AALBORG UNIVERSITY

Nonlinear approximation in α-modulation spaces

by

Lasse Borup and Morten Nielsen

May 2003 R-2003-11

DEPARTMENT OF MATHEMATICAL SCIENCES
AALBORG UNIVERSITY
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 96 35 80 80 ▪ Telefax: +45 98 15 81 29
URL: www.math.auc.dk/research/reports/reports.htm

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

NONLINEAR APPROXIMATION IN α-MODULATION SPACES

LASSE BORUP AND MORTEN NIELSEN

ABSTRACT. The α-modulation spaces are a family of spaces that contain the Besov and modulation spaces as special cases. In this paper we prove that brushlet bases can be constructed to form unconditional and even greedy bases for the α-modulation spaces. We study m-term nonlinear approximation with brushlet bases, and give complete characterizations of the associated approximation spaces in terms of α-modulation spaces.

1. INTRODUCTION

Two classical families of smoothness spaces are the Besov spaces $B_q^s(L_p(\mathbb{R}))$ and the modulation spaces $M_q^s(L_p(\mathbb{R}))$. These two families of spaces are constructed from the same type of scheme arising from segmentations of the frequency axis. The modulation spaces are built by making a uniform partition of the frequency axis, while the Besov spaces arise from a dyadic partition.

For the Besov space family, wavelets form nice unconditional bases and the correspondence between $m-$term nonlinear approximation in $L_p(\mathbb{R})$ and Besov spaces has been studies in great detail. It was also proved by the authors in [3] that so-called brushlet bases form unconditional bases for the Besov spaces. In fact, we will see in the present paper that brushlet systems form unconditional bases for the more general α-modulation spaces, a property not shared by wavelets.

The brushlet systems, which will be defined in Section 2, are based on local Fourier bases as introduced by Coifman and Meyer in [4], and by Malvar in [17] for applications in signal processing. These systems were further developed by Wickerhauser in [23]. An atom from a local Fourier basis has perfect localization in time and is well localized in frequency. Laeng noticed in [16] that it is possible to map a local Fourier basis by the Fourier transform to a new basis with compact support in the frequency domain. In [18], Coifman and Meyer studied similar systems, called bruslets, using the bases introduced by Wickerhauser. In [3] brushlets were considered from an approximation theoretical point of view. It was proven that brushlet systems, just like wavelet systems, form so-called greedy bases for $L_p(\mathbb{R})$, which is an even stronger condition than being unconditional. Recently it has been proved in [10] that greedy bases are the right bases to work with when one is interested in characterizing m-term nonlinear approximation with elements from the basis in terms of generalized smoothness.

2000 *Mathematics Subject Classification.* Primary 41A17; Secondary 41A30, 41A25, 41A46.

Key words and phrases. Modulation spaces, α-modulation spaces, Besov spaces, brushlet bases, nonlinear approximation.

This work is in part supported by the Danish Technical Science Foundation, Grant no. 9701481.

Wavelets are not suited to analyze the modulation spaces which is a direct consequence of the fact that the wavelets basically form a dyadic Littlewood-Paley decomposition of the frequency axis, and as mentioned above, the modulation spaces are defined by considering a uniform segmentation of the frequency axis. However, it was proved in [12] that local Fourier bases form unconditional bases for the modulation spaces. Jackson and Bernstein inequalities for nonlinear approximation with local Fourier bases were also studied in [12].

It was pointed out by Feichtinger and Gröbner in the papers [8, 7] that Besov and modulation spaces are special cases of the decomposition type Banach spaces $D(Q,B,Y)$ introduced in [8]. One can also use the methods in [8] to define "intermediate" spaces in between the modulation and Besov spaces, and it is this type of spaces that will be the focus of the present paper. In his Ph.D. thesis [11], Gröbner introduced the so-called α-modulation spaces, a family of intermediate spaces between the classical modulation spaces $M^s_q(L_p(\mathbb{R}))$ and the Besov spaces $B^s_q(L_p(\mathbb{R}))$. The (univariate) Besov spaces are based on coverings of the frequency axis consisting of intervals $[a,b]$ satisfying $|a| \asymp |b-a|$, that is to say there exist constants $c,C \in (0,\infty)$ such that $c|a| \le |b-a| \le C|a|$ for all the intervals. Likewise, modulation spaces are given by uniform coverings, i.e., coverings satisfying $|a| \asymp |b-a|^0$. Gröbner therefore suggested to define "intermediate" spaces corresponding to coverings based on the rule $|a| \asymp |b-a|^\alpha$, $0 \le \alpha \le 1$. The precise definition of an α-modulation space will be given in Section 3.

In the present paper we will show that brushlet systems are the "right" type of bases to analyze the α-modulation spaces, in the sense that it is possible to construct brushlet bases that form greedy bases for the α-modulation spaces. The α-modulation spaces can be characterized entirely by the brushlet coefficients, and this is done in Section 2. Best m-term approximation with brushlets is studied and a complete characterization of the approximation spaces are derived in Section 4. The approximation error is measured in the α-modulation norm in Section 4. However, for nonlinear approximation with wavelet systems one usually measures the error in the L_p-norm or more generally in a Triebel-Lizorkin norm. In Section 5 we define α-Triebel-Lizorkin spaces, and derive several results on nonlinear approximation with brushlet systems with the approximation error measure in the α-Triebel-Lizorkin norm.

Finally, we should mention that Fornasier has studied Gabor frames in α-modulation spaces in his Ph.D. thesis [9].

2. Brushlet systems

In this section we define the brushlet orthonormal bases for $L_2(\mathbb{R})$ that will be our main tool to analyze the α-modulation spaces introduced in Section 3. We use the shorthand notation L_p for the univariate Lebesgue space $L_p(\mathbb{R})$, $1 \le p \le \infty$.

Each brushlet basis is associated with a partition of the frequency axis. The partition can be chosen with almost no restrictions, but in order to have good properties of the associated basis we need to impose some growth conditions on the partition. We introduce the following definition.

