal{C}$ then $F(\sigma)$ is an isomorphism of $\mathcal{D}$, there is a unique functor $G : \mathcal{C}[\Sigma^{-1}] \longrightarrow \mathcal{D}$ such that*

[2] Note that the assumptions in 1. are not necessary for the existence of a category of fractions. Considering only those $\Sigma$ that are subcategories of $\mathcal{C}$ will make things simpler in the rest of the paper, and we do not lose generality by this.

*the following diagram commutes :*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad F \quad} & \mathcal{D} \\ & {\scriptstyle P_\Sigma}\searrow \quad \nearrow{\scriptstyle G} & \\ & \mathcal{C}[\Sigma^{-1}] & \end{array}$$

In fact, each morphism of $\mathcal{C}[\Sigma^{-1}]$ can be represented in the form $\sigma_1^{-1} \circ a_1 \circ \cdots \circ \sigma_{k-1}^{-1} \circ a_k$ where each $a_i$ is a morphism of $\mathcal{C}$ and $\sigma_i^{-1}$ denotes the formal inverse of $\sigma_i \in \Sigma$, cf. [6, 1].

**Example 1.** *In algebraic topology, one considers the category of CW-complexes or of simplicial sets with formal inverses to the system of "weak equivalences", i.e., those maps which induce isomorphisms of all homotopy groups. This category of fractions is called the* homotopy category *or the category of "homotopy types"* [6].

3.2. **Component categories.** Any morphism of the form $s_1^{-1} \circ s_2 \circ \cdots \circ s_{2k-1}^{-1} \circ s_{2k}$, $s_j \in \Sigma, k \in \mathbb{N}$ is called a $\Sigma$-*zig-zag* morphism. The set $ZZ(\Sigma)$ of all $\Sigma$-*zig-zag* morphisms forms a system of morphisms contained in the invertibles of the category of fractions, denoted $Inv(\mathcal{C}[\Sigma^{-1}])$. Equality holds if $\Sigma$ contains the invertibles $Inv(\mathcal{C})$ of the original category $\mathcal{C}$. In fact, $\mathcal{C}[(\Sigma \cup Inv(\mathcal{C}))^{-1}] = \mathcal{C}[\Sigma^{-1}]$. The subcategory of $\mathcal{C}[\Sigma^{-1}]$ with all objects, the morphisms of which are given by the zig-zag morphisms $ZZ(\Sigma)$, forms in fact a *groupoid*.

Two objects $x, y \in Ob(\mathcal{C})$ are called $\Sigma$-*related* – $x \simeq_\Sigma y$ – if there exists a zig-zag-morphism from $x$ to $y$. This definition corresponds to usual path connectedness *with respect to paths in* $\Sigma$ *only – but regardless of orientation.* Being $\Sigma$-related is an equivalence relation; the equivalence classes will be called the $\Sigma$-*connected components* – the path components with respect to $\Sigma$-zig-zag paths, i.e., the components of the groupoid above.

Next, consider the smallest equivalence relation on the morphisms of $\mathcal{C}[\Sigma^{-1}]$ generated (under composition) by

$$\alpha \simeq \alpha \circ s^j, \alpha \simeq t^j \circ \alpha \text{ for } \alpha \in Mor(x, y), s \in \Sigma(x', x), t \in \Sigma(y, y'), j = \pm 1.$$

Remark that equivalent morphisms no longer need to have the same source or target. In particular, every morphism in $\Sigma$ is equivalent to the identities in both its source and its target; hence, all zig-zag morphisms within a component are equivalent to each other.

Dividing out the morphisms in $\Sigma$ within $\mathcal{C}$, we arrive at a *component category*: The objects of the component category $\pi_0(\mathcal{C}; \Sigma)$ are by definition the $\Sigma$-connected components of $\mathcal{C}$; the morphisms from $[x]$ to $[y]$, $x, y \in Ob(\mathcal{C})$, are the equivalence classes of morphisms in $\bigcup_{x' \simeq_\Sigma x, y' \simeq_\Sigma y} Mor_{\mathcal{C}[\Sigma^{-1}]}(x', y')$. The composition of $[\beta] \circ [\alpha]$ for $\alpha \in Mor_{\mathcal{C}[\Sigma^{-1}]}(x, y)$ and $\beta \in Mor_{\mathcal{C}[\Sigma^{-1}]}(y', z)$ is given by $[\beta \circ s \circ \alpha]$ with $s$ any zig-zag morphism from $y$ to $y'$. The equivalence class of that composition is independent of the choices of representatives $\alpha$ and $\beta$ (by definition) and of the choice of the zig-zag path $s$ by the preceeding remark.

The overall idea is thus as follows: Having fixed a suitable system $\Sigma$ of "weakly invertible" morphisms, we decompose the study of $\mathcal{C}$ into the study of

- the component category encompassing the global effects of irreversibility and
- the components with a *groupoid* structure given by the $\Sigma$-zig-zags.

The original category $\mathcal{C}$ and the component category $\pi_0(\mathcal{C};\Sigma)$ are related by a functor $\pi_0(\Sigma) : \mathcal{C} \xrightarrow{q_\Sigma} \mathcal{C}[\Sigma^{-1}] \to \pi_0(\mathcal{C};\Sigma)$; the last arrow is the quotient functor.

3.3. **Functors.** Let $\Sigma$ denote a system of morphisms in the category $\mathcal{C}$ and $\Upsilon$ a system of morphisms in the category $\mathcal{D}$. To ensure that a functor $F : \mathcal{C} \to \mathcal{D}$ induces a well-defined functor between the categories of fractions $\mathcal{C}[\Sigma^{-1}] \to \mathcal{D}[\Upsilon^{-1}]$ and then between the categories of components $\pi_0(\mathcal{C},\Sigma)$ and $\pi_0(\mathcal{D},\Upsilon)$, we need to assume that $F(\Sigma) \subseteq \Upsilon$. This is not at all automatically satisfied in easy geometric examples with systems of weakly invertible morphisms. But one can always refine a given system to ensure this condition:

**Lemma 1.** *$F$ induces*

- *a functor $F_{\Sigma,\Upsilon}$ from $\mathcal{C}[(\Sigma \cap F^{-1}(\Upsilon))^{-1}]$ to $\mathcal{D}[\Upsilon^{-1}]$.*
- *a functor $\pi_0 F_{\Sigma,\Upsilon}$ from $\pi_0(\mathcal{C}, \Sigma \cap F^{-1}(\Upsilon))$ to $\pi_0(\mathcal{D},\Upsilon)$.*

*Proof.* Obvious. □

Particularly important is the case of an inclusion $i : \Sigma_1 \hookrightarrow \Sigma_2$ of systems of morphisms within the *same* category $\mathcal{C}$. The identity on $\mathcal{C}$ leads immediately to the functor $i_{\Sigma_1\,\Sigma_2} : \mathcal{C}[\Sigma_1]^{-1} \to \mathcal{C}[\Sigma_2]^{-1}$ and to the functor $\pi_0 i_{\Sigma_1\,\Sigma_2} : \pi_0(\mathcal{C},\Sigma_1) \to \pi_0(\mathcal{C},\Sigma_2)$ which reflects an inverse to a *refinement*. In general, it is useful to understand the structure of wide sub-categories of fractions of $\mathcal{C}[\Sigma^{-1}]$ where we invert less morphisms than the ones of $\Sigma$.

