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Inverse Acoustic Wave Equation.

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Inverse Acoustic Wave Equation.

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Abstract. The paper addresses the scattering of acoustic waves from an inhomogeneous compactly supported scatterer. It is proved that the knowledge of one incident wave and the corresponding scattered field at a plane outside the scatterer for all times is sufficient for the unique determination of the field and wave velocity. A layer stripping method is used. In each strip the wave velocity is uniquely determined by Neumann to Dirichlet mappings defined on the boundary of the strips. The field in a strip is uniquely determined by integral equations. The stability of the ill posed field problems is discussed.

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1 Introduction

This work deal with the reconstruction of the wave velocity of a three dimensional layered medium in the half-space $x_3 \geq 0$. The inverse problem is: The differential equation is the wave equation. Dirichlet and Neumann data is specified at the plane $x_3 = 0$ for any time for one scattering experiment. The inverse problem is to reconstruct the wave velocity and the field such that the wave equation and the Dirichlet and Neumann conditions hold. The uniqueness of this problem was discussed by M Lassas, M Cheney and G.Uhlmann (1998). A layer stripping method is used in this paper. Various multidimensional layer stripping methods have been suggested. Cheney and Kristensson (1988), Weston (1990, 1992), Sailing He and Weston (1992,1994) and Cheney and Isaacson (1995). An important input in these theories is a Dirichlet to Neumann or a Neumann to Dirichlet mapping:

$$w + K\frac{\partial w}{\partial x_3} = 0. \tag{1}$$

In Weston (1990, 1992) the Neumann to Dirichlet mapping for the up and down moving waves was used to construct the operator equations for the up and down moving field components. In Cheney and Isaacson (1995) the Dirichlet to Neumann operator was used to formulate the basic Riccati equation. Generally it is not possible to find an explicit expression for the operators K or K^{-1}. It is an important observation of this paper that locally in a space-time neighborhood of the wavefront it is indeed possible explicitly to construct the operator K of (1) at the boundary of any layer. In addition this construction will uniquely determine the wave velocity in the q'th layer. A short review of the method of the paper is: Let the q'th layer be the strip $x_3 \in]a_q, a_{q+1}[$. It is assumed that the wave velocity is independent of x_3 in this strip. It is proved that there exist an operator $K(\xi)$ depending on a parameter ξ such that the Neumann to Dirichlet mapping (1) hold in a space time neighbourhood of the wavefront of the field. The operator K is explicitly constructed in a small time interval. Let x_0 denote a point at the plane $x_3 = a_q$. It is proved that the equation $w(x_0) + K(\xi)w = 0$ has a unique solution ξ which equals the wave velocity at x_0 limit from obove. That is the local Neumann to Dirichlet mapping (1) determine uniquely the wave velocity in the q'th layer. The second step is a reconstruction of the field in the q'th strip from given wave velocity of the strip and known Dirichlet and Neumann data at $x_3 = a_q$. This problem is formulated as integral equations, with kernels which depend on the known wave velocity of the q'th strip and with the Dirichlet and Neumann data at $x_3 = a_q$ as the inhomogeneous terms. It is proved that the integral equations have a unique solution. It is well known that the Cauchy problem for the field in the q'th layer is ill posed: If the metric is based on a Sobolev norm, then the mapping from Dirichlet and Neumann data at the plane $x_3 = a_q$ onto the Dirichlet and Neumann data at $x_3 = a_{q+1}$ is not continuous. It is an important problem see Cheney and Isaacson (1995) how unstable/stable the inverse problem is. In this paper we prove with the assumptions of corrolar 1 that there exist Hilbert spaces of Dirichlet and Neumann data at the planes $x_3 = a_q$ for which the mapping of Dirichlet and Neumann data at $x_3 = a_q$ onto Dirichlet and Neumann data at the plane $x_3 = a_{q+1}$ is a continuous (isometric) mapping.

The important kind of assumptions needed for the proof of these results is: The q'th layer may be divided in subsets Ω_{qs} where the wave velocity is a constant c_{qs} in Ω_{qs}. The wavefront is assumed to be upp-going. The field has a wavefront with a jump of the field at the wavefront. Actually the wave velocity problem is very overdetermined, the velocities are determined by infinitely many equations, each of which has a unique solution. The continuity result of the Cauchy problems is based on the assumption that the wave velocity is a nonincreasing function of x_3 for any fixed (x_1, x_2) .

2 The Inverse Boundary Value Problem.

In this section the inverse boundary value problem for an inhomogeneous medium exited by an incident wave will be defined. The geometry of the inverse problem is an inhomogeneous medium imbedded in the halfspace above the plane $x_3 = 0$. The field is assumed to satisfy the differential equation equivalent to the wave equation:

$$[\nabla^2(\partial_t)^{-2} - c(x)^{-2}]u = 0 \qquad (x,t) = (\underline{x}, x_3, t) \in R^4, \tag{2}$$

with ∂_t^{-1} defined by:

$$\partial_t^{-1} u(x,t) = \int_{-\infty}^{t} u(x,\tau) d\tau. \tag{3}$$

Here and in the following three dimensional vectors is denoted by x and two dimensional vectors by $\underline{x} = (x_1, x_2)$. The wave velocity below the plane $x_3 = 0$ and outside the scatterer in the upper half-space is assumed to be a constant c_-. The scatterer is compactly supported in a set Ω. The field u is a sum of a known incident field and the scattered field $u = u^{in} + u^{sc}$. The incident field is assumed to be generated by sources in the halfplane $x_3 < 0$. The initial condition of the field is that the field arrive at the scatterer at a time $t \geq 0$.

$$\forall (x,t) \in R^2 \times [0,\infty[\times]-\infty,0[\quad u(x,t) = 0. \tag{4}$$

Definition: The forward problem is for given wave velocity $c(x)$ and given incident field generated by sources in the halfplane $x_3 < 0$ to construct the solution of the wave equation (2) which satifies the initial condition (4), and the condition that the field u^{sc} has compact support in space in any compact time interval.

The boundary data of the inverse problems to be used in the present paper is: The incident field is given, and the scattered field is measured at the boundary $x_3 = 0$. An equivalent information is that Dirichlet and Neumann data is given at the plane $x_3 = 0$. (Obviously there is a one to one relation between these data and the set of data $(u^{in}(\underline{x},0,t), u^{sc}(\underline{x},0,t))$). The inverse problem is for given boundary data to find the wave velocity such that u is a solution of the wave equation (2) which satifies the initial condition (4) and the boundary conditions. The Cauchy data of the inverse problem will be:

$$\partial_t^{-2} u(\underline{x},0,t) = b(\underline{x},t) \quad \text{and} \quad \partial_3 \partial_t^{-2} u(\underline{x},0,t) = b_\nu(\underline{x},t) \quad (\underline{x},t) \in R^3. \tag{5}$$

The inverse boundary value problem is defined by:

Let the Cauchy data (b, b_ν) be given functions which vanishes for $t < 0$. Let the wavefront of the incident field be up-going. Construct the wave velocity $c(x)$ and the field u such that u is a solution of the wave equation (2) with Cauchy data (5), u^{sc} has compact support in space in any finite time interval and u satisfies the initial condition (4).