Definition 2.1. *A family $\mathbb{I}$ of intervals is called a disjoint covering of $\mathbb{R}$ if it consists of a countable set of pairwise disjoint half-open intervals $I = [\alpha_I, \alpha_I')$, $\alpha_I < \alpha_I'$, such that $\cup_{I\in\mathbb{I}} I = \mathbb{R}$. If, furthermore, each interval in $\mathbb{I}$ has a unique adjacent interval in $\mathbb{I}$ to the left and to the right, and there exists a constant $A > 1$ such that*

$$A^{-1} \le \frac{|I|}{|I'|} \le A, \qquad \text{for all adjacent } I, I' \in \mathbb{I}, \tag{2.1}$$

we call $\mathbb{I}$ a moderate disjoint covering of $\mathbb{R}$.

Given a moderate disjoint covering $\mathbb{I}$ of $\mathbb{R}$, assign to each interval $I \in \mathbb{I}$ a cutoff radius $\varepsilon_I > 0$ at the left endpoint and a cutoff radius $\varepsilon_I' > 0$ at the right endpoint, satisfying

$$\begin{cases} \text{(i)} & \varepsilon_I' = \varepsilon_{I'} \text{ whenever } \alpha_I' = \alpha_{I'} \\ \text{(ii)} & \varepsilon_I + \varepsilon_I' \le |I| \\ \text{(iii)} & \varepsilon_I \ge c|I|, \end{cases} \tag{2.2}$$

with $c > 0$ independent of I.

Example 2.2. *If we let $\varepsilon_I = \frac{1}{2A}|I|$ and ε_I' be given by (i) in (2.2) then (ii) and (iii) are clearly satisfied.*

We are now ready to define the brushlet system. For each $I \in \mathbb{I}$, we will construct a smooth bell function localized in a neighborhood of this interval. Take a non-negative ramp function $\rho \in C^r(\mathbb{R})$, for some $r \ge 1$, satisfying

$$\rho(\xi) = \begin{cases} 0 & \text{for } \xi \le -1, \\ 1 & \text{for } \xi \ge 1, \end{cases} \tag{2.3}$$

with the property that

$$\rho(\xi)^2 + \rho(-\xi)^2 = 1 \qquad \text{for all } \xi \in \mathbb{R}. \tag{2.4}$$

Define for each $I = [\alpha_I, \alpha_I') \in \mathbb{I}$ the *bell function*

$$b_I(\xi) := \rho\left(\frac{\xi - \alpha_I}{\varepsilon_I}\right)\rho\left(\frac{\alpha_I' - \xi}{\varepsilon_I'}\right). \tag{2.5}$$

Notice that $\operatorname{supp}(b_I) \subset [\alpha_I - \varepsilon_I, \alpha_I' + \varepsilon_I']$ and $b_I(\xi) = 1$ for $\xi \in [\alpha_I + \varepsilon_I, \alpha_I' - \varepsilon_I']$. Now the set of local cosine functions

$$\hat{w}_{n,I}(\xi) = \sqrt{\frac{2}{|I|}}\, b_I(\xi) \cos\left(\pi\big(n + \tfrac{1}{2}\big)\frac{\xi - \alpha_I}{|I|}\right), \quad n \in \mathbb{N}_0, \quad I \in \mathbb{I}, \tag{2.6}$$

constitute an orthonormal basis for L_2, see e.g. [1]. We call the collection $\{w_{n,I} \colon I \in \mathbb{I}, n \in \mathbb{N}_0\}$ a *brushlet system*. The brushlets also have an explicit representation in the time domain. Define the set of *central bell functions* $\{g_I\}_{I\in\mathbb{I}}$ by

$$\hat{g}_I(\xi) := \rho\left(\frac{|I|}{\varepsilon_I}\xi\right)\rho\left(\frac{|I|}{\varepsilon_I'}(1 - \xi)\right), \tag{2.7}$$

such that $b_I(\xi) = \hat{g}_I\big(|I|^{-1}(\xi-\alpha_I)\big)$, and let for notational convenience

$$e_{n,I} := \frac{\pi\big(n+\frac{1}{2}\big)}{|I|}, \qquad I \in \mathbb{I},\ n \in \mathbb{N}_0.$$

Then,

$$w_{n,I}(x) = \sqrt{\frac{|I|}{2}}e^{i\alpha_I x}\big\{g_I\big(|I|(x+e_{n,I})\big)+g_I\big(|I|(x-e_{n,I})\big)\big\}. \tag{2.8}$$

By a straight forward calculation it can be verified (see [3]) that there exists a constant $C<\infty$ independent of $I \in \mathbb{I}$, such that

$$|g_I(x)| \le C(1+|x|)^{-r}, \tag{2.9}$$

with $r \ge 1$ given by the smoothness of the ramp function. Thus a brushlet $w_{n,I}$ essentially consists of two humps at $\pm e_{n,I}$.

Given a bell function b_I, define an operator $P_I : L_2 \to L_2$ by

$$\widehat{P_I f}(\xi) := b_I(\xi)\big[b_I(\xi)\hat{f}(\xi)+b_I(2\alpha_I-\xi)\hat{f}(2\alpha_I-\xi)-b_I(2\alpha_I'-\xi)\hat{f}(2\alpha_I'-\xi)\big].$$

It can be verified that P_I is an orthogonal projection, mapping L_2 onto $\overline{\text{span}}\{w_{n,I} : n \in \mathbb{N}_0\}$ (see e.g. [13]). Moreover, P_I extends to a bounded operator on Lebesgue spaces. This can be seen by the following arguments.