**Lemma 2.** *Let $L_\Sigma$ be the poset of categories of the form $\mathcal{C}[\Lambda^{-1}]$ where $\Lambda \subseteq \Sigma$, with the inclusion of morphisms as partial order. $L_\Sigma$ is a complete lattice.*

*Proof.* Let $(\Sigma_i)_{i\in I}$ a family of system of morphisms. It is easy to see that $C[\bigcap_{i\in I}\Sigma_i]$ is the greatest lower bound in $L_\Sigma$.

Let the least system of morphisms stable under composition of the underlying category, containing all $\Sigma_i$ and identities on all objects of $\Sigma_i$ be denoted by $\uplus_{i\in I}\Sigma_i$. Then $\mathcal{C}[(\uplus_{i\in I}\Sigma_i)^{-1}]$ is the least upper bound of the families of categories $\mathcal{C}[\Sigma_i^{-1}]$. □

In fact, the induced functor of Lemma 1 is the largest functor agreeing with $F$, meaning that it is couniversal with respect to inclusion maps $\mathcal{C}[\Lambda^{-1}] \hookrightarrow \mathcal{C}[\Sigma^{-1}]$ (which are the maps induced by set-theoretic inclusion maps $\Lambda \hookrightarrow \Sigma$). If one uses the components with respect to a system $\Sigma$ of morphisms as the basis for a topology on the objects, then Lemma 1 states that we can always take the greatest such topology making $F$ continuous.

## 4. Calculi of fractions

4.1. **Weakly invertible morphisms.** In the case of the fundamental category $\mathcal{C} = \vec{\pi}_1(X)$ of a po-space $(X, \leq)$, we want to define morphisms to be "weakly invertible" if no "decision" is taken. This means that composition on the left and on the right with such morphims induce (natural) bijections between sets of morphisms. This idea can be formulated for general small categories:

Let $\mathcal{C}_{\to x}$ (respectively $\mathcal{C}_{y\to}$) denote the full subcategory of $\mathcal{C}$ consisting of objects $z$ such that $\mathcal{C}(z,x) \neq \emptyset$ (respectively $\mathcal{C}(y,z) \neq \emptyset$), and consider first the Yoneda functor: $\mathcal{Y}_\mathcal{C} : \mathcal{C} \to \hat{\mathcal{C}}$, where $\hat{\mathcal{C}}$ is the category of presheaves over $\mathcal{C}$.

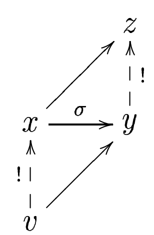
**Definition 3.** *We say that a morphism $\sigma : x \to y$ in $\mathcal{C}$ is* weakly invertible *on the left (respectively on the right) if for all objects $z$, $\mathcal{Y}_\mathcal{C}(\sigma)$ (respectively $\mathcal{Y}_{\mathcal{C}^{op}}(\sigma)$) is a natural isomorphism when restricted to $\mathcal{C}_{\to x}$ (respectively on $\mathcal{C}^{op}_{\to y} = \mathcal{C}_{y\to}$). We say that $\sigma$ is weakly invertible if $\sigma$ is weakly invertible both on the left and on the right*[3].

Less abstractly formulated, we ask that all maps (for all $v, z \in \mathcal{C}$):

$$\mathcal{C}(y,z) \longrightarrow \mathcal{C}(x,z) \qquad\qquad \mathcal{C}(v,x) \longrightarrow \mathcal{C}(v,y)$$

$$g \longmapsto g \circ \sigma \qquad\qquad h \longmapsto \sigma \circ h$$

are set-theoretic bijections:



whenever $\mathcal{C}(y,z) \neq \emptyset$ (respectively, on the right-hand side, $\mathcal{C}(v,x) \neq \emptyset$). Obviously,

**Lemma 3.** *The weakly invertible morphisms in $\mathcal{C}$ form a system of morphisms (cf. Def. 1).*

As an example, consider the po-space of Figure 5, which is $\vec{I} \times \vec{I}$ minus the interior of a square.

All morphisms with end-points within the closed square region $A$ or $D$ are weakly invertible in the sense above. Similarly, all morphisms with both end points within the open regions $B$, resp. $C$, are weakly invertible.

[3] The fact, that we look only at *restrictions* of the Yoneda functor on $\mathcal{C}_{\to x}$ and $\mathcal{C}_{y\to}$ is of primary importance: otherwise we would define the weakly invertible morphism to be the isomorphisms in the *original* category, by Yoneda's Lemma.

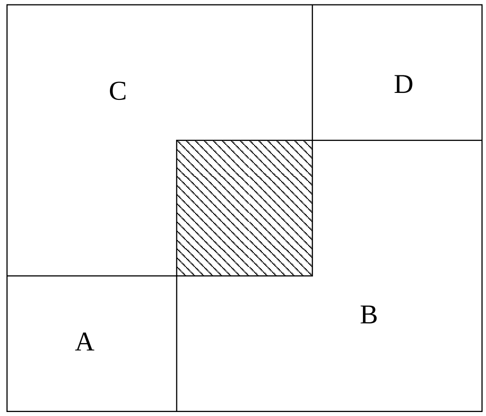



FIGURE 5. A simple po-space and components containing only weakly invertible morphisms

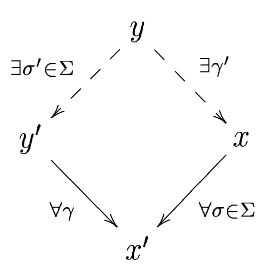
4.2. **Calculi of left and right fractions.** Whether a morphism $s \in \mathcal{C}(x, y)$ is weakly invertible or not depends only on the morphisms with a target reachable from $y$, resp. with a source that can reach $x$. This condition is thus, in general too weak to compare objects with respect to all ingoing and outgoing morphisms.

**Example 2.** *In Fig. 6, the "vertical" morphism $\sigma$ is weakly invertible, but "taking" this morphism represents a decision (in particular to end in a deadlock or to have the possibility of ending in the final state).*

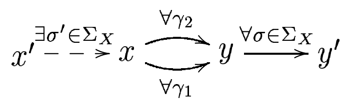
This defect can be repaired by an extra condition to the system $\Sigma$ to be chosen. This "*lr*" condition moreover allows us to represent morphisms in the category of fractions and in the component category in a much easier way:

**Definition 4.** [1] *Let $\mathcal{C}$ be a category. A system $\Sigma$ of morphisms in $\mathcal{C}$ is said to* admit a right calculus of fractions *(for short: is an lr-system) if it satisfies (in addition to properties (i) and (ii) from Def.4):*

(iii) $\forall \gamma : y' \longrightarrow x'$, $\forall \sigma : x \longrightarrow x' \in \Sigma$, $\exists \sigma' : y \longrightarrow y'$, $\exists \gamma' : y \longrightarrow x$ *such that* $\sigma \circ \gamma' = \gamma \circ \sigma'$, *i.e. the following diagram is commutative:*



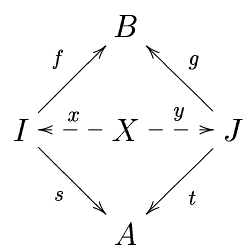
(iv) $\forall \gamma_1, \gamma_2 : x \longrightarrow y$, $\forall \sigma : y \longrightarrow y' \in \Sigma$ *such that* $\sigma \circ \gamma_1 = \sigma \circ \gamma_2$, $\exists \sigma' : x' \longrightarrow x \in \Sigma$ *such that* $\gamma_1 \circ \sigma' = \gamma_2 \circ \sigma'$



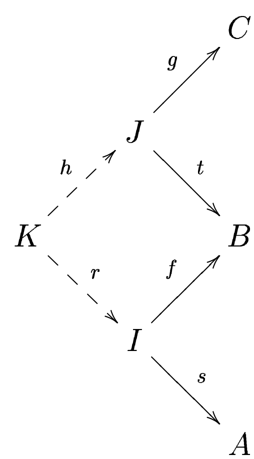

*Property (iii) will be called the extension property for calculi of right fractions. A left calculus of fractions is defined similarly. We will also call such a system* $\Sigma$ *an r-system, respectively l-system respectively lr-system for left and right fractions).*

A straightforward consequence of the extension property for calculi of right fractions – and explaining the name – is that every morphism of $\mathcal{C}[\Sigma^{-1}]$ can be written as $[(a\sigma^{-1})]$ for certain morphisms $a$ of $\mathcal{C}$ and $\sigma \in \Sigma$.