In an actual reconstruction of the field and wave velocity we need to specify in addition the spaces for the wave velocity and the space of the field. The spaces will be chosen such that we may prove uniqueness for the inverse problem, and find an explit construction of c and u. In addition the choice should not be two restrictive. The space of velocities is chosen such that any given physical media may be approximated with any given error with a velocity in the chosen space. It is assumed that the scatterer consist of N layers each bounded by two planes $x_3 = a_q$ and $x_3 = a_{q+1}$, with $a_{q+1} > a_q$. The q'th layer is a finite set of homogeneous cylinders Ω_{qs}. Let $c_{qs}, q = 0, \cdots, N; s = 0, \cdots, N_q$. denote the wave velocity velocity in the sets Ω_{qs} then we may formulate "the velocity space assumptions" by:

$$\Omega = \bigcup_{q,s} \overline{\Omega_{qs}} \quad \Omega_{qs} = D_{qs} \times [a_q, a_{q+1}] \quad c_{qs}(x) = c_{qs} \quad x \in \dot{\Omega}_{qs}. \tag{6}$$

The boundary $\partial\Omega_{qs}$ consist of points in the planes $x_3 = a_q$, and points above this plane:

$$\partial\Omega_{qs}^- = \{x \in \partial\Omega_{qs} \mid x_3 = a_q\} \qquad \partial\Omega_{qs}^+ = \partial\Omega_{qs} \setminus \partial\Omega_{qs}^-. \tag{7}$$

We are going to use a "layer stripping method". Then the general inverse problem will be formulated as a finite set of inverse problems. Denote the given Cauchy data of the q'th problem at the plane $x_3 = a_q$ by:

$$\partial_t^{-2} u(\underline{x}, a_q, t) = b_q(\underline{x}, t) \text{ and } \partial_3 \partial_t^{-2} u(\underline{x}, a_q, t) = b_{\nu q}(\underline{x}, t); \quad (\underline{x}, t) \in R^3. \tag{8}$$

It may easily be proved that the following problem: (u, c) satisfies the differential equation (2) and the initial and boundary conditions (4) and (8) has not a unique solution. Thus we need additional information on the wave. We will restrict ourselves to formulate the inverse problem for a wave which has an up-going wavefront in that part of space where we wont to reconstruct the wave velocity. Define the arrival time of the wave at the point x by:

$$t_a(x) = \sup_t \{t \mid \forall \tau \in]-\infty, t[\;\; u(x, \tau) = 0\}. \tag{9}$$

Define: A wave has an up-going wavefront in the strip $]a_q, a_{q+1}[$ if there exists a positive k_q such that:

$$\partial_3 t_a(x) \geq k_q > 0 \quad \text{for} \quad x \in \dot{\Omega}_{qs}, \quad s = 0, \cdots, N_q. \tag{10}$$

The q'th layer inverse boundary value problem is defined by:

Let $(b_q, b_{\nu q})$ be given Cauchy data. Construct the wave velocities c_{qs} and the field u such that the wave equation (2) with c of the

form (6), the initial (4) and the boundary condition (8) and the up-going wavefront condition (10) hold. Construct the boundary data of the $q+1$'th layer problem: $(b_{q+1}, b_{\nu q+1})$.

The original inverse problem may be treated by the set of "q'th inverse problems" for $q = 0, \cdots, N$. We will see that the q'th inverse problem decompose in two independent problems. First the wave velocities of the q'th strip may be reconstructed from the boundary data at $x_3 = a_q$. With appropriate assumptions the wave velocity equations have unique solutions. The second step is the reconstruction of the field in the q'th strip from the Cauchy data at the plane $x_3 = a_q$ and the known wave velocity of this strip. Using this reconstruction the boundary data of the $q+1$'th strip may be found.

3 The Forward Problem.

The algebraic equations which the velocities c_{qs} of the forward problem satisfies will be found in the first part of this section. The point will be that the equations will be independent of the field inside the strip $x_3 \in]a_q, a_{q+1}[$, that is the equations will relate the velocities c_{qs} to the Cauchy data $(b_q, b_{q\nu})$. The second problem to be treated is to formulate the integral equations for the field in the strip $x_3 \in]a_q, a_{q+1}[$ which uniquely determine the field in this strip expressed by $(b_q, b_{q\nu})$.

3.1 Local Neumann to Dirichlet Mapping.

We will prove that the local Neumann to Dirichlet mapping (1) has the form (29). It is well known, see Weston (1992), that if c is a constant for $x_3 > a_q$ then the local K given by (29) coincide with the K operator (1) of the Neumann to Dirichlet mapping for all times. The application of the local Neumann to Dirichlet mapping is to construct the wave velocity and we will refer to (29) as "The wave velocity equation". The wave equation in any of the sets Ω_{qs} is:

$$[\nabla^2[\partial_t]^{-2} - c_{qs}^{-2}]u = 0. \tag{11}$$

The time interval for the wave velocity equation, associated with a point $(\underline{x}_0, a_q) \in \partial\Omega_{qs}^-$, will be defined by:

$$I_{qsm}(\underline{x}_0) = [t_a(\underline{x}_0, a_q), t_{qsm}(\underline{x}_0)[, \tag{12}$$

with

$$t_{qsm}(\underline{x}_0) = \inf_{y \in \partial\Omega^+_{qs}} \{t_a(y) + \frac{1}{c_{qs}} d((\underline{x}_0, a_q), y)\}. \tag{13}$$

Let $H^2(\Omega, T)$ denote the Sobolev space of functions with second order derivaties in $L^2(\Omega\times]-\infty, T])$. The basic " wave velocity equation" result is:

Lemma 1 *Let u, with $\partial_t^{-2} u \in H^2(\Omega, T)$, be a solution of the forward scattering problem with given wave velocity c_{qs} in the sets Ω_{qs}. Assume that there exist a point $(\underline{x}_0, a_q) \in \partial\dot{\Omega}^-_{qs}$ for which $t_{qsm}(\underline{x}_0) >\ t_a(\underline{x}_0, a_q)$. Then $\xi = c_{qs}$ is a solution of the wave velocity equation (29) in the time interval $I_{qsm}(\underline{x}_0)$.*

Proof:

"The regularized Greens function" is defined by:

$$\Phi_m(x - x_m, t, \tau, \xi) = \frac{1}{4\pi\|x - x_m\|}\delta_m(\tau - t + \|x - x_m\|\xi^{-1}), \tag{14}$$

where δ_m is a sequence of $\mathcal{C}^\infty$ functions converging to the Dirac delta, ξ is any constant in the interval $]0, c_{qs}]$ and $x_{3m} < a_q$. We assume that the support of δ_m is compact and $\delta_m(t) = 0$ if $\mid t \mid \geq 1/m$. The wave equation for Φ_m is:

$$[\nabla^2 - \xi^{-2}\partial_t^2]\Phi_m(x - x_m, t, \tau, \xi) = \ 0 \quad \text{if} \quad x_3 > a_q. \tag{15}$$