Notice that $(\xi-\alpha_I)\frac{d}{d\xi}b_I(\xi) = y\frac{d}{dy}\hat{g}_I(y)$, where $y = |I|^{-1}(\xi-\alpha_I)$, and since $\varepsilon_I \asymp |I| \asymp \varepsilon_I'$, (2.7) implies that

$$\big|(\xi-\alpha_I)\tfrac{d}{d\xi}b_I(\xi)\big| \le C < \infty,$$

with C independent of I. Now, by the Hörmander-Mihlin multiplier theorem [2] the operator $S_I, I \in \mathbb{I}$, given by $\widehat{S_I f} := b_I\hat{f}$, extends to a bounded operator on $L_p(\mathbb{R})$, $1<p<\infty$, with norm independent of I. Notice that,

$$P_I f = S_I\big[S_I f + e^{i2\alpha_I}S_I f(-\cdot) - e^{i2\alpha_I'}S_I f(-\cdot)\big].$$

Thus, two applications of the Hörmander-Mihlin multiplier theorem give that P_I, $I \in \mathbb{I}$, extends to a bounded operator on L_p for $1<p<\infty$, with norm independent of I.

Lemma 2.3. *Given a moderate disjoint covering* $\mathbb{I}$ *with associated brushlet system* $\{w_{n,I}\}_{n,I}$, *we have*

$$\|P_I f\|_{L_p} \asymp |I|^{\frac{1}{2}-\frac{1}{p}}\Big(\sum_{n\in\mathbb{N}_0}|\langle f, w_{n,I}\rangle|^p\Big)^{1/p}, \qquad 1 \le p \le \infty, \tag{2.10}$$

for any $f \in L_p$.

Proof. From (2.9) we have that $\|g_I\|_{L^1} \le C$ with C independent of $I \in \mathbb{I}$. This bound, together with the representation (2.8), imply that

$$\sup_{x\in\mathbb{R}}\sum_{n\in\mathbb{N}_0}|w_{n,I}(x)| \le C|I|^{\frac{1}{2}} \quad \text{and} \quad \sup_{n\in\mathbb{N}_0}\|w_{n,I}\|_{L_1} \le C'|I|^{-\frac{1}{2}}.$$

With these two properties (2.10) follows by Hölder's inequality (see e.g. [19]). □

3. Modulation spaces

In this section we define the α-modulation spaces. The α-modulation spaces, first introduced by Gröbner in [11], are a family of spaces that contain the classical modulation and Besov spaces as special "extremal" cases. The spaces are defined by a parameter α, belonging to the interval $[0,1]$. This parameter determines a segmentation of the frequency axis from which the spaces are built. Let us be more specific. First we define an α-covering of $\mathbb{R}$.

Definition 3.1. *A family $\mathbb{I}$ of intervals $I \in \mathbb{R}$ is called an admissible covering of $\mathbb{R}$ if $\cup_{I\in\mathbb{I}} I = \mathbb{R}$ and $\#\{I \in \mathbb{I} \colon x \in I\} \leq 2$ for all $x \in \mathbb{R}$. Furthermore, if there exists a constant $0 \leq \alpha \leq 1$, such that $|I| \asymp (1+|\xi|)^\alpha$ for all $I \in \mathbb{I}$, and all $\xi \in I$, then $\mathbb{I}$ is called an α-covering of $\mathbb{R}$.*

Remark 3.2. Notice that for $\mathbb{I}$ a disjoint covering, both $\mathbb{I}$ and the set $\{\mathrm{supp}(b_I)\}_{I\in\mathbb{I}}$ are an admissible covering. Moreover, if $\mathbb{I}$ is an α-covering too (called a disjoint α-covering), it is automatically a moderate covering.

Definition 3.3. *Given an admissible covering $\mathbb{I}$ of $\mathbb{R}$, a family $\Psi = \{\psi_I\}_{I\in\mathbb{I}}$ of nontrivial functions is called a bounded admissible partition of unity subordinate to $\mathbb{I}$, if the following conditions are satisfied: $\sup_{I\in\mathbb{I}} \|\psi_I\|_{\mathcal{F}L_1} < \infty$, $\mathrm{supp}(\psi_I) \subset I$ for all $I \in \mathbb{I}$, and $\sum_{I\in\mathbb{I}} \psi_I(\xi) = 1$ for all $\xi \in \mathbb{R}$.*

We can now define the α-modulation spaces.

Definition 3.4. *Given $1 \leq p,q \leq \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha \leq 1$, let $\mathbb{I}$ be an α-covering of $\mathbb{R}$ and let Ψ be a corresponding partition of unity. Then we define the α-modulation space, $M_q^{s,\alpha}(L_p)$ as the set of distributions $f \in S'(\mathbb{R})$ satisfying*

$$\|f\|_{M_q^{s,\alpha}(L_p)} := \left(\sum_{I\in\mathbb{I}} (1+|\xi_I|)^{qs} \left\|\mathcal{F}^{-1}(\psi_I \mathcal{F} f)\right\|_{L_p}^q\right)^{1/q} < \infty,$$

with $\{\xi_I\}_{I\in\mathbb{I}}$ a sequence satisfying $\xi_I \in I$. For $q = \infty$ we have the usual change of the sum to sup *over $I \in \mathbb{I}$.*

Remark 3.5. Notice that for $\alpha > 0$, we have

$$\|f\|_{M_q^{s,\alpha}(L_p)} \asymp \left(\sum_{I\in\mathbb{I}} |I|^{qs/\alpha} \left\|\mathcal{F}^{-1}(\psi_I \mathcal{F} f)\right\|_{L_p}^q\right)^{1/q} < \infty,$$

and thus, in particular, $M_q^{s,1}(L_p) = B_q^s(L_p)$ for $1 \leq p,q \leq \infty$, and $s \in \mathbb{R}$. The other "extreme" $M_q^{s,0}(L_p)$ is the classical modulation space $M_q^s(L_p)$, so in this sense the α-modulation spaces are intermediate between the Besov spaces and the modulation spaces.

It is possible to rewrite the $M_q^{s,\alpha}(L_p)$-norm, for $0 < \alpha \leq 1$, using the projection operators P_I associated to a disjoint α-covering $\mathbb{I}$.