As for ordinary fractions, $fs^{-1}$ and $gt^{-1}$ can represent equivalent morphisms in the category of fractions $\mathcal{C}[\Sigma^{-1}]$. In fact they do if one can find morphisms $x : X \to I$ and $y : X \to J$ in $\mathcal{C}$ such as in the following commutative diagram:




and such that $sx$ ($=ty$) is in $\Sigma$. Now the composite of equivalence classes of $fs^{-1} : A \to B$ with $gt^{-1} : B \to C$ is the class of morphism $(g \circ h)(s \circ r)^{-1}$ as pictured in the following diagram:




The object $K$ and the morphisms $h$ and $r$ arise from the extension property of calculi of right fractions.

For the properties of the component category with respect to an $lr$-system, cf. Sect. 5.2.

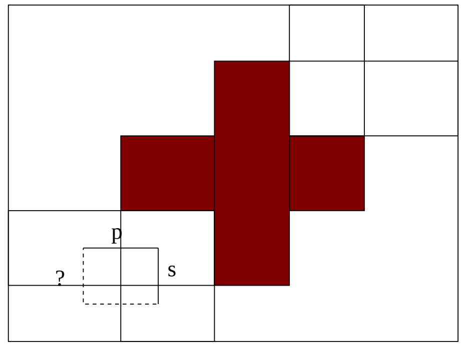



FIGURE 6. How to find the components with lr conditions

**Lemma 4.** *The class of weakly invertible morphisms on the right satisfy property (iv) of calculi of right fractions. The class of weakly invertible morphisms on the left satisfy property (iv) of calculi of left fractions.*

*Proof.* Let $s : z \to x$ be a weakly invertible on the right and $f, g : x \to y$ such that $f \circ s = \mathcal{Y}_{\mathcal{C}^{op}}(s)(f) = g \circ s = \mathcal{Y}_{\mathcal{C}^{op}}(s)(g)$.

As $s$ is weakly invertible on the right, $\mathcal{Y}_{\mathcal{C}}op(s)$ is a bijection from $\mathcal{C}(x, y)$ to $\mathcal{C}(z, y)$ so we must have $f = g$. Just take $t = Id_y$ which is weakly invertible on the right (by (i)): this gives property (iv) of right-fractions.

The dual of (iv) is proven similarly by using $\mathcal{Y}_{\mathcal{C}}(s)$. □

It is *not* true in general that the class of weakly invertible morphisms is a calculus of left or right fractions. An example is the fundamental category of the swiss flag again, Figure 1. Every morphism is weakly invertible in regions 1, 2, 3, 5, 6, 8, 9 and 10 of Figure 4 but there is no way to "detect" regions 4 and 7. Look at Figure 6, if we suppose $s$ to be weakly invertible, $p$ is the dipath shown on this figure, we cannot find a way that property (i) is satisfied. So if we impose the lr properties, then we are bound to subdivide furthermore the regions, to find regions 4 and 7. There are examples for which the only left and right calculus of fractions included in the weak-invertibles is the set of identities, making the retract of the fundamental category no simpler than the fundamental category itself (see Figure 9).

4.3. **Pure systems.** Why not just stop here? If we look at simple examples, the category of components seems alright. For instance the component category of the weakly invertible morphisms (defining here a left and right category of fractions) of Definition 3) for the po-space from Fig. 5 is just the free category on the graph delineated in Figure 8.

But now, consider the po-space consisting of the left part of Figure 7, i.e., $\vec{I} \times \vec{I} \times \vec{I}$ minus the interior of a cube. Then all morphisms in the interior of the 26 regions delineated in the right hand side of the same figure are weakly invertible. But any dipath from the initial to the final point is weakly invertible as well, composed of the composition of a number of

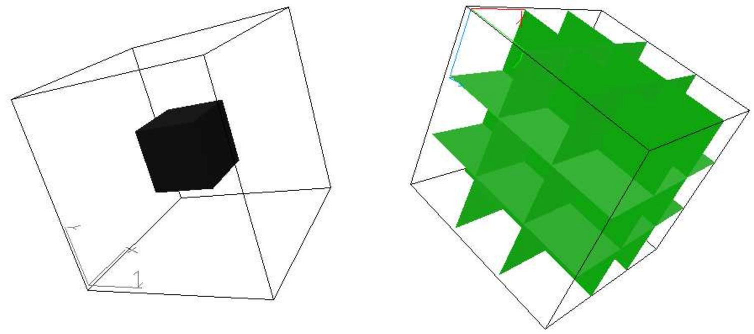


FIGURE 7. Weakly invertible morphisms need not be pure.

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

FIGURE 8. The category of components of a simple po-space.

non weakly invertible morphisms going from one of the 26 regions to a neighbouring one (we will come back to the full calculation in Section 6). This means that the component category given by inverting the weakly invertible morphisms would actually have some endomorphisms which would not be the identity. In the concrete case here, when we impose the *lr*-conditions, the problem disappears, but it is not at all clear, that this is true in general.

So we try to eliminate such pathologies by imposing some extra condition at the calculus of invertibles, that we now define:

**Definition 5.** *A system of morphisms $\Sigma$ within a category $\mathcal{C}$ is called* pure *if*

(v) *for all $\sigma \in \Sigma$, if we can decompose $\sigma$ as $g \circ f$ then $g$ and $f$ must be in $\Sigma$ too.*

Another way to phrase this property is that no invertibles should be decomposed using a non-invertible, or that $Mor(\mathcal{C}) \setminus \Sigma$ is closed under composition. In some ways, a real inessential morphism should be a morphism that does not make any decision, not only from start to end point but also on the way: some decisions cancel out but we want to have the "atomic" ones. This technical condition will also prove extremely useful in the proofs of most propositions in Sections 5.2 and 5.3.

4.4. **Maximal systems.** In the following, we will mainly be concerned with pure lr-systems of morphisms. A system of morphisms always contains a largest subsystem which is lr, but there is no reason that there should be a maximal pure lr subsystem.