From (11) and (15) we have the identity:

$$0 = \int_{-\infty}^{\infty} d\tau \int_{\Omega_{qs}} dx \{\Phi_m(x - x_m, t, \tau, \xi)[\nabla^2[\partial_\tau]^{-2} - c_{qs}^{-2}]u(x, \tau) - \tag{16}$$

$$\{[\partial_\tau]^{-2}u(x, \tau)\}[\nabla^2 - \xi^{-2}\partial_\tau^2]\Phi_m(x - x_m, t, \tau, \xi)\}. \tag{17}$$

The function Φ_m has for any fixed t compact support in τ. Partial integration of the term containing $[\partial_\tau]^{-2}u\partial_\tau^2\Phi_m$, and application of the Greens formula in the remaining terms leads to:

$$[c_{qs}^{-2} - \xi^{-2}] \int_{-\infty}^{\infty} d\tau \int_{\Omega_{qs}} dx \{\Phi_m(x - x_m, t, \tau, \xi) u(x, \tau)\} = \eta_{qsm}(t, \xi, x_m), \quad (18)$$

where the right hand side is an integral over the boundary $\partial\Omega_{qs}$:

$$\eta_{qsm}(t, \xi, x_m) = \int_{-\infty}^{\infty} d\tau \int_{\partial\Omega_{qs}} dA(x) \{\Phi_m(x - x_m, t, \tau, \xi) \partial_\nu [\partial_\tau]^{-2} u(x, \tau) \quad (19)$$

$$-[\partial_\tau]^{-2} u(x, \tau) \partial_\nu \Phi_m(x - x_m, t, \tau, \xi)\}. \quad (20)$$

Let $(\underline{x}_0, a_q) \in \partial\dot{\Omega}_{qs}^-$ and $x_m = (\underline{x}_0, a_q - m^{-1})$. Then the limit of (18) as $m \to \infty$ is easily found:

$$\frac{1}{4\pi}[c_{qs}^{-2} - \xi^{-2}] L_{qs} u = \eta_{qs}(t, \xi, \underline{x}_0) + \eta_{qs, x_3 > a_q}(t, \xi, \underline{x}_0), \quad (21)$$

with

$$L_{qs} u = \int_{\Omega_{qs}} \|x - (\underline{x}_0, a_q)\|^{-1} u(x, t^{ret})\} dx \quad t^{ret} = t - \frac{\|x - (\underline{x}_0, a_q)\|}{\xi}. \quad (22)$$

The first term in the right hand side of (21) η_{qs} is the contribution from the boundary $\partial\Omega_{qs}^-$:

$$\eta_{qs}(t, \xi, \underline{x}_0) = -\frac{1}{2}[\partial_t]^{-2} u((\underline{x}_0, a_q), t) - K_{qs}[\partial_{y_3}(\partial_t)^{-2} u](t, \xi, \underline{x}_0), \quad (23)$$

$$K_{qs}\psi(t, \xi, \underline{x}_0) = \frac{1}{4\pi} \int_{\partial\Omega_{qs}^-} \{\|\underline{x} - \underline{x}_0\|^{-1} \psi(\underline{x}, a_q, t^{ret})\} d\underline{x}. \quad (24)$$

And the second term is the integral over $\partial\Omega_{qs}^+$:

$$4\pi\eta_{qs, x_3 > a_q}(t, \xi, \underline{x}_0) = [\partial_t]^{-2} \{ \int_{\partial\Omega_{qs}^+} \{\|x - (\underline{x}_0, a_q)\|^{-1} \partial_{\nu(y)} u(y, t^{ret}(x)) \quad (25)$$

$$-\partial_{\nu(y)}[\|y-(\underline{x}_0,a_q)\|^{-1}u(x,t-\frac{\|y-(\underline{x}_0,a_q)\|}{\xi})]\}_{y=x}dA(x)\} \qquad (26)$$

If $t_{qsm}(\underline{x}_0)$ defined by (13) is larger than $t_a(\underline{x}_0,a_q)$ and if $\xi\in]0,c_{qs}]$ then the boundary integral (26) vanishes in the time interval I_{qsm}, and (21) reduce to:

$$\frac{1}{4\pi}[c_{qs}^{-2}-\xi^{-2}]L_{qs}u=\eta_{qs}(t,\xi,\underline{x}_0), \quad t\in I_{qsm}(\underline{x}_0) \quad \xi\leq c_{qs}. \qquad (27)$$

The left hand side of this equation vanishes for $\xi=c_{qs}$:

$$\eta_{x_3=a_q}(t,c_{qs},\underline{x}_0)=0, \quad t\in I_{qsm}(\underline{x}_0). \qquad (28)$$

That is $\xi=c_{qs}$ is a solution of the local Neumann to Dirichlet mapping:

$$\partial_t^{-2}u(\underline{x}_0,a_q,t)=-2K_{qs}[\partial_{y_3}\partial_t^{-2}u](t,\xi,\underline{x}_0), \quad t\in I_{qsm}(\underline{x}_0), \quad \xi\in]0,c_{qs}]. \qquad (29)$$

This conclude the proof of the lemma.

We wont to show that if the conditions of the q'th inverse boundary value problem and a few additionel assumptions hold then there exist is a proper time interval where the wave velocity equation has a solution. If (10) hold then we have for any $x_3\in[a_q,a_{q+1}]$:

$$t_a(\underline{x},x_3)-t_a(\underline{x},a_q)\geq(x_3-a_q)k_q. \qquad (30)$$

If $t_a\in C^1(\partial\Omega_{qs}^-)$ the mean value theorem apply to show that:

$$|\ t_a(\underline{x},a_q)-t_a(\underline{x}_0,a_q)\ |\leq\|\underline{\nabla}t_a\|_{\infty qs}\|\underline{x}-\underline{x}_0\|. \qquad (31)$$

With the assumptions (10) and (31) there exist a proper time interval where $\xi=c_{qs}$ is a solution of the wave velocity equation:

Lemma 2 *Let u, with $\partial_t^{-2}u\in H^2(\Omega\times]0,T[)\cap C^0(\overline{\Omega}\times[0,T])$ be a solution of the wave equation with an up-going wave front and constant wave velocity c_{qs} in Ω_{qs}. Assume that (31) hold. Let ξ_{qs}^m be any constant in the interval $]c_{qs},(\|\underline{\nabla}t_a\|_{\infty qs})^{-1}]$, let $(\underline{x}_0,a_q)\in\partial\dot{\Omega}_{qs}^-$. Then there exists a time interval I defined by (36) such that $\xi=c_{qs}$ is a solution of the wave velocity equation (29) and $\eta_{qs,x_3>a_q}=0$ for any $(\xi,t)\in]0,\xi_{qs}^m[\times I$.*

Proof: The lemma is proved if we can show that the boundary term $\eta_{qsx_3>a_q}$ vanishes in the set $(\xi,t)\in]0,\xi^m_{qs}[\times I$. For any y in a set $\{y\in\partial\Omega^+_{qs}\mid y_3<a_{q+1}\}$ we find from (30) and the definition of t_a that:

$$t^{ret}<t_a(\underline{y},a_q)\le t_a(y)\quad\Rightarrow\quad u(y,t^{ret})=0,\quad t^{ret}=t-\xi^{-1}\|y-(\underline{x}_0,a_q)\|.(32)$$