Theorem 3.6. *Given $0 < \alpha \leq 1$, let $\mathbb{I}$ be a disjoint α-covering with associated projection operators P_I, $I \in \mathbb{I}$. Then for $1 < p,q < \infty$, $s \in \mathbb{R}$, and for any $f \in$*

$M_q^{s,\alpha}(L_p)$ we have

$$\|f\|_{M_q^{s,\alpha}(L_p)} \asymp \Big(\sum_{I\in\mathbb{I}} \big(|I|^{s/\alpha}\|P_I f\|_{L_p}\big)^q\Big)^{1/q}. \tag{3.1}$$

In order to prove Theorem 3.6 we need the following technical lemma.

Lemma 3.7. *Given $0<\alpha<1$, let $\mathbb{I}$ and $\mathbb{I}'$ be two α-coverings. For each $I\in\mathbb{I}$ let*

$$A_I = \{I'\in\mathbb{I}' \colon I'\cap I \neq \emptyset\}.$$

Then there exists a constant d_A such that $\#A_I \le d_A$ independent of I.

Proof. We can, without loss of generality, assume that $\mathbb{I}$ and $\mathbb{I}'$ are disjoint coverings. Fix $I=[a,b)\in\mathbb{I}$ and let $I_i=[a_i,b_i)\in\mathbb{I}'$, $i=1,2,\dots,N=\#A_I$, be given such that $a_1<a\le b_1=a_2<b_2=a_3<\cdots<a_N\le b<b_N$. Notice that on one hand $a_N-b_1\le b-a\le Ca^\alpha$ and on the other hand

$$a_N-b_1=\sum_{i=2}^{N-1}|I_i| \ge c\sum_{i=2}^{N-1} a_i^\alpha \ge ca^\alpha(N-2).$$

Thus, $\#A_I-2\le C/c$, with C and c independent of I. □

Proof of Theorem 3.6. The case $\alpha=1$ was proved in [3]. We consider the case $0<\alpha<1$. Take $f\in M_q^{s,\alpha}(L_p)$. Let Ψ be a bounded admissible partition of unity subordinate to an α-covering $\mathbb{I}'$. Then, according to Lemma 3.7,

$$P_I f = \sum_{I'\in A_I} P_I(\mathcal{F}^{-1}(\psi_{I'}\hat{f})), \qquad I\in\mathbb{I}$$

in $\mathcal{S}'(\mathbb{R})$ and since $\mathcal{F}^{-1}(\psi_{I'}\hat{f})\in L_p$ for any $I'\in\mathbb{I}'$ and P_I is a bounded operator on L_p uniform in $I\in\mathbb{I}$, we have

$$\|P_I f\|_{L_p} \le C_p \sum_{I'\in A_I} \|\mathcal{F}^{-1}(\psi_{I'}\hat{f})\|_{L_p}.$$

Clearly, $|I|\asymp|I'|$ for any $I'\in A_I$. Hence,

$$\sum_{I\in\mathbb{I}}\big(|I|^{s/\alpha}\|P_I f\|_{L_p}\big)^q \le C\sum_{I\in\mathbb{I}}\Big(\sum_{I'\in A_I}|I'|^{s/\alpha}\|\mathcal{F}^{-1}(\psi_{I'}\hat{f})\|_{L_p}\Big)^q.$$

Furthermore, Hölder's inequality with $1=1/q+1/q'$ implies

$$\begin{aligned}\sum_{I\in\mathbb{I}}\Big(\sum_{I'\in A_I}|I'|^{s/\alpha}\|\mathcal{F}^{-1}(\psi_{I'}\hat{f})\|_{L_p}\Big)^q &\le \sum_{I\in\mathbb{I}}\Big(\sum_{I'\in\mathbb{I}'}(\mathbf{1}_{A_I}(I'))^{q'}\Big)^{q/q'} \\ &\qquad \Big(\sum_{I\in\mathbb{I}'}\big(\mathbf{1}_{A_I}(I')|I'|^{s/\alpha}\|\mathcal{F}^{-1}(\psi_{I'}\hat{f})\|_{L_p}\big)^q\Big) \\ &\le d_A^{q-1}\sum_{I\in\mathbb{I}}\sum_{I'\in\mathbb{I}'}\mathbf{1}_{A_I}(I')\big(|I'|^{s/\alpha}\|\mathcal{F}^{-1}(\psi_{I'}\hat{f})\|_{L_p}\big)^q,\end{aligned}$$

where $\mathbf{1}_{A_I}(I') = 1$ for $I' \in A_I$ and 0 for $I' \in \mathbb{I}' \setminus A_I$. Finally, since $\mathbf{1}_{A_I}(I') = \mathbf{1}_{A_{I'}}(I)$, for any $I \in \mathbb{I}$ and $I' \in \mathbb{I}'$, this gives

$$\sum_{I\in\mathbb{I}} \big(|I|^s \|P_I f\|_{L_p}\big)^q \le d_A^{(q-1)} \sum_{I'\in\mathbb{I}'} \Big(\sum_{I\in\mathbb{I}} \mathbf{1}_{A_{I'}}(I)\Big) \big(|I'|^{s/\alpha} \|\mathcal{F}^{-1}(\psi_{I'}\hat{f})\|_{L_p}\big)^q,$$
$$\le C' d_A^q \sum_{I'\in\mathbb{I}'} \big(|I'|^{s/\alpha} \|\mathcal{F}^{-1}(\psi_{I'}\hat{f})\|_{L_p}\big)^q.$$

The upper bound in (3.1) can be proved in a similar fashion. □

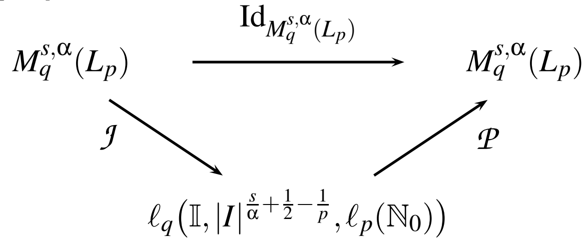
Remark 3.8. Let $\mathbb{I}$ be a disjoint α-covering. Given $\Lambda \subset \mathbb{I}$, let $T_\Lambda = \sum_{I\in\Lambda} P_I$. By the equivalence (3.1) and since the operators P_I, $I \in \mathbb{I}$, are orthogonal projections in L_2, T_Λ extends to a bounded operator on $M_q^{s,\alpha}(L_p)$ for $1 < p,q < \infty$, $s \in \mathbb{R}$, and $0 < \alpha \le 1$, with norm independent of Λ.