Given a system of morphisms $\Sigma$ in a category $\mathcal{C}$. The least subsystem $\Sigma' \subset \Sigma$ that has the l-, r-, lr-property, consists just of the identity morphisms $Id_x$. But there is also always a greatest such system:

**Lemma 5.** *Let* $(\Sigma_i)_{i\in I}$ *a family of systems of morphisms so that* $\forall i \in I$, $\Sigma_i$ *satisfies all the conditions of the right (left, respectively) calculus of fractions and* $\Sigma_i \subset \Sigma$. *Then* $\uplus_{i\in I}\Sigma_i \subset \Sigma$ *satisfies all the conditions of the right (left, respectively) calculus of fractions.*

*Proof.* The first and second conditions are clearly satisfied. The rest is easily done by induction on the number of compositions of morphisms. □

If we are lucky enough to start with a pure subcategory $\Sigma$ (meaning that factorization of morphisms are always within $\Sigma$) of weakly invertible morphisms, then it is the case that the greatest lr-subcategory is still pure. This is of course in that case the "maximal pure lr-system" in $\Sigma$:

**Proposition 1.** *Let* $\Sigma$ *be any pure subcategory of the weakly invertible morphisms in a category* $\mathcal{C}$ *(with unique identity endomorphisms). Then the greatest left and right calculus of fractions in* $\Sigma$ *for* $\mathcal{C}$ *is a pure calculus of fractions.*

*Proof.* We take any sub-lr-system $\Sigma'$ of $\Sigma$ and we suppose it is not pure. Then there exists $\sigma = f_1 \circ f_2$ with $f_1$ or $f_2$ not in $\Sigma'$. Consider $\Sigma''$ the category generated by $\Sigma'$ and $f_1$ (similarly for $f_2$). Then it can be shown that it is a lr-system, including strictly $\Sigma'$, but included in $\Sigma$. This proves that the greatest lr-system (which always exists by Lemma 5) in $\Sigma$ has to be pure. □

In the general situation, one may proceed as follows: We work with pairs $(\mathcal{C}, \Sigma)$ of categories and admissible systems of weakly invertible morphisms (admissible means lr, pure lr etc.) Call a functor $F : (\mathcal{C}_1, \Sigma_1) \to (\mathcal{C}_2, \Sigma_2)$ with $F(\Sigma_1) \subseteq \Sigma_2$ an elementary equivalence if

- $\mathcal{C}_1 = \mathcal{C}_2$ or
- $\pi_0 F_{\Sigma_1\Sigma_2} : \pi_0(\mathcal{C}_1, \Sigma_1) \to \pi_0(\mathcal{C}_2, \Sigma_2)$ is an equivalence of categories.

the first of which reflects refinements, cf. Sect. 3.3. Regard the equivalence relation on these pairs generated by elementary equivalences. Then the component category with respect to *any* admissible system of morphisms is equivalent to the original category with the system of identities. Of course, it is still interesting to ask for as large as possible admissible systems (as coarse as possible component categories), although these might not be unique.

## 5. Properties of systems of weakly invertible morphisms and corresponding components

5.1. **Weakly invertible morphisms and histories.** There is a strong link between factorization properties in $\mathcal{C}$ and weak invertibility (as categories of fractions and factorization systems do have in general, see [1]), which will have a strong geometric and concurrency theoretical meaning: the homotopy histories of [5]. First, we need a definition:

**Definition 6.** *Given two objects $x_0$, $x_1$ in $\mathcal{C}$ and $f : x_0 \to x_1$ a morphism in $\mathcal{C}$, the history $h_{x_0,x_1}[f]$ of $f$ is defined as*

$$h_{x_0,x_1}[f] = \{x \in \mathcal{C} | \ \exists f_0 : x_0 \to x, f_1 : x \to x_1 \text{ with } f = f_1 \circ f_0\}.$$

*Two objects $x, y \in \mathcal{C}$ are history equivalent (for a given $x_0$ and $x_1$, noted $x \sim_{x_0,x_1} y$, if $x \in h_{x_0,x_1}[f] \Leftrightarrow y \in h_{x_0,x_1}[f]$ for all $f : x_0 \to x_1$.*

**Lemma 6.** *Let $\mathcal{C}$ be a category where the only endomorphisms are identities. Then, $\sigma : x \to y$ has surjective composition on the left (as weakly invertible on the left, in Def.3, except we only require surjectivity) and on the right if and only if,*

- *$x \sim_{x_0,x_1} y$ for all $x_0$ and $x_1$ such that $\mathcal{C}(x_0, x) \neq \emptyset$ and $\mathcal{C}(y, x_1) \neq \emptyset$,*
- *$\sigma$ is the only morphism from $x$ to $y$ in $\mathcal{C}$.*

*Proof.* Easy diagram chasing. The second condition of the lemma is of course needed geometrically: take for instance the square minus one square in Figure 5. Regions A and D are in the same homotopy-history equivalence classes. This is implied by the surjectivity of left composition with $\sigma$ by the same argument as the one of Proposition 2. □

This implies that weak invertibility refines the notion of homotopy history equivalence of [5]. In the case where $\mathcal{C}$ is the fundamental category of a sub-pospace of $\mathbb{R}^2$, these are in turn equivalent to $\sigma : x \to y$ is weakly invertible (see [16]).

When applied to the fundamental category of po-spaces, this means that the essential schedules in a concurrent system are separated out by the notion of component. Notice that this is true also for partial schedules and not just the maximal ones as in e.g. [10] and [5].

5.2. **General properties of systems of weakly invertible morphisms.** In this section, we state and prove essential properties of the partition into components of a small category $\mathcal{C}$ that a system $\Sigma$ of weakly invertible morphisms induces. The most important case we have in mind is the fundamental category $\mathcal{C} = \vec{\pi}_1(X)$ of a po- or $d$-space $X$ together with a maximal pure lr-system of weakly invertible morphisms in $\mathcal{C}$ from Sect. 4.4. Several properties are true for more general systems, and they will thus be stated with a minimal set of conditions.

First an easy formal consequence of the definitions:

**Lemma 7.** *The weakly invertible morphisms in $\mathcal{C} \times \mathcal{D}$ are products of weakly invertible morphisms in $\mathcal{C}$ with weakly invertible morphisms in $\mathcal{D}$.*

**Proposition 2.** *Let $\Sigma$ consist of weakly invertible morphisms and let $s \in \Sigma(x, y)$. Then the maps*

$$\mathcal{C}(x,x) \xrightarrow{s\#} \mathcal{C}(x,y) \xleftarrow{s\#} \mathcal{C}(y,y)$$

*are bijections.*

*If, in particular,* $\mathcal{C}(x,x) = \{Id_x\}$[4]*, then* $\mathcal{C}(x,y) = \Sigma(x,y) = \{s\}$*. In other words: The components in the component category* $\pi_0(\mathcal{C};\Sigma)$ *have unique endomorphisms.*

*Proof.* Immediate from definitions. □

**Proposition 3.** *Let* $\Sigma$ *denote an* $l$*-system of morphisms in* $\mathcal{C}$*.*

(1) *For every* $f \in \mathcal{C}(x,y)$ *and every* $x' \sim_\Sigma x$ *there exists* $y' \sim_\Sigma y$ *and* $f' \in \mathcal{C}(x',y')$ *such that* $f' \sim_\Sigma f$*.*
(2) *Let* $[f]_\Sigma \in \pi_0(\mathcal{C};\Sigma)([x]_\Sigma,[y]_\Sigma)$ *and let* $x' \in [x]_\Sigma$*. Then there exists* $y' \in [y]_\Sigma$ *and* $f' \in \mathcal{C}(x',y')$ *such that* $[f']_\Sigma = [f]_\Sigma$*.*

Statement (2) should be interpreted as a lifting property, lifting morphisms from the component category $\pi_0(\mathcal{C};\Sigma)$ to the original category $\mathcal{C}$.

*Proof.* Immediate from the definition of an $l$-system. □

There is of course an analogous statement for liftings in $r$-families of morphisms.