Using (31) in this inequality we find for $\{y\in\partial\Omega^+_{qs}\mid y_3<a_{q+1}\}$ and $\xi\in]0,\xi^m_{qs}]$:

$$t<t_a(\underline{x}_0,a_q)+\delta_1\quad\Rightarrow\quad u(y,t^{ret})=0. \tag{33}$$

$$\delta_1=\inf_{(\underline{y},a_q)\in\partial\overline{\Omega}^-_{qs}\cap\partial\overline{\Omega}^+_{qs}}\|\underline{y}-\underline{x}_0\|[[\xi^m_{qs}]^{-1}-\|\underline{\nabla}t_a\|_{\infty qs}]>0. \tag{34}$$

For any point y in the set $\{y\in\partial\Omega^+_{qs}\mid y_3=a_{q+1}\}$ we use (30) and find with $t^{ret}=t-\xi^{-1}\|y-(\underline{x}_0,a_q)\|$ and $\delta_2=k_q(a_{q+1}-a_q)$:

$$t^{ret}<t_a(\underline{y},a_q)+\delta_2\le t_a(\underline{y},a_{q+1})\quad\Rightarrow\quad u(\underline{y},a_{q+1},t^{ret})=0. \tag{35}$$

Finally (31) and (35) show that $u(\underline{y},a_{q+1},t^{ret})=0$ for any $\xi<\mid\underline{\nabla}t_a\mid^{-1}$ if t is less than $t_a(\underline{x}_0,a_q))+\delta_2$ for any ξ less than $[\|\underline{\nabla}t_a\|_{\infty qs}]^{-1}$. This proves that with $\delta:=\min(\delta_1,\delta_2)$ we have:

$$\forall\xi\in]0,\xi^m_{qs}[\ \ \forall t\in I:=]t_a(\underline{x}_0,a_q),t_a(\underline{x}_0,a_q)+\delta[\ \Rightarrow\eta_{qsx_3>a_q}=0. \tag{36}$$

Then (27) hold for any $(\xi,t)\in]0,\xi^m_{qs}[\times I$ and the lemma is proved.

3.2 Integral Equations for the Cauchy Problem.

The Cauchy problem of the inverse problem will be to construct the solution of the wave equation if we know Dirichlet and Neumann boundary data at a plane $x_3=a_q$. We will formulate the integral equations for a solution of the wave equation in the strip $x_3\in]a_q,a_{q+1}[$ with given wave velocity. Introduce the Greens function Φ as the unique solution of the differential equation (37) for $x,y\in R^3$ which in any finite time interval has compact support in space and satisfy the initial condition (38):

$$[\nabla_x^2 - [c_q(\underline{x})]^{-2}[\partial_t]^2]\Phi(x,y,t) = -\delta(x-y)\delta(t), \tag{37}$$

where $c_q(\underline{x})$ is the wave velocity in the strip $x_3 \in]a_q, a_{q+1}[$. And the Greens function vanishes for negative values of t:

$$\Phi(x,y,t) = 0, \quad x,y \in R^3 \;\; t \in]-\infty, 0[. \tag{38}$$

The following function is a solution of the wave equation (2) in the strip $x_3 \in]a_q, a_{q+1}[$:

$$\partial_t^{-2}u(x,t) = \partial_t^{-2}u_{q+}(x,t) + \partial_t^{-2}u_{q+1,-}(x,t), \tag{39}$$

with $u_{q,\pm}$ and the Single layer operators S_q given by:

$$\partial_t^{-2}u_{q,+}(x,t) = S_q f_{q,+} \;\text{ and }\; [\partial_t]^{-2}u_{q+1,-}(x,t) = S_{q+1}f_{q+1,-}, \tag{40}$$

$$S_q f(x,t) = \int_{\infty}^{\infty}\int_{R^2}[\partial_t]^{-2}\Phi(x,\underline{y},a_q,t-\tau)f(\underline{y},\tau)d\underline{y}d\tau, \tag{41}$$

with

$$[\partial_t]^{-2}\Phi = -\mathcal{F}_\omega^{-1}\{(\omega + i0_+)^{-2}\mathcal{F}_t\Phi\}. \tag{42}$$

Let u be a solution of the wave equation (11) with trace fields:

$$\partial_3\partial_t^{-2}u(\underline{x},a_q,t) = b_{q\nu}(\underline{x},t) \quad \partial_3\partial_t^{-2}u(\underline{x},a_{q+1},t) = b_{q+1\nu}(\underline{x},t). \tag{43}$$

The potentials f_{q+} and f_{q+1-} are related to the boundary value problem of the wave equation (11) with boundary values (43). From the jump relation of the derivative of the Greens function for $x_3 \to a_q$ follows that f_q is a solution of the Fredholm integral equations:

$$-\frac{1}{2}f_{q+} + \partial_3\{S_q f_{q+} + S_{q+1}f_{q+1-}\}\,|_{x_3=a_q} = b_{q\nu}, \tag{44}$$

and from the boundary $x_3 = a_{q+1}$ limit from below:

$$\frac{1}{2} f_{q+1-} + \partial_3 \{ S_q f_{q+} + S_{q+1} f_{q+1-} \} \mid_{x3=a_{q+1}} = b_{q+1\nu}. \tag{45}$$

Assume that the trace fields $b_{q+1\nu}$ and $b_{q\nu}$ are in $L^2(\partial\Omega_q, T)$. Then the operator problem (44), (45) is a second kind Fredholm integral equation with $\partial_3 S_q$ and $\partial_3 S_{q+1}$ compact operators in L^2. The coupled Fredholm integral equations (44) and (45) have at most one solution and by the Fredholm alternative this solution is unique. This proves that a solution of the wave equation in the strip $]a_q, a_{q+1}[$ may be represented as a sum of single layer potentials of the form (39), (40) and (41). The potentials (f_q, f_{q+1}) are unique. Obviously (f_{q+}, f_{q+1-}) is a solution of the coupled integral equations (44) and

$$S_q f_{q+}(\underline{x}, a_q, t) + S_{q+1} f_{q+1-}(\underline{x}, a_q, t) = b_q(\underline{x}, a_q, t). \tag{46}$$

We will now show that the field u in the strip $x_3 \in]a_q, a_{q+1}[$ may be constructed as a sum of single layer potentials of the form (39), (40), (41) with (f_{q+}, f_{q+1-}) the unique solution of the coupled integral equations (44), (46).