Using Lemma 2.3 we can derive the following result from Theorem 3.6.

Proposition 3.9. *Let $\mathcal{B} = \{w_{n,I}\}_{I\in\mathbb{I},n\in\mathbb{N}_0}$ be a brushlet system associated with an α-covering $\mathbb{I}$ for some $0 < \alpha \le 1$. Then $\mathcal{B}$ constitutes an unconditional basis for the α-modulation spaces $M_q^{s,\alpha}(L_p)$, $1 < p,q < \infty$, $s \in \mathbb{R}$, and we have the characterization*

$$\|f\|_{M_q^{s,\alpha}(L_p)} \asymp \Big(\sum_{I\in\mathbb{I}} \Big(\sum_{n\in\mathbb{N}_0} \big(|I|^{\frac{s}{\alpha}+\frac{1}{2}-\frac{1}{p}} |\langle f, w_{n,I}\rangle|\big)^p\Big)^{q/p}\Big)^{1/q}.$$

Remark 3.10. We can also state Proposition 3.9 in the form that the brushlet basis makes $M_q^{s,\alpha}(L_p)$ a retract of the weighted sequence space $\ell_q\big(\mathbb{I}, |I|^{\frac{s}{\alpha}+\frac{1}{2}-\frac{1}{p}}, \ell_p(\mathbb{N}_0)\big)$. Indeed, if we define $\mathcal{J} : M_q^{s,\alpha}(L_p) \to \ell_q\big(\mathbb{I}, |I|^{\frac{s}{\alpha}+\frac{1}{2}-\frac{1}{p}}, \ell_p(\mathbb{N}_0)\big)$ by $f \to \{\langle f, w_{n,I}\rangle\}_{n,I}$, and $\mathcal{P} : \ell_q\big(\mathbb{I}, |I|^{\frac{s}{\alpha}+\frac{1}{2}-\frac{1}{p}}, \ell_p(\mathbb{N}_0)\big) \to M_q^{s,\alpha}(L_p)$ by $\{c_{n,I}\} \to \sum_{n,I} c_{n,I} w_{n,I}$, then we have $\mathcal{P} \circ \mathcal{J} = \mathrm{Id}_{M_q^{s,\alpha}(L_p)}$.



4. Nonlinear approximation with brushlet systems

Recent results [14, 10] have shown that to characterize nonlinear m-term approximation with elements from a Schauder basis for a Banach space, it is advantageous to deal with so-called greedy bases. In this section we show that (normalized) brushlet systems from greedy bases for the α-modulation spaces, and from this fact we deduce several direct and inverse estimates for nonlinear m-term approximation with brushlet systems.

First, let us define a greedy basis. A greedy basis is an unconditional basis that also satisfies the so-called *democracy* condition.

Definition 4.1. *A system $\{g_k\}_{k\in\mathbb{N}}$ in a Banach space X is called* democratic *if there exists a constant $C>0$ such that*

$$\Big\|\sum_{k\in P} g_k\Big\|_X \le C\Big\|\sum_{k\in Q} g_k\Big\|_X,$$

for any two finite sets of indices P and Q with the same cardinality, $\#P=\#Q$.

We let $\mathcal{B}=\{w_{n,I}\}_{I\in\mathbb{I},n\in\mathbb{N}_0}$ be a brushlet system associated with a disjoint α-covering $\mathbb{I}$ for some $0<\alpha\le 1$. Consider the normalized functions

$$\tilde{w}_{n,I}=\frac{w_{n,I}}{\|w_{n,I}\|_{M_p^{s,\alpha}(L_p)}}, \qquad I\in\mathbb{I}, n\in\mathbb{N}.$$

Notice that

$$\|f\|_{M_p^{s,\alpha}(L_p)}\asymp\Big(\sum_{I\in\mathbb{I},n\in\mathbb{N}}|\langle f,\tilde{w}_{n,I}\rangle|^p\Big)^{1/p},$$

by Proposition 3.9. Thus, for a finite subset $\Lambda\subset\mathbb{I}\times\mathbb{N}$, we have the uniform estimate

$$\Big\|\sum_{(n,I)\in\Lambda}\tilde{w}_{n,I}\Big\|_{M_p^{s,\alpha}(L_p)}\asymp(\#\Lambda)^{1/p}, \tag{4.1}$$

which shows that $\{\tilde{w}_{n,I}\}_{n,I}$ constitutes a greedy basis for $M_p^{s,\alpha}(L_p)$.

Let us introduce some notation that will be needed to explore nonlinear approximation with brushlet bases. Let $\mathcal{B}=\{g_k\}_{k\in\mathbb{N}}$ be a Schauder basis in a Banach space X. We consider the collection of all possible m-term expansions with elements from $\mathcal{B}$:

$$\Sigma_m(\mathcal{B}):=\Big\{\sum_{i\in\Lambda}c_ig_i\,\Big|\,c_i\in\mathbb{C},\#\Lambda\le m\Big\}.$$

The error of the best m-term approximation to an element $f\in X$ is then

$$\sigma_m(f,\mathcal{B})_X:=\inf_{f_m\in\Sigma_m(\mathcal{B})}\|f-f_m\|_X.$$

Definition 4.2 (Approximation spaces)**.** *The approximation space $\mathcal{A}_q^\gamma(X,\mathcal{B})$ is defined by*

$$|f|_{\mathcal{A}_q^\gamma(X,\mathcal{B})}:=\Big(\sum_{m=1}^\infty\big(m^\gamma\sigma_m(f,\mathcal{B})_X\big)^q\frac{1}{m}\Big)^{1/q}<\infty,$$

and (quasi)normed by $\|f\|_{\mathcal{A}_q^\gamma(X,\mathcal{B})}=\|f\|_X+|f|_{\mathcal{A}_q^\gamma(X,\mathcal{B})}$ for $0<q,\gamma<\infty$, with the ℓ_q norm replaced by the sup-norm, when $q=\infty$.