**Proposition 4.** *Let* $\Sigma$ *denote a* pure l-*system of weakly invertible morphisms. Let* $\mathcal{C}(x,x) = \{Id_x\}$ *and let* $x \sim_\Sigma y$*. If*

$$x \xrightarrow{f} z \xrightarrow{g} y,$$

*then* $f, g \in \Sigma$ *and* $z \sim_\Sigma x$*.*

*Proof.* Since $x \sim_\Sigma y$ with $\Sigma$ an $l$-system, there exist morphisms $\sigma \in \Sigma(x,u), \tau \in \Sigma(y,u)$ in the diagram

$$\begin{array}{ccccc} x & \xrightarrow{f} & z & \xrightarrow{g} & y \\ & \searrow^{\sigma} & & \swarrow_{\tau} & \\ & & u & & \end{array} .$$

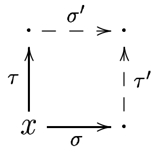
By Prop. 2, these are the only morphisms beween $x$ and $u$, resp. $y$ and $u$. In particular, $\tau \circ g \circ f = \sigma$. Since $\sigma$ is pure, we conclude that $f, g \in \Sigma$. □

Again, there is an analogous property (with an analogous proof) for pure r-families of weakly invertible morphisms.

The result can be understood as a "diconvexity" property of the components. Here is a negative formulation of the result: If $x \not\sim_\Sigma z$ and $\mathcal{C}(z,y) \neq \emptyset$, then $x \not\sim_\Sigma y$. You never return to a component that you have left.

[4]This condition is always satisfied for the fundamental category of a po-space.

**Proposition 5.** *Let $\Sigma$ denote a* pure *$l$-system of morphisms in $\mathcal{C}$, let $\sigma, \tau \in \Sigma(x, -)$. There exists a solution of the extension problem*




*with* both *morphisms $\sigma', \tau' \in \Sigma$.*

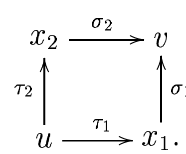
*Proof.* From the extension condition we get $\sigma' \in \Sigma$ and $\tau' \in \mathcal{C}$ such that $\tau' \circ \sigma = \sigma' \circ \tau \in \Sigma$. Since $\Sigma$ is pure, $\tau'$ has to be in $\Sigma$, as well. □

Again, there is an analogous statement for extensions with respect to pure $r$-families of morphisms.

**Corollary 1.** *Let $\Sigma$ denote a* pure *$l$-system (resp. $r$-system) of morphisms in $\mathcal{C}$. Every morphism in the subcategory generated by $\Sigma$ in $\mathcal{C}[\Sigma]^{-1}$ can be represented in the form $\sigma_1^{-1} \circ \sigma_2$ (resp. $\sigma_1 \circ \sigma_2^{-1}$), $\sigma_i \in \Sigma$, $1 \le i \le 2$.*

*Proof.* The same proof as for the expression of general morphisms using the $\Sigma$-extension property from Prop. 5. □

**Proposition 6.** *Let $\Sigma$ be a pure $l$-system (or pure $r$-system or an $lr$-system) and suppose $x_1 \sim_\Sigma x_2$. Then there exist objects $u, v$ and $\Sigma$-morphisms as in the following diagram:*




*Proof.* For an $l$-system, $x_1 \sim_\Sigma x_2$ ensures the existence of $v, \sigma_1$ and $\sigma_2$ as in the diagram. Proposition 5 allows to extend this part of the diagram with $\Sigma$-morphisms $\tau_i$. For a pure $r$-system, the proof proceeds in the reverse sequence. □

For a pure system $\Sigma$, the lifing property from Prop. 3 can be sharpened:

**Proposition 7.** *Let $\Sigma$ be a pure $l$-system (or pure $r$-system) within $\mathcal{C}$. Let $C_1, C_2 \subset Ob(\mathcal{C})$ denote two components such that the set of morphisms (in $\pi_0(\mathcal{C};\Sigma)$) is* finite. *Then, for every $x_1 \in C_1$ there exists $x_2 \in C_2$ such that the quotient map*

$$\mathcal{C}(x_1, x_2) \to \pi_0(\mathcal{C};\Sigma)(C_1, C_2),\ f \mapsto [f]$$

*is* onto. *If $\Sigma$ is a pure $lr$-system of weakly invertible morphisms with $\mathcal{C}(x,x) = \{Id_x\}$ for all $x \in Ob(\mathcal{C})$ , the quotient map is even a* bijection.

*Proof.* By repeated application of Prop. 3, all $n$ morphisms from $C_1$ to $C_2$ can be lifted to morphisms $f_1, \ldots, f_n$ with source in $x_1$ and targets in $y_1, \ldots, y_n \in C_2$. By repeated application of Prop. 6, there exist morphisms $\sigma_i \in \Sigma$ from $y_i$ to the *same* target $x_2 \in C_2$, $1 \le i \le n$. The quotient map is onto, since $\sigma_i \circ f_i \simeq f_i$, $1 \le i \le n$.

To prove injectivity, assume $f_i \in \mathcal{C}(x_1, x_2)$ with

$$[f_1] = [f_2] \in \pi_0(\mathcal{C}; \Sigma)(C_1, C_2)$$

Then, there exist $x_0 \in C_1, x_3 \in C_2$ and morphisms $\sigma_i \in \Sigma(x_0, x_1), \tau_i \in \Sigma(x_2, x_3),\ 1 \leq i \leq 2$, such that $\tau_1 \circ f_1 \circ \sigma_1 = \tau_2 \circ f_2 \circ \sigma_2 \in \mathcal{C}(x_0, x_3)$. By Prop. 2, $\sigma_1 = \sigma_2$ and $\tau_1 = \tau_2$. Since $\sigma_1$ and $\tau_1$ are weakly invertible, we conclude: $f_1 = f_2 \in \mathcal{C}(x_1, x_2)$. □

Another application of Prop. 6 shows that any component can possess at most one maximal, resp. minimal object:

**Definition 7.** *Let $D \subseteq Ob(\mathcal{C})$. An object $m \in D$ is called a* minimal *element in $D$, if $\mathcal{C}(x, m) \neq \emptyset \Rightarrow x = m$ or $x \notin D$. A maximal element is defined similarly.*

**Corollary 2.** *Let $\Sigma$ be a pure l-system (or pure r-system) within $\mathcal{C}$. Every component with respect to $\Sigma$ can at most have one maximal and one minimal element.*

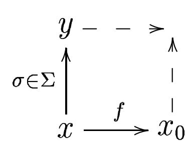
**Remark 1.** *From easy geometric examples as Ex. 5, we know that a component in general need not possess a minimal or maximal element. Question: Is there always an infimum (supremum) for every component? Are those unique? We conjecture that some of the results of* [13] *could be useful for proving this.*

In the presence of maximal or minimal elements for the objects of the whole category, several nice properties can be proved to hold without the pureness assumption. The first lemmas are relevant for the unsafe regions in deadlock analysis (cf. [4]):

**Definition 8.** *For $x \in Ob(\mathcal{C})$, let $x^{\rightarrow}$, resp. $x^{\leftarrow}$ denote the set of maximal (minimal) elements $x_0, y_0 \in Ob(\mathcal{C})$ with $\mathcal{C}(x, x_0) \neq \emptyset$ ($\mathcal{C}(y_0, x) \neq \emptyset$), i.e., the maximal elements reachable from $x$ (minimal elements that can reach $x$).*

**Lemma 8.** *Let $\Sigma$ denote an r-system of morphisms in $\mathcal{C}$, and let $\Sigma(x, y) \neq \emptyset$. Then $x^{\rightarrow} = y^{\rightarrow}$. If $\Sigma$ is an l-system, then $x^{\leftarrow} = y^{\leftarrow}$.*

*Proof.* Consider an extension problem




with $x_0$ a maximal element. The right vertical arrow has to be an endomorphism of $x_0$, and hence $x_0$ can be reached from $y$. □

**Lemma 9.** *Let $\Sigma$ denote an r-system of morphisms in $\mathcal{C}$ and let $x_0$ denote a maximal element of $Ob(\mathcal{C})$. If $\Sigma(x, x_0) \neq \emptyset$ and $\mathcal{C}(x, y) \neq \emptyset$, then $\Sigma(y, x_0) \neq \emptyset$.*

In other words: If $x$ and $x_0$ are $\Sigma$-equivalent, then every $y$ reachable from $x$ is $\Sigma$-equivalent to $x$.