Lemma 3 *Let u be a solution of the wave equation in the strip $]a_q, a_{q+1}[$ with given wave velocity c of the form (6). Assume that the trace field of u at the plane $x_3 = a_q$ $(\partial_t^{-2} u_{x3=a_q}, \partial_3 \partial_t^{-2} u_{x3=a_q}) = (b_q, b_{q\nu})$ are square integrable in $L^2(R^2 \times [0,T])$. Then the coupled integral equations (44), (46) has a unique solution (f_{q+}, f_{q+1-}). And the field u in the strip $x_3 \in]a_q, a_{q+1}[$ is constructed as a sum of the single layer potentials (39), (40), (41). The Cauchy data of the $q+1$'th inverse problem $(b_{q+1}, b_{q+1\nu})$ is constructed by (45) and (48).*

Proof: All integral equations in the following are understood as integral equations in L^2. We have seen that (44), (46) has a solution. We are going to prove uniqueness. Let (f^h_{q+}, f^h_{q+1-}) be a solution of the coupled homogeneous integral equations corresponding to (44), (46). Let the field u^h be defined for all x by $\partial_t^{-2} u^h = S_q f^h_{q+} + S_{q+1} f^h_{q+1-}$. Then $\partial_t^{-2} u^h = \partial_3 \partial_t^{-2} u^h = 0$ for $x_3 = a_q$, and u^h is a solution of the wave equation with wave velocity $c(\underline{x})$ in the strip $x_3 \in]a_q, a_{q+1}[$. This Cauchy problem has the unique solution $u_h = 0$. Below the plane $x_3 = a_q$ the field u^h is a solution of the same wave equation which for any fixed time vanish at $x_3 = a_q$ and at $\|x\| = b$ if b is sufficient large. Then u^h vanishes below the plane $x_3 = a_q$. By the same kind of argument u^h

is zero for all (x,t). The jump relations at $x_3 = a_q$ apply for the calculation of f^h_{q+}:

$$f^h_{q+}(\underline{x},t) = \partial_3\partial_t^{-2}u^h(\underline{x},a_q-0_+,t) - \partial_3\partial_t^{-2}u^h(\underline{x},a_q+0_+,t) = 0. \qquad (47)$$

In a similar way we verify that $f^h_{q+1-}(\underline{x},t) = 0$. This proves that (44) and (46) has a unique solution $(f_{q+}, f_{q+1-}) \in (L^2(R^2\times[0,T]))^2$. Let u in the strip $x_3 \in]a_q, a_{q+1}[$ be defined by (39), (40), (41). Then this u is the unique solution of the wave equation in the strip $]a_q, a_{q+1}[$ with the given wave velocity c and with the given Cauchy data $(b_q, b_{q\nu})$ at the plane $x_3 = a_q$. The Cauchy data at the plane $x_3 = a_{q+1}$ is constructed from (45) and (48) which proves the lemma.:

$$b_{q+1}(\underline{x},t) = S_q f_{q+}(\underline{x},a_{q+1},t) + S_{q+1}f_{q+1-}(\underline{x},a_{q+1},t). \qquad (48)$$

3.3 Stability of Cauchy Problem.

The Cauchy problem of the lemma is ill posed if the metric is a Sobolev norm. The illposedness of the problem may, see the comments on the instability of Cheyney and Isaacson (1995), depend on the mathematical representation used for the Cauchy problem. We wont to introduce Hilberspaces with a modified Sobolev norm such that the mapping $(b_q, b_{q,\nu}) \to (b_{q+1}, b_{q+1,\nu})$ with appropriately chosen assumptions is continuous in the metric of the Hilberspaces. First a bilinear functional B_q defined on the space of trace functions at the plane $x_3 = a_q$ is introduced. With appropriate assumptions this functional is an inner product in a Hilberspace.

Let u and v be solutions of the wave equation in the strip $]a_q, a_{q+1}[$. Denote time Fourier Transform of the fields by $U = \mathcal{F}_t u$ and $V = \mathcal{F}_t v$. Denote the wave velocity defined in the strip $x_3 \in]a_q, a_{q+1}[$ by c_q. The fields U and V are solutions of the Helmholtz equation:

$$\nabla^2 U + \frac{\omega^2}{c_q^2}U = 0. \quad a_q < x_3 < a_{q+1} \qquad (49)$$

Let σ be a (small) positive constant. Define the bilinear form of the trace fields at $x_3 = a_q$:

$$< U,V >_{q,w} = \int_{-\infty}^{\infty} d\omega \int_{|\underline{x}|\leq w} U(\underline{x}, a_q, \omega)\overline{V(\underline{x}, a_q, \omega)} e^{-\sigma\omega^2} d\underline{x}. \qquad (50)$$

Let $L^2_{q,w}$ denote the space of trace functions at the plane $x_3 = a_q$ for which the bilinear form $< U,V >_{q,w}$ is defined.. Define the space of locally square integrable functions:

$$X_{q+} = \{(U, \nabla U) \mid \forall w \in R_+ : \ (\omega c_q^{-1} U, \partial_s U)(x_3 = a_q) \in L^2_{qw}\}. \qquad (51)$$

Define the bilinear form $B_{q,w}$ on the space X_{q+}:

$$B_{q,w}\{(U, \nabla U), (V, \nabla V)\} =< \partial_3 U, \partial_3 V >_{q,w} + \qquad (52)$$

$$+ < \omega c_q^{-1} U, \omega c_q^{-1} V >_{q,w} - < \underline{\nabla} U, \underline{\nabla} V >_{q,w} . \qquad (53)$$

We will formulate conditions for the fields which are sufficient for the existence of the limit of $B_{q,w}$ for $w \to \infty$. Assume that the incident field $U^{in} = \mathcal{F}_t \mathcal{F}_{\underline{x}} u^{in}$ satisfy the condition

$$\omega U^{in} \in L^\infty(R^3) \cap C^\infty(R^3) \quad \wedge \quad U^{in}(\underline{p}, \omega) = 0 \ \text{ for } \ c_- \mid \underline{p} \mid > \mid \omega \mid . \qquad (54)$$

Then the following far field expansion hold for $x_3 \geq 0$:

$$U(x, \omega) = \frac{1}{\|x\|} \exp\{i \frac{\mid \omega \mid}{c_-} \|x\|\} U_\infty(\frac{x}{\|x\|}, \omega) + O(\|x\|^{-2}), \qquad (55)$$

where $O(\|x\|^{-2})$ denote a function of x and ω for which $\|x\|^2$ times the function is bounded for all x and ω. Then the following assymptotic relations hold for fields which satisfy (55):

$$[\omega^2 c_-^{-2} U \overline{V} - \underline{\nabla} U \overline{\underline{\nabla} V}](x_3 = a_q) = O(\|\underline{x}\|^{-3}), \qquad (56)$$

$$\partial_3 U(x_3) = O(\|\underline{x}\|^{-2}). \tag{57}$$

Obviously the inner products $< \omega c_q^{-1} U, \omega c_q^{-1} V >_{q,w}$ and $< \underline{\nabla} U, \underline{\nabla} V >_{q,w}$ diverges for $w \to \infty$; but $B_{q,w}\{(U, \nabla U), (V, \nabla V)\}$ converges for $w \to \infty$ for fields which are locally square integrable and satisfies the assymptotic relations (55), (56) and (57). In that case we define the bilinear form:

$$B_q\{(U, \nabla U), (V, \nabla V)\} = \lim_{w \to \infty} B_{q,w}\{(U, \nabla U), (V, \nabla V)\}. \tag{58}$$