We also need to define smoothness spaces in order to characterize the approximation spaces. We give the definition in an abstract setting, but later in this section (Proposition 4.4) it is proved that the smoothness spaces corresponding to brushlet systems can be identified with certain α-modulation spaces.

For $\tau\in(0,\infty)$ and $s\in(0,\infty]$, we let $\mathcal{K}_s^\tau(\mathcal{B},M)$ denote the set

$$\operatorname{clos}_X\Big\{f\in X\,|\,\exists\Lambda\subset\mathbb{N},\#\Lambda<\infty, f=\sum_{k\in\Lambda}c_kg_k, \|\{c_k\}\|_{\ell_{\tau,s}}\le M\Big\}.$$

Then we define

$$\mathcal{K}_s^\tau(X,\mathcal{B}) := \bigcup_{M>0} \mathcal{K}_s^\tau(\mathcal{B},M), \tag{4.2}$$

with

$$\|f\|_{\mathcal{K}_s^\tau(X,\mathcal{B})} = \inf\{M : f \in \mathcal{K}_s^\tau(\mathcal{B},M)\}.$$

For a democratic basis $\mathcal{B} = \{g_k\}_{k\in\mathbb{N}}$ in X, we define $\varphi(n) := \big\|\sum_{k=1}^n g_k\big\|_X$. The following theorem was proved in [10].

Theorem 4.3. *Assume $\mathcal{B}$ is a greedy basis for X with $\varphi(n) \asymp n^{1/p}$. Then*

$$\mathcal{A}_q^\gamma(X,\mathcal{B}) = \mathcal{K}_q^\tau(X,\mathcal{B}), \qquad \tau^{-1} = \gamma + p^{-1}, \gamma > 0,$$

with equivalent norms.

We can now apply this machinery to the brushlet bases, using the estimate (4.1).

Proposition 4.4. *Let $\{w_{n,I}\}_{I\in\mathbb{I},n\in\mathbb{N}_0}$ be a brushlet system associated with a disjoint α-covering $\mathbb{I}$ for some $0 < \alpha \le 1$, and let $\mathcal{B} = \big\{w_{n,I}/\|w_{n,I}\|_{M_p^{s,\alpha}(L_p)}\big\}_{I\in\mathbb{I},n\in\mathbb{N}}$ for some $s \ge 0$ and $1 < p < \infty$. Then*

$$\mathcal{A}_q^\gamma\big(M_p^{s,\alpha}(L_p),\mathcal{B}\big) = \mathcal{K}_q^\tau\big(M_p^{s,\alpha}(L_p),\mathcal{B}\big), \qquad \tau^{-1} = \gamma + p^{-1}, \gamma > 0, 0 < q \le \infty,$$

with equivalent norms. Moreover, for $\tau > 0$,

$$\mathcal{K}_\tau^\tau\big(M_p^{s,\alpha}(L_p),\mathcal{B}\big) = M_\tau^{\beta,\alpha}(L_\tau), \qquad \text{with} \quad \beta = \frac{\alpha}{\tau} - \frac{\alpha}{p} + s.$$

Proof. The first part of the Proposition follows at once from Theorem 4.3 and the estimate (4.1). To prove the second claim, we use the fact that $\|w_{n,I}\|_{M_p^{s,\alpha}(L_p)} \asymp |I|^{s/\alpha+1/2-1/p}$, which can easily be deduced from Proposition 3.9. Again using Proposition 3.9, we deduce the following equation for β,

$$|I|^{(\beta/\alpha+1/2-1/\tau)-(s/\alpha+1/2-1/p)} = |I|^0,$$

from which we obtain $\beta = \frac{\alpha}{\tau} - \frac{\alpha}{p} + s$. ☐

5. TRIEBEL-LIZORKIN TYPE SPACES

For wavelet systems, the standard approach to m-term nonlinear approximation is to measure the error in a Triebel-Lizorkin space for functions with smoothness measured on the Besov scale, see e.g. [5, 15]. In this section we present the natural extension of this result to approximation with brushlet systems. The classical Triebel-Lizorkin spaces are only adapted to the case $\alpha = 1$, so first we introduce spaces adapted to α-partitions of the frequency axis.

Definition 5.1. *Given $0 \le \alpha \le 1$. Let $\mathbb{I}$ be an α-covering of $\mathbb{R}$ and let Ψ be a bounded admissible partition of unity subordinate to $\mathbb{I}$. For $0 < p,q \le \infty$ and $s \in \mathbb{R}$ we define the α-Triebel-Lizorkin space, $F_q^{s,\alpha}(L_p)$ as the set of distributions $f \in S'(\mathbb{R})$ for which the quasi-norm*

$$\|f\|_{F_q^{s,\alpha}(L_p)} := \Big\|\Big(\sum_{I\in\mathbb{I}}\big((1+|\xi_I|)^s|\mathcal{F}^{-1}(\psi_I\mathcal{F}f)|\big)^q\Big)^{1/q}\Big\|_{L_p} < \infty,$$

with $\{\xi_I\}_{I\in\mathbb{I}}$ *a sequence satisfying* $\xi_I \in I$*. For* $q=\infty$ *we have the usual change of the sum to* sup *over* $I\in\mathbb{I}$*.*

Remark 5.2. Clearly, for the case $\alpha = 1$, $F_q^{s,1}(L_p) = F_q^s(L_p)$, the standard Triebel-Lizorkin space.