*Proof.* Consider an extension problem

$$\begin{array}{ccc} y & \dashrightarrow & \\ {\scriptstyle f}\uparrow & & \uparrow \\ x & \xrightarrow{\sigma\in\Sigma} & x_0. \end{array}$$

Again, the right vertical arrow has to be an endomorphism of $x_0$, and hence there is a $\Sigma$-morphism from $y$ to $x_0$. □

**Proposition 8.** *Let $\Sigma$ denote an r-system of weakly invertible morphisms in $\mathcal{C}$ and let $x_0$ denote a maximal element of $Ob(\mathcal{C})$ with $\mathcal{C}(x_0, x_0) = \{Id_{x_0}\}$. If $\Sigma(x, x_0) \neq \emptyset$ and $\mathcal{C}(x, y), \mathcal{C}(y, z), \mathcal{C}(y, x_0)$ are all non-empty, then all these sets consist of a single element.*

In other words: If $x$ and $x_0$ are $\Sigma$-equivalent, the category is trivial between $x$ and $x_0$ (no non-trivial dihomotopy between $x$ and $x_0$ in the fundamental category).

*Proof.* Any composition $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} x_0$ is equal to the unique morphism $\sigma \in \Sigma(x, x_0)$ by Prop. 2. By Lemma 9, $\Sigma(y, x_0) \neq \emptyset \neq \Sigma(z, x_0)$, and by Prop. 2, $\mathcal{C}(z, x_0) = \Sigma(z, x_0) = \{h\}$, $\mathcal{C}(y, x_0) = \Sigma(y, x_0) = \{h \circ g\}$. Since $h \circ g$ is weakly invertible, $f$ is the only element of $\mathcal{C}(x, y)$; since $h$ is weakly invertible, $g$ is the only element of $\mathcal{C}(y, z)$. □

5.3. **Topological properties.** More precise information on components relies on topological properties on top of the categorical ones. From now on we investigate a *topological category* $\mathcal{C}$, i.e., the objects $Ob(\mathcal{C})$ form a *topological space* $X$. Additionally, a system $\Sigma$ of morphisms in $\mathcal{C}$ is given.

**Definition 9.** (1) *Let $U$ denote an open set in $X$ and $x, y \in U$. A morphism $f \in \mathcal{C}(x, y)$ is called a $U$*-morphism *if*

$$f = f_2 \circ f_1, f_1 \in \mathcal{C}(x, z), f_2 \in \mathcal{C}(z, y) \Rightarrow z \in U.$$

*The set of all such $U$-morphisms from $x$ to $y$ will be denoted $\mathcal{C}_U(x, y)$.*

(2) *An open set $U \subset X$ is called $\Sigma$-simple if*
   (a) $x, y \in U \Rightarrow |\mathcal{C}_U(x, y)| \leq 1$.
   (b) *For all $x \sim_\Sigma y \in U$ there exists $z \in U$ such that $\Sigma(x, z) \neq \emptyset \neq \Sigma(y, z)$.*

**Definition 10.** *Two $\Sigma$-components $C_1$ and $C_2$ are called* neighbours *if there are*

(1) *a morphism with source in $C_1$ and target in $C_2$ and*
(2) *a $\Sigma$-simple open set $U$ containing an element $p \in C_1 \cap \partial C_2$ such that every $g \in \mathcal{C}(p, x_2)$, $x_2 \in C_2$ decomposes as $g = g_2 \circ g_1$ with $g_1 \in \mathcal{C}_U(p, z)$ and $z \in C_2$ (or the symmetric condition with $p \in \partial C_1 \cap C_2$).*

**Proposition 9.** *Let $(\mathcal{C}, \Sigma)$ denote a category with a pure lr-system of weakly invertible morphisms. Let $C_1$ and $C_2$ denote two neighbouring $\Sigma$-components such that $\mathcal{C}(x_i, x_i) = \{Id_{x_i}\}$ for some $x_i \in C_i$. Furthermore assume there is a $\Sigma$-simple open set containing an element $p \in \partial C_1 \cap \partial C_2$. Then, $Mor(C_1, C_2)$ has exactly one element in the component category $\pi_0(\mathcal{C}; \Sigma)$.*

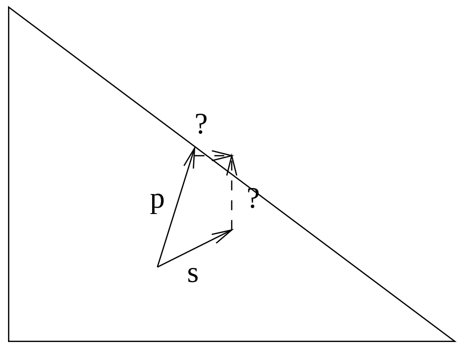



FIGURE 9. A non-cubical po-space

*Proof.* Choose an element $p$ in the intersection of the boundaries of the components as in Def. 10 and assume $p \in C_1$ without loss of generality. By Prop 3, every morphism from $C_1$ to $C_2$ is equivalent to one with source $p$; by Def. 10.1, there exists such a morphism. By assumption in Def. 10.2, every such morphism decomposes as $s \circ f$ with $f$ a $U$-morphism and $s$ a $\Sigma$-morphism within $C_2$ (use Prop. 2) and is hence equivalent to the $U$-morphism $f$.

Consider two $U$-morphisms $f, f'$ with source $p$. By Def. 9, there are $U$-morphisms $s, s' \in \Sigma$ such that $s \circ f = s' \circ f'$. Thus $f \simeq s \circ f = s' \circ f' \simeq f'$. □

The result allows to interpret the component category as a directed graph (rather than a multigraph) with relations. If, in particular, every morphism decomposes into morphisms between neighbour components (as for the fundamental category of a po-space), one may use the classes of these unique morphisms between neighbour components as *generators* for the component category.

## 6. EXAMPLES

In the case of Figure 9, the *only* left and right calculus of fractions included in the weakly invertible morphisms is easily shown to consist of the identities only. As a consequence, the category of components with respect to this greatest lr-system of weakly invertible morphisms in this case is isomorphic to the original category!

The po-spaces arising from *2-dimensional mutual exclusion models*, i.e., a square, from which a number of isothetic rectangles (with edges parallel to the square) have been deleted (as the forbidden region), are handled completely in [16] and [10]: A system of morphisms is a pure lr-system of weakly invertible morphisms if no such morphism crosses a system of line segments emerging from (certain of) the minima, resp. maxima of the rectangles that constitute the forbidden region.