This proves that the bilinear form (58) is defined for all fields U which are bounded and locally square integrable and satisfies the asymptotic relation (55). It is easy to prove that if there exist a vector U for which $B_q\{(U, \nabla U), (U, \nabla U)\}$ is positive then there exist a maximal subspace of X_q which contain U and is a Hilberspace with inner product defined by the bilinear form B_q. Denote this Hilbertspace by $H_{q,+}$. Then the continuity of the mapping of $(U, \nabla U) \in H_{q,+}$ onto the boundary data on the plane $x_3 = a_{q+1}$ result is:

Theorem 1 *Let c_q be independent of x_3 for $x_3 \in]a_q, a_{q+1}[$. Assume that u is a solution of the wave equation in the strip $a_q < x_3 < a_{q+1}$ and let the Fourier Transform trace data $(U, \nabla U)(x_3 = a_q) \neq 0$ live in a Hilberspace $H_{q,+}$ with inner product B_q. Assume that the trace data $(\omega c_q^{-1} U, \partial_s U)(x_3 = a_{q+1})$ are vectors in the spaces L^2_{q+1w} for all $w > 0$. Then the trace data of U at $x_3 = a_{q+1}$ is a vector in a Hilbertspace $H_{q+1,-}$ with inner product defined by (62). And the mapping $(U, \nabla U) \in H_{q,+} \to (U, \nabla U)(x_3 = a_{q+1})) \in H_{q+1,-}$ is isometric.*

Proof: Let V be a solution of (49) for which $(V, \nabla V) \in H_{q,+}$. Helmholtz equation for U and $\overline{V}$ leads to the identity:

$$\int_{-\infty}^{\infty} d\omega e^{-\sigma\omega^2} \int_{a_q}^{a_{q+1}} dx_3 \int_{\|\underline{x}\|<w} \{\nabla \cdot \{\partial_3 U \nabla \overline{V} + \overline{\partial_3 V} \nabla U\}\} d\underline{x} \tag{59}$$

$$= \int_{-\infty}^{\infty} d\omega e^{-\sigma\omega^2} \int_{a_q}^{a_{q+1}} dx_3 \int_{\|\underline{x}\|<w} \partial_3 \{\nabla U \cdot \nabla \overline{V} - \frac{\omega^2}{c_q^2} U \overline{V}\} d\underline{x}. \tag{60}$$

We apply the divergence theorem on the left hand side and calculate the x_3 integral on the right hand side of this equation, and let $w \to \infty$. The equation found contain one integral over the plane $x_3 = a_q$, this integral reduce to $B_q\{(U, \nabla U), (V, \nabla V)\}$. The integral of remaining terms in (60) reduce to:

$$B_{q+1-}\{(U, \nabla U), (V, \nabla V)\} = \lim_{w\to\infty}\{< \partial_3 U, \partial_3 V >_{q+1,w} + \tag{61}$$

$$+ < \omega c_q^{-1} U, \omega c_q^{-1} V >_{q+1,w} - < \underline{\nabla} U, \underline{\nabla} V >_{q+1,w}\}. \tag{62}$$

and we have proved:

$$B_q\{(U, \nabla U), (V, \nabla V)\} = B_{q+1-}\{(U, \nabla U), (V, \nabla V)\}. \tag{63}$$

By the assumptions of the theorem $B_{q+1-}\{(U, \nabla U), (U, \nabla U)\}$ is positive. The Hilbertspace $H_{q+1,-}$ is defined as the maximal closed subspace of trace fields defined on the plane $x_3 = a_{q+1}$ which contain $(U, \nabla U)(x_3 = a_{q+1})$ with inner product defined by (62). Then the two Hilbertspaces are isomorphic and:

$$\|(U, \nabla U)\|^2_{H,q} = B_{q,+}\{(U, \nabla U), (U, \nabla U)\} = \|(U, \nabla U)\|^2_{H,q+1,-}. \tag{64}$$

Which proves the theorem. The important question is in which cases the bilinear form B_q is positive for given Cauchy data at the boundary $x_3 = a_q$. The first example is: Let the medium below $x_3 = 0$ be homogeneous with wave velocity c_-. The scatterer is exited by an incident field which satify (54). We may then formulate the stability result:

Theorem 2 *Let the incident field satisfy (54). Assume that $c_- \geq c(x) = c_0(\underline{x})$ for all $x \in R^2 \times]a_0, a_1[$. Assume that all the trace data at the two planes $x_3 = 0$ and $x_3 = a_1$ live in the spaces $L^2_{0,w}$ and $L^2_{1,w}$. Assume that $U, \partial_s U \in C^0(R^3) \cap H^1(\Omega)$. Then there exist Hilbertspaces H_0 and H_1 with inner products B_{0+} and B_{1-} for which the boundary data of u at these planes are vectors in the Hilbertspaces H_{0+} and H_{1-}. And the mapping $(U, \nabla U) \in H_0 \to (U, \nabla U)(x_3 = a_1)) \in H_{1-}$ is isometric.*

Proof: Below the plane $x_3 = 0$ the Fourier transform of the field has the form:

$$\mathcal{F}_t \mathcal{F}_{\underline{x}} u(\underline{p}) = \phi^i(\underline{p}, \omega) e^{ip_3 x_3} + \phi^r(\underline{p}, \omega) e^{-ip_3 x_3}, \quad p_3 = \sqrt{\frac{\omega^2}{c_-^2} - \mid \underline{p} \mid^2}. \tag{65}$$

Define the bilinear functional $B_{0,-,w}$ by (53) substituting a_q by $a_0 = 0$ and and c_q by the velocity outside the scatterer c_-. In the same way as previously we prove that the limit of $B_{0,-,w}$ as w tend to infinity exist. Using (54) and (65) the limit is found:

$$B_{0,-}\{(U, \nabla U), (U, \nabla U)\} = 2 \int_{|\underline{p}| < \frac{|\omega|}{c_-}} \mid p_3 \mid^2 \{\mid \phi^i \mid^2 + \mid \phi^r \mid^2\} e^{-\sigma\omega^2} d\underline{p} d\omega. \tag{66}$$

We conclude that $B_{0,-}\{(U, \nabla U), (U, \nabla U)\}$ is positive. Finally we use the assumption $c_- \geq c_0(\underline{x})$ in the expression for the bilinear form $B_{0,-}$ and find:

$$0 < B_{0,-}\{(U, \nabla U), (U, \nabla U)\} \leq B_{0+}\{(U, \nabla U), (U, \nabla U)\}. \tag{67}$$

Then H_{0+} is the maximal Hilbertspace with inner product B_{0+} which contain $(U, \underline{\nabla} U)$. Theorem 1 show that there exist a Hilbertspace H_{1-} with inner product $B_{1,-}$ and the mapping $(U, \nabla U) \in H_{0+} \to (U, \nabla U)(x_3 = a_1)) \in H_{1-}$ is isometric. Which proves the theorem. An application of this theorem is that the set of Cauchy problems for media which satisfies the condition:

$$x_3 < y_3 \Rightarrow c_- \geq c(\underline{x}, x_3) \geq c(\underline{x}, y_3), \quad \underline{x} \in R^2, \tag{68}$$

are stable in the sense that the mapping from the boundary data of the plane a_q onto the boundary data of the plane a_{q+1} is continuous:

Corrolar 1 *Let the incident field satisfy (54). Assume that $c_- \geq c(x) = c(\underline{x})$ and that (68) hold. Assume that the trace data $(U, \nabla U)$ at any plane $x_3 = a_q$ live in the space $L^2_{q,w}$. Assume that $U, \partial_s U \in C^0(R^3) \cap H^1(\Omega)$. Then there exist Hilbertspaces $H_{q\pm}$ with inner products $B_{q,+}$ and $B_{q-1,-}$, such that the boundary data of U at any plane $x_3 = a_q$ are vectors in the Hilbertspaces $H_{q\pm}$. And all mappings $(U, \nabla U) \in H_{q,+} \to (U, \nabla U)(x_3 = a_{q+1}) \in H_{q+1,-}$ are isometric.*

Proof: We use induction. Theorem 2 show that the statement of the corrolar hold for $q = 0$, and $B_{1,-}$ is positive. Then the corrolar is proved if we can show that if $B_{q,-}$ is positive then $B_{q+1,-}$ is positive and the mapping $(U, \nabla U) \in H_{q,+} \rightarrow (U, \nabla U)(x_3 = a_{q+1}) \in H_{q+1,-}$ is isometric. A simple estimate show that if $c_q(\underline{x}) \leq c_{q-1}(\underline{x})$ for all $\underline{x}$, then the following inequality hold:

$$B_{q,-}\{(U, \nabla U), (U, \nabla U)\} \leq B_{q,+}\{(U, \nabla U), (U, \nabla U)\}, \tag{69}$$

Then $B_{q,+}\{(U, \nabla U), (U, \nabla U)\}$ is positive and $(U, \nabla U) \in H_{q,+}$. From theorem 1 follows that $(U, \nabla U)(x_3 = a_{q+1}) \in H_{q+1,-}$ and $B_{q+1,+}\{(U, \nabla U), (U, \nabla U)\}$ is positive. This proves the corrolar.

It is emphasized that the Cauchy problem is basically ill posed. Small errors in $\mathcal{F}u(\underline{x}, a_q)(\underline{p})$ for large values of $\mid \underline{p} \mid$ will generate large errors for the field at the next plane a_{q+1}. Thus for numerical solutions regularization of the problem is needed. Such a proceedure should involve a cut off of the $\mathcal{F}u(\underline{x}, a_q)(\underline{p})$ for large values of p. In cases where the waves moves up and down and only a sufficient small amount of energy moves horisontal the regularized term $< \underline{\nabla} U, \underline{\nabla} U >$ in the bilinear form will be small compared to the two other terms of B and the bilinear functional could be expected to be positive. In that case the regularized Cauchy problem will be stable. But if a larger part of the energy moves horisontal B could be negative, and theorem 1 does not apply for the discussion of stability of the Cauchy problem.

4 The Inverse Problem.

In this section the existence and uniqueness problems of the inverse problem will be discussed. The proof of uniqueness will be based on the assumption that there exists points in any of the sets Ω_{qs} where the wavefront has a jump. In addition we need the assumptions (10) and (31). Thus for any $x \in \Omega_{qs}$:

$$t_a(x) - t_a(\underline{x}_0, a_q) \geq t_a(\underline{x}, a_q) - t_a(\underline{x}_0, a_q) \geq -\|\underline{\nabla} t_a\|_{\infty qs} \|\underline{x} - \underline{x}_0\|. \tag{70}$$

In a similar way $t_a \in C^1(\dot{\Omega}_{qs}) \cap C^0(\overline{\Omega_{qs}})$ is a sufficient condition to show by the mean value theorem that:

$$t_a(x) - t_a(\underline{x}_0, a_q) \leq \|\nabla t_a\|_\infty \|x - (\underline{x}_0, a_q)\|. \tag{71}$$

The basic uniqueness - reconstruction result for a field with an up-going wavefront for which the field has a jump at the wavefront is:

Theorem 3 *Let* $b_q, b_{q\nu} \in \mathcal{C}^0(\partial\dot{\Omega}_{qs}^-)$ *be boundary data of a forward scattering problem with constant unknown wave velocities* c_{qs} *in the sets* Ω_{qs} *and an up-going wavefront in* Ω_{qs}*. Assume that* t_a *satisfies (10), (70) and (71). Assume that there exists a point* $(x_{0s}, a_q) \in \partial\dot{\Omega}_{qs}^-$ *for which the function* $u(x, t_a(x) + t)$ *has a nonzero limit for* $(x,t) \in \Omega_{qs} \times]0, \infty[$ *converging to* $(\underline{x}_0, a_q, 0_+)$*. Then there exist a time interval* I *such that the unknown wave velocity* c_{qs} *of the forward problem is the unique solution of the wave velocity equation (29) for any* $t \in I$ *and* $\xi \in]0, (\|\underline{\nabla} t_a\|_{\infty qs})^{-1}[$ *.*

Proof: With the assumptions of the theorem we proved in lemma 2 that the wave velocity equation has the solution $\xi = c_{qs}$ for any $t \in I = I_{qsm}(\underline{x}_0)$. Thus the theorem is proved if we can show that (29) has at most one root in the interval $\xi \in]0, (\|\underline{\nabla} t_a\|_{\infty qs})^{-1}[$. The boundary term $\eta_{qs,x_3>a_q}$ defined in (26) vanishes for $\xi \in I_{qsm}$ and equation (27) hold. If we can show that the time function $L_{qs}u$ defined by (22) is nonzero for $t - t_a(x_{0s}, a_q)$ sufficient small positive, then (27) and (29) or:

$$(4\pi)^{-1}[c_{qs}^{-2} - \xi^{-2}]L_{qs}u = \eta_{qs} = 0, \quad t \in I, \tag{72}$$

has the unique solution $\xi = c_{qs}$ in the interval $\xi \in I_\xi :=]0, (\|\underline{\nabla} t_a\|_{\infty qs})^{-1}[$, and the theorem is proved. Let ξ be any fixed number in this interval. For such a ξ we will investigate where the function $u(x, t - \|x - (\underline{x}_0, a_q)\|\xi^{-1})$ which appear in (22) is zero and where it is nonzero. The following three equations and (70) and (71) are used for that purpose. A simple application of (71) leads to the estimate:

$$\|x - (\underline{x}_0, a_q)\| < \rho_{min}(t) := \frac{t - t_a(\underline{x}_0, a_q)}{\xi^{-1} + \|\nabla t_a\|_\infty} \Rightarrow t^{ret} - t_a(x) > 0, \tag{73}$$

with $t^{ret} = t - \frac{\|x - (\underline{x}_0, a_q)\|}{\xi}$. And (70) apply for proving that:

$$\xi \in I_\xi \quad \wedge \quad x \in \Omega_{qs} : \quad t^{ret} - t_a(x) \leq t - t_a(\underline{x}_0, a_q). \tag{74}$$