Remark 5.3. Notice that $\|f\|_{F_q^{s,\alpha}(L_p)} = \|f_\psi^q\|_{L_p}$, where

$$f_\psi^q := \|\{\mathcal{F}^{-1}(\psi_I \mathcal{F} f)\}_{n,I}\|_{\ell_q(\mathbb{N}_0\times\mathbb{I},(1+|\xi_I|)^s)}.$$

Since $(L_p)' = L_{p'}$, $1 = 1/p + 1/p'$, and $(\ell_q(\mathbb{N}_0\times\mathbb{I},(1+|\xi_I|)^s))' = \ell_{q'}(\mathbb{N}_0\times\mathbb{I},(1+|\xi_I|)^{-s})$, $1 = 1/q+1/q'$, the Banach dual $(F_q^{s,\alpha}(L_p))' = F_{q'}^{-s,\alpha}(L_{p'})$ (use [8, Theorem 2.8] combined with the Fourier transform).

It is immediate to verify that $F_p^{s,\alpha}(L_p) = M_p^{s,\alpha}(L_p)$, and generally we have the embedding given by[1]

Proposition 5.4. *For* $1 < p,q < \infty$ *and* $s\in\mathbb{R}$*,*

$$M^{s,\alpha}_{\min\{p,q\}}(L_p) \hookrightarrow F_q^{s,\alpha}(L_p) \hookrightarrow M^{s,\alpha}_{\max\{p,q\}}(L_p).$$

Proof. The result is verified using the same type of estimates as given in [21, Section 2.3.2]. □

Similar to the modulation space case, it is possible to rewrite the α-Triebel-Lizorkin norms using an expansion in a brushlet system.

Proposition 5.5. *Given* $0<\alpha\le 1$*, let* $\{w_{n,I}\}_{I\in\mathbb{I},n\in\mathbb{N}_0}$ *be a brushlet system associated with a disjoint* α*-covering* $\mathbb{I}$*. For* $1<p,q<\infty$ *and* $s\in\mathbb{R}$*, we have*

$$\|f\|_{F_q^{s,\alpha}(L_p)} \asymp \big\|S_q^{s,\alpha}(f,\cdot)\big\|_{L_p},$$

where

$$S_q^{s,\alpha}(f,\cdot) := \bigg(\sum_{I\in\mathbb{I},n\in\mathbb{Z}} \big(|I|^{s/\alpha+1/2}|\langle f, w_{n,I}\rangle|\chi_{E_{(n,I)}}(\cdot)\big)^q\bigg)^{1/q},$$

and $E_{(n,I)} := \big\{x\in\mathbb{R} : |I|x - \pi\big(n+\tfrac12\big)\in(-1,1)\big\}$*.*

[1]The notation $V\hookrightarrow W$ means that the two (quasi)normed spaces V and W satisfy $V\subset W$ and there is a constant $C<\infty$ such that $\|\cdot\|_W \le C\|\cdot\|_V$.

Proof. Since $E_{(n,I)}$, $n \in \mathbb{N}_0$, are disjoint intervals for a fixed $I \in \mathbb{I}$, given $x \in \mathbb{R}$ there is at most one $m \in \mathbb{N}_0$ such that $x \in E_{(m,I)}$. Using (2.8), we have

$$\begin{aligned}
&|I|^{1/2}\chi_{E_{(m,I)}}(x)|\langle f, w_{m,I}\rangle| \\
&\qquad \le \sum_{I'\in A_I} |I|^{1/2}\chi_{E_{(m,I)}}(x)|\langle(\psi_{I'}\hat{f})^\vee, w_{m,I}\rangle| \\
&\qquad \le \sum_{I'\in A_I} 2^{1/2}\chi_{E_{(m,I)}}(x)\Big(\int_{-\infty}^{\infty} |(\psi_{I'}\hat{f})^\vee(y)||I|\big|g_I\big(|I|y-\pi(m+\tfrac{1}{2})\big)\big|\,dy \\
&\qquad\qquad + \int_{-\infty}^{\infty} |(\psi_{I'}\hat{f})^\vee(y)||I|\big|g_I\big(|I|y+\pi(m+\tfrac{1}{2})\big)\big|\,dy\Big) \\
&\qquad \le C\sum_{I'\in A_I} \big[\mathcal{M}((\psi_{I'}\hat{f})^\vee)(x)+\mathcal{M}((\psi_{I'}\hat{f})^\vee)(-x)\big],
\end{aligned}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator, and we used [20, p. 57] in the last step. By the arguments given in the proof of Theorem 3.6, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{I\in\mathbb{I}}\Big(|I|^{s/\alpha}\sum_{I'\in A_I}\mathcal{M}((\psi_{I'}\hat{f})^\vee)\Big)^q\Big)^{1/q}\Big\|_{L_p} \\
&\qquad\qquad \le Cd_A\Big\|\Big(\sum_{I'\in\mathbb{I}'}\big(|I'|^{s/\alpha}\mathcal{M}((\psi_{I'}\hat{f})^\vee)\big)^q\Big)^{1/q}\Big\|_{L_p}.
\end{aligned}$$

Furthermore, by the Fefferman-Stein maximal inequality,

$$\Big\|\Big(\sum_{I'\in\mathbb{I}'}\mathcal{M}(|I'|^{s/\alpha}|(\psi_{I'}\hat{f})^\vee|)^q\Big)^{1/q}\Big\|_{L_p} \le C_p\Big\|\Big(\sum_{I'\in\mathbb{I}'}(|I'|^{s/\alpha}|(\psi_{I'}\hat{f})^\vee|)^q\Big)^{1/q}\Big\|_{L_p},$$

and we may conclude, $\|S_q^{s,\alpha}(f,\cdot)\|_{L_p} \le C\|f\|_{F_q^{s,\alpha}(L_p)}$ for all $f \in F_q^{s,\alpha}(L_p)$.