6.1. **The surface of a 3-cube.** Now for a more intricate example, treated in full details. The faces of the 3-cube $C$, or equivalently, the 3-cube minus an interior 3-cube, has 26 components. Points on the faces of the 3-cube are $\{(x,y,z) \in \vec{I}^3 \;|\{x,y,z\} \cap \{0,1\} \neq \emptyset\}$. Let $\mathcal{C} = \vec{\pi}_1(C)$. Observe that

- There are two elements in $\mathcal{C}\left((0,0,a),(1,1,a)\right)$, $\mathcal{C}\left((0,a,0),(1,a,1)\right)$ and $\mathcal{C}\left((a,0,0),(a,1,1)\right)$ when $a \neq 0$. For instance the composition of arrows from $(0,0,a)$ to $(0,1,a)$ and then to $(1,1,a)$ is different from going from $(0,0,a)$ to $(1,0,a)$ and then to $(1,1,a)$. They are different, since all dipaths from $(0,0,a)$ to $(1,1,a)$ has the third coordinate $a \in ]0,1[$, and since the interior of the cube is missing.
- Between other pairs of points, there are at most one morphism.
- All morphisms $\alpha : (x_1,x_2,x_3) \to (y_1,y_2,y_3)$ such that $x_i = 0 \Rightarrow y_i = 0$ and $x_i = 1 \Rightarrow y_i = 1$ are weakly invertible. This is easy to see by the geometry of the cube - the future and past of $(x_1,x_2,x_3)$ and $(y_1,y_2,y_3)$ have the same geometry.

By the last property, we can restrict attention to the 26 classes of points represented by $(x,y,z) \in \{0,-,1\}^3 \setminus (-,-,-)$ where the coordinate $-$ just means an interior point of $]0,1[$. We will see, that none of the morphisms between these points are in the system $\Sigma$. We will omit the commas and write $(0-1)$ for $(0,-,1)$.

6.1.1. *The weakly invertible morphisms:* We will find the arrows which are *not* weakly invertible. Since $\mathcal{C}((00-),(11-))$ has 2 elements and $\mathcal{C}((00-),(1--))$ has one element, the arrow $(1--) \to (11-)$ is not weakly invertible. Similarly, there is only one element in $\mathcal{C}((00-),(111))$, so the arrow $(11-) \to (111)$ is not weakly invertible. Hence (the lack of) weak invertibility implies that no arrow from an upper face $(1--)$, $(-1-)$ or $(--1)$ to an upper edge $(11-)$, $(1-1)$ or $(-11)$ is in $\Sigma$, and similarly for all maps from lower edges to lower faces. Lack of weak invertibility also implies that maps from upper edges to $(1,1,1)$ or from $(0,0,0)$ to lower edges are not in $\Sigma$.

Similarly, $\Sigma((xy-),(11-)) = \emptyset$, when $xy \neq 11$ and $\Sigma((00-),(xy-)) = \emptyset$, when $xy \neq 00$ since there are no weakly invertible morphisms. Permuting the coordinates gives 24 other instances of this.

Notice that the system of weakly invertible morphisms is not pure.

6.1.2. *The maximal lr-system in the weakly invertible morphisms.*

(1) We study maps from any $(abc) \neq (111)$, which is not an upper edge, to $(111)$. These are weakly invertible, but they are not in $\Sigma$: Suppose $s : (abc) \to (111)$ is in $\Sigma$ and suppose $c \neq 1$. Let $f : (abc) \to (11-)$. Then the lr-property implies that we can complete the diagram

$$\begin{array}{ccc} (111) & \overset{g}{\dashrightarrow} & (xyz) \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle \sigma} \\ (abc) & \xrightarrow{f} & (11-) \end{array}$$

with $\sigma \in \Sigma$. Since $(xyz)$ has to be $(111)$, $g$ is the identity.

But $\Sigma((11-),(111)) = \emptyset$, so the diagram cannot be completed with $\sigma \in \Sigma$. Hence $\Sigma((abc),(111)) = \emptyset$ when $(abc) \neq (111)$ and similarly for maps from $(000)$. Notice that this is a concrete example of Prop.8

(2) Any morphism $s : (ab0) \to (11-)$ is weakly invertible, since $(ab0)$ is not reachable from $(00-)$. But suppose $s \in \Sigma$. Then the lr-property is violated: Let $f : (00-) \to (11-)$ be one of the two morphisms. Then the lr-property says that there are maps $\sigma \in \Sigma$ and $g$ completing the diagram:

$$\begin{array}{ccc} (00-) & \xrightarrow{f} & (11-) \\ {\scriptstyle\sigma}\uparrow & & \uparrow{\scriptstyle s} \\ (xyz) & \overset{g}{\dashrightarrow} & (ab0) \end{array}$$

but since $(xyz)$ is below $(00-)$ and $(ab0)$, we conclude that $(xyz) = (000)$ and we know that $\Sigma((000),(00-)) = \emptyset$. Hence $s \notin \Sigma$. Together with what we found in (1), we have seen that no map to an upper edge, and symmetrically, no map from a lower edge is in $\Sigma$.

(3) Now for maps from and to intermediate edges: Suppose first, we map to an upper face. The morphism $t : (1-0) \to (1--)$ is weakly invertible. If it is in $\Sigma$, then we can complete the diagram

$$\begin{array}{ccc} (1-0) & \xrightarrow{t} & (1--) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ (110) & \overset{\sigma}{\dashrightarrow} & (xyz) \end{array}$$

with $\sigma \in \Sigma$. But then $(xyz) \in \{(11-),(111)\}$ and we know there is no such $\sigma$. So $t \notin \Sigma$. The only other option, which is not already covered above, is to map to an intermediate vertex $s : (1-0) \to (110)$; use the diagram above - now assuming $f = s \in \Sigma$. Symmetrically, no map from or to an intermediate edge is in $\Sigma$.

(4) Maps from and to faces: Maps *from* an upper face or *to* a lower face are covered above. Now suppose $s : (abc) \to (1--)$ is in $\Sigma$, and suppose $(abc)$ is not an edge - these are covered above. Then suppose $(abc) < (10-)$ (else $(abc) < (01-)$, so this case is similar). Let $f : (abc) \to (10-)$ and do the diagram. There are no (nontrivial) morphisms from $(10-)$, so $s \notin \Sigma$. The other cases follow in a similar way.

(5) The last case we have to check is maps between intermediate vertices, since maps to and from all other types is covered above: Let $s : (100) \to (110)$. If $s \in \Sigma$, the diagram with $f : (100) \to (1-0)$ should have a completion with $\sigma : (1-0) \to (xyz) \neq (1-0)$ and there are no such morphisms from an intermediate edge.

Hence, in this case, the biggest lr-system in the weakly invertible morphisms is in fact pure, since the morphisms between the 26 types of points are all in the complement of $\Sigma$.

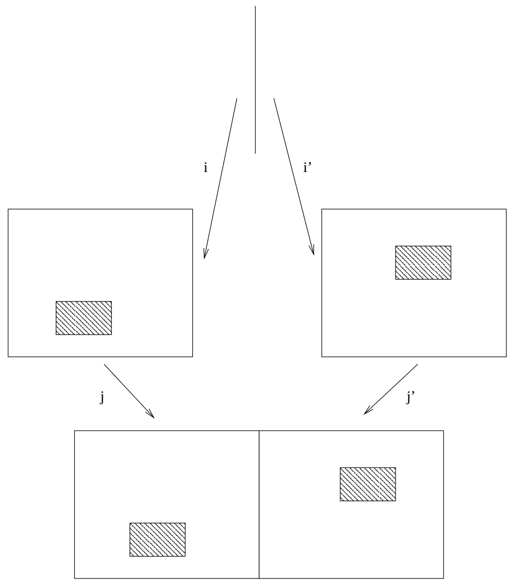



FIGURE 10. A pushout of two po-spaces.