Assume that the limit $u(\underline{x}_0, a_q, t_a(\underline{x}_0, a_q) + 0_+)$ is positive. Then there exist positive numbers (h, δ_1, γ) such that:

$$\forall (x,t) \in \Omega_{qs} \cap \mathcal{B}_h(\underline{x}_0, a_q) \times]0, \delta_1[) : u(x, t_a(x) + t) > \gamma > 0, \quad (75)$$

where $\mathcal{B}_h(\underline{x}_0, a_q)$ denote the ball with center at $(\underline{x}_0, a_q)$ and radius h. By the definition of t_a the function $u(x, t^{ret})$ vanishes if $t^{ret} < t_a(x)$. Using (70) in this expression we easily find that $u(x, t^{ret})$ vanishes if:

$$t^{ret} < t_a(\underline{x}_0, a_q) - \|x - (x_0, a_q)\| \|\underline{\nabla} t_a\|_{\infty qs} \leq t_a(x), \quad (76)$$

or equivalently that:

$$\|x - (\underline{x}_0, a_q)\| > \rho_{max}(t) := \frac{t - t_a(\underline{x}_0, a_q)}{\xi^{-1} - \|\underline{\nabla} t_a\|_{\infty qs}} \Rightarrow \quad u(x, t^{ret}) = 0. \quad (77)$$

That is the support of the function $u(x, t^{ret})$ is a subset of the ball $\mathcal{B}_{\rho_{max}}(\underline{x}_0, a_q)$. We will investigate Lu in the time interval:

$$I_2 =]t_a(\underline{x}_0, a_q), t_a(\underline{x}_0, a_q) + \delta_2[\quad \delta_2 := \min(\delta_1, [\xi^{-1} - \|\underline{\nabla} t_a\|_{\infty qs}]h). \quad (78)$$

For any fixed t in this time interval we find from (75) that $\mathcal{B}_{\rho_{max}(t)} \subseteq \mathcal{B}_h$. From the identity $u(x, t^{ret}) = u(x, t_a(x) + [t^{ret} - t_a(x)])$ and (73), (74) and (75) follows that for any $(x,t) \in \Omega_{qs} \times I_2$:

$$t^{ret} < t_a(x) \Rightarrow u(x, t^{ret}) = 0, \quad t^{ret} > t_a(x) \Rightarrow u(x, t^{ret}) > 0. \quad (79)$$

And from the same equations is seen that $u(x, t^{ret})$ is larger than γ for all $(t,x) \in I_2 \times \mathcal{B}_{\rho_{min}(t)}(\underline{x}_0, a_q)$. That is the support of $u(x, t^{ret})$ is a proper set and for any $(x,t) \in \Omega_{qs} \times I_2$:

$$\mathcal{B}_{\rho_{min}(t)}(\underline{x}_0, a_q) \subseteq \text{supp } u(x, t^{ret}) \subseteq \overline{\mathcal{B}}_{\rho_{max}(t)}(\underline{x}_0, a_q) \subseteq \overline{\mathcal{B}}_h(\underline{x}_0, a_q). \quad (80)$$

We conclude that if $(x,t) \in \Omega_{qs} \times I_2$ then $u(x, t^{ret})$ is positive in the open set $\dot{supp}\ u$. If the limit $u(\underline{x}_0, a_q, t_a(\underline{x}_0, a_q) + 0_+)$ is negative the function $-u$ will satisfy (75), and we may easily prove that is $u(x, t^{ret})$ is negative in the open set $\dot{supp}\ u$. This proves that:

$$\forall \xi \in]0, (\|\underline{\nabla} t_a\|_{\infty qs})^{-1}[\quad \forall t \in I_2 \quad : Lu(\xi, t) \neq 0. \tag{81}$$

That is the wave velocity equation (29) has the unique solution $\xi = c_{qs}$ in the time interval which proves the theorem.

In cases where we have given boundary conditions which is not known to be data of a forward problem we may still show existence and uniqueness of the q'th inverse problem in the following case:

Definition: The Cauchy data $(b_q, b_{q\nu})$ are " consistant boundary data of a wave with upgoing wavefront" if the conditions 1,2 and 3 hold.

1. For any $s = 0, \cdots, N_q$ there exists a point $x_{qs} = (x_{0s}, a_q) \in \partial\dot{\Omega}^-_{qs}$, a constant c_{qs}, and a time interval $I_{qs} =]t_a(x_{qs}), t_a(x_{qs}) + d_{qs}]$ with d_{qs} positive such that $\xi = c_{qs}$ is the unique solution in the interval $\xi \in]0, \|\underline{\nabla} t_a\|^{-1}_{\infty qs}[$ of the wave velocity equation for any time t in the interval I_{qs}.

2. The coupled integral equations (44),46) with the Greens function corresponding to the wave velocities c_{qs} has for any positive T a unique solution $(f_q, f_{q+1}) \in L^2(\Omega, [0, T])$.

3. Let u be given by (39),(40) and (41). Then for any $s = 0, \cdots, N_q$ there exists a k_s such that (10) hold.

We have proved that if $(b_q, b_{q\nu})$ is boundary data of a forward problem with an up-going wave front then 1. 2. and 3. hold. We will prove that the three conditions are sufficient for a unique solution of the q'th inverse problem:

Theorem 4 *Assume that $(b_q, b_{q\nu})$ are consistant boundary data of waves with up-going wavefront. Then the q'th inverse problem has a unique solution with the wave velocity c_{qs} in the interval $\xi \in]0, \|\underline{\nabla} t_a\|^{-1}_{\infty qs}[$. The wave velocities c_{qs} and the field u is uniquely found from the points 1.2 and 3 in the definition of " consistant boundary data of a wave with upgoing wavefront". The boundary data at the plane $x_3 = a_{q+1}$ is constructed by (45), (48).*

Proof. The existence: The field u constructed from 1,2 and 3 is a solution of the q'th inverse problem. Uniqueness: Let f_{qs} and v be wave velocities and a field solving the q'th inverse problem. Then $\xi = f_{qs}$ is a solution of the wave velocity equation associated with the point $(x_{0s}, a_q) \in \partial\dot{\Omega}^-_{qs}$ in a proper time interval. Condition 1 show that $c_{qs} = f_{qs}$ is the unique solution of the wave velocity equation. Then v and u equals the unique solution of the Cauchy problem with Cauchy data $(b_q, b_{q\nu})$. Proving that $v = u$ in the strip $]a_q, a_{q+1}[$. Which proves the theorem.

References.

Cheney M and Kristensson G 1988 Three -dimensional inverse scattering: layer stripping formulae and ill-posedness results Inverse Problems 4 625 -42.

Cheney M and Isaacson D 1995 Inverse problems for a dissipative half-space. Inverse Problems 865 - 888.

M Lassas, M Cheney and G.Uhlmann 1998 Uniqueness for a wave propagation inverse problem in a half-space Inverse Problems 14 697-684.

Sailing H and Weston V H 1992 Inverse problem for the dissipative wave equation in a stratisfied half-space and linearization of the imbedding equations Inverse problems 8 435 - 455.

Sailing H and Weston V H 1994 Determination of the permittivity and conductivity in R^3 using wave splitting of Maxwell's equations J.Math.Phys. 36 1776-1791.

Weston V H 1990 Invariant imbedding for the wave equation in three dimensions and applications to direct and inverse problems Inverse Problems 6 1075-1105.

Weston V H 1992 Invariant imbedding and wave splitting in R^3:II. The Green function approach to inverse scattering. Inverse Problems 8 919 - 947.