## 7. Conclusion and future work

We hope to achieve an effective calculation of the component categories of the fundamental category of reasonable po-spaces by applying Marco Grandis' directed version [11] of a van Kampen theorem for directed spaces. More precisely, let $X = X_1 \cup X_2$ and let $\Sigma_0$, $\Sigma_1$ and $\Sigma_2$ denote admissible (lr, pure lr) systems of weakly invertible morphisms in the fundamental categories $\vec{\pi}_1(X_1 \cap X_2), \vec{\pi}_1(X_1)$ and $\vec{\pi}_1(X_2)$. The task is to derive an admissible system $\Sigma_{12}$ of weakly invertible morphisms in $\vec{\pi}_1(X)$ – and thus derive a suitable component category for the union. For the time being, we can only state the following conjecture:

Let $i_1 : X_1 \cap X_2 \to X_1$ and $i_2 : X_1 \cap X_2 \to X_2$ be the canonical inclusion morphisms (respectively $i_1^* : \vec{\pi}_1(X_1 \cap X_2) \to \vec{\pi}_1(X_1)$ and $i_2^* : \vec{\pi}_1(X_1 \cap X_2) \to \vec{\pi}_1(X_2)$ the induced functors between the corresponding fundamental categories).

Our claim is:

**Conjecture 1.** *(van Kampen on components) The greatest left and right calculus of fractions (cf. Lemma 5) in the pushout $\Delta$ of $\Sigma_1$ and $\Sigma_2$ above $I_{12} := i_1^{*-1}(\Sigma_1) \cap i_2^{*-1}(\Sigma_2) \cap \Sigma_0$ as below[5]:*

$$\begin{array}{ccc} I_{12} & \xrightarrow{i_1^*} & \Sigma_1 \\ \downarrow{\scriptstyle i_2^*} & & \downarrow \\ \Sigma_2 & \longrightarrow & \Delta \end{array}$$

[5]The induced functors from $i_1^*$ and $i_2^*$ on the invertible morphisms, still denoted the same way, are the ones of Lemma 1.

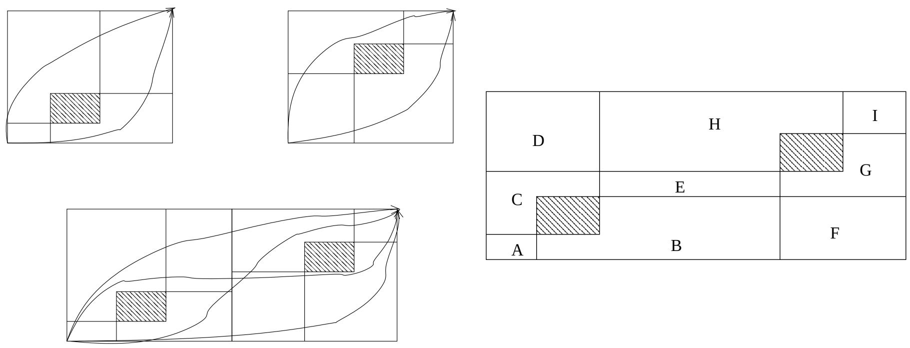



FIGURE 11. The inessential morphims after pushout.

*is denoted* $\bar{\Sigma}_{12}$. *Then the system*

$$\Sigma_{12} = \{s \in \vec{\pi}_1(X) | \; P_{\bar{\Sigma}_{12}}(s) \in ZZ(\bar{\Sigma}_{12})\},$$

*containing all morphisms that are identified with zig-zag morphisms in the category of fractions with respect to* $\bar{\Sigma}_{12}$*, is the admissible system describing the "inessential" morphisms of the fundamental category* $\vec{\pi}_1(X)$.

As an example of this conjectural van Kampen theorem, consider the situation of Figure 10 with two copies of Figure 5 glued together along a common boundary. Figure 10 shows the corresponding pushout diagram of po-spaces $X_1$ and $X_2$.

The left part of Figure 11 shows the union of the components of $X_1$ and $X_2$. Extension properties imposed by the property to be left and right systems imply that some of the inessential morphisms should *no longer* be considered as inessential in the union of the two spaces. The greatest left and right system (which is pure) is shown in the right part of Figure 11.

As a second example, consider a rectangle $X$ as the union of $X_1$, a rectangle without an inner square (Fig. 5), and $X_2$ filling in that inner square (with a collar). The intersection $X_1 \cap X_2$ is dihomeomorphic to $X_1$. This example shows that it is necessary to "complete" $\bar{\Sigma}_{12}$ in the category of fractions to arrive at the result $\Sigma_{12} = \pi_1(X)$.

The system $\Sigma_{12}$ in the conjecture is an lr-system almost by definition. Probably, one needs additional assumptions (e.g., $X_1$ "below" $X_2$ or vice versa) to make sure that it consists of weakly invertible morphisms and/or satisfies pureness.

For application purposes, we would like to exploit the van Kampen conjecture to arrive at a geometrically based algorithm detecting the components in a mutual exclusion model (cf. Sect. 1) from a description of the forbidden region, as a generalisation of our algorithm detecting deadlocks, unsafe and unreachable regions [4].

## REFERENCES

1. F. Borceux, *Handbook of categorical algebra 1 : Basic category theory*, Cambridge University Press, 1994.

2. R. Brown, *Elements of modern topology*, McGraw Hill, Maidenhead, 1968.
3. E.W. Dijkstra, *Cooperating sequential processes*, Academic Press, 1968.
4. L. Fajstrup, E. Goubault, and M. Raussen, *Detecting deadlocks in concurrent systems*, Proceedings of the 9th International Conference on Concurrency Theory, also available at http://www.dmi.ens.fr/~goubault, Springer-Verlag, 1998.
5. ______, *Algebraic topology and concurrency*, submitted to Theoretical Computer Science, also technical report, Aalborg University (1999).
6. P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, no. 35, Springer Verlag, 1967.
7. P. Gaucher, *A convenient category for the homotopy theory of concurrency*, preprint available at math.AT/0201252, 2002.
8. P. Gaucher and E. Goubault, *Topological deformation of higher dimensional automata*, Tech. report, arXiv:math.AT/010760, to appear in HHA, 2001.
9. E. Goubault, *Some geometric perspectives in concurrency theory*, to appear in Homology Homotopy and Applications (2002).
10. É. Goubault and M. Raussen, *Dihomotopy as a tool in state space analysis*, LATIN 2002: Theoretical Informatics (Cancun, Mexico) (S. Rajsbaum, ed.), Lect. Notes Comput. Sci., vol. 2286, Springer-Verlag, April 2002, pp. 16 – 37.
11. M. Grandis, *Directed homotopy theory, I. the fundamental category*, Cahiers Top. Gom. Diff. Catg, to appear, Preliminary version: Dip. Mat. Univ. Genova, Preprint 443 (2001).
12. P. T. Johnstone, *Stone spaces*, Cambridge University Press, 1982.
13. G. M. Kelly, S. Lack, and R. F. C. Walters, *Coinverters and categories of fractions for categories with structure*, Applied Categorical Structures (1993).
14. S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag, 1971.
15. L. Nachbin, *Topology and order*, Van Nostrand, Princeton, 1965.
16. M. Raussen, *On the classification of dipaths in geometric models for concurrency*, Mathematical Structures in Computer Science **10(4)** (2000).
17. M. Raussen, *State Spaces and Dipaths up to Dihomotopy*, Homology Homotopy Appl, to appear (2002).

*E-mail address*: `fajstrup@math.auc.dk, raussen@math.auc.dk`

Department of Mathematical Sciences, Aalborg University, Denmark

*E-mail address*: `Eric.Goubault@cea.fr, haucourt@aigle.saclay.cea.fr`

CEA/Saclay, France