Given any $K \in (1,\infty)$ let $\Lambda_K = \{(n,I) \in \mathbb{N}_0\times\mathbb{I}\colon n \le K, I \subset [-K,K]\}$. Suppose $f \in F_q^{s,\alpha}(L_p)$, let $f_K := \sum_{(n,I)\in\Lambda_K}\langle f, w_{n,I}\rangle w_{n,I}$. Then it is straight forward to see that $f_K \in L_2 \cap F_q^{s,\alpha}(L_p)$, and $f_K \to f$ in $F_q^{s,\alpha}(L_p)$ as $K \to \infty$. Thus, it suffices to prove the upper bound for $f \in L_2 \cap F_q^{s,\alpha}(L_p)$.

Consider the linear operator $W : L_2 \to \ell_2(\mathbb{N}_0\times\mathbb{I})$ defined by

$$Wh = \big\{\langle h, w_{n,I}\rangle|I|^{1/2}\chi_{E_{(n,I)}}\big\}_{I\in\mathbb{I},n\in\mathbb{N}_0}.$$

By a direct calculation using Parsevals relation, we have

$$\int_{-\infty}^{\infty}\langle Wf, Wg\rangle_{\ell_2}(x)\,dx = 2\langle f, g\rangle, \qquad f,g \in L_2.$$

Suppose $f \in L_2 \cap F_q^{s,\alpha}(L_p)$ and $g \in L_2 \cap F_{q'}^{-s,\alpha}(L_{p'})$. Then,

$$\begin{aligned}
2|\langle f,g\rangle| &= \Big|\int_{-\infty}^{\infty}\langle Wf, Wg\rangle_{\ell_2}(x)\,dx\Big| \\
&\le \|S_q^{s,\alpha}(f,\cdot)\|_{L_p}\|S_{q'}^{-s,\alpha}(g,\cdot)\|_{L_{p'}} \le C_{p'}\|S_q^{s,\alpha}(f,\cdot)\|_{L_p}\|g\|_{F_{q'}^{-s,\alpha}(L_{p'})}.
\end{aligned}$$

Taking the sup over g with $\|g\|_{F_{q'}^{-s,\alpha}(L_{p'})} \le 1$, yields $\|f\|_{F_q^{s,\alpha}(L_p)} \le C\big\|S_q^{s,\alpha}(f,\cdot)\big\|_{L_p}$ for all $f \in L_2 \cap F_q^{s,\alpha}(L_p)$. □

Remark 5.6. The proposition can be restated as saying that the space $F_q^{s,\alpha}(L_p)$ can be considered a retract of the space $L_p(\ell_q(\mathbb{N}_0 \times \mathbb{I}, |I|^{s/\alpha+1/2}))$, just as in the modulation space case, through the maps $\mathcal{J}$ and $\mathcal{P}$ from Remark 3.10.

It is clear that $F_2^{0,\alpha}(L_2) = L_2$, and generalizing this observation we obtain

Proposition 5.7. *The space $F_2^{0,\alpha}(L_p)$, $1 < p < \infty$, is exactly the family of functions $f \in L_p$, for which the brushlet expansion converges unconditionally in L_p.*

Proof. First, suppose the brushlet expansion for $f \in L_p$ converges unconditionally. It follows that $f = \sum_{I\in\mathbb{I}} P_I f$ also converges unconditionally in L_p and by Khintchine's inequality,

$$\Big\|\Big(\sum_{I\in\mathbb{I}} |P_I f|^2\Big)^{1/2}\Big\|_{L_p} \le C\|f\|_{L_p} < \infty.$$

Using arguments similar to those in the first half of the proof of Proposition 5.5, we obtain

$$\|S_2^{0,\alpha}(f,\cdot)\|_{L_p} \le C\Big\|\Big(\sum_{I\in\mathbb{I}} |P_I f|^2\Big)^{1/2},$$

and thus $\|f\|_{F_2^{0,\alpha}(L_p)} \le C\|f\|_{L_p} < \infty$, by Proposition 5.5.

Conversely, suppose that $\|f\|_{F_2^{0,\alpha}(L_p)} < \infty$. Define $\tilde{f} = \sum_{n,I} \beta_{n,I}\langle f, w_{n,I}\rangle w_{n,I}$, where $\{\beta_{n,I}\} \in \{-1,0,1\}^{\mathbb{Z}\times\mathbb{I}}$. Take g to be a function which is C_c^∞ in the frequency domain with the properties that $2|\langle \tilde{f}, g\rangle| \ge \|\tilde{f}\|_{L_p}$ and $\|g\|_{L_{p'}} = 1$. The brushlet expansion of g converges unconditionally in $L_{p'}$, which can be seen from the following estimate: let $\tilde{g} = \sum_{I,n} = \gamma_{n,I}\langle g, w_{n,I}\rangle w_{n,I}$ for any sequence $\{\gamma_{n,I}\} \in \{-1,0,1\}^{\mathbb{Z}\times\mathbb{I}}$. Then

$$\begin{aligned}\|\tilde{g}\|_{L_{p'}} &= \Big\|\sum_{I\in\mathbb{I}} P_I\tilde{g}\Big\|_{L_{p'}}\\ &\le \sum_{I\in\mathbb{I}} \|P_I\tilde{g}\|_{L_{p'}} \asymp \sum_{I\in\mathbb{I}} |I|^{1/2-1/p'}\Big(\sum_{n\in\mathbb{Z}} |\langle g, w_{n,I}\rangle|^{p'}\Big)^{1/p'}.\end{aligned}$$

Now, by arguments similar to those given in the last part of the proof of Proposition 5.5, we get $\|\tilde{f}\|_{L_p} \le C\|S_2^{0,\alpha}(f,\cdot)\|_{L_p}\|g\|_{L_{p'}}$, independent of $\{\beta_{n,I}\}$. This proves the unconditional convergence of the brushlet expansion for f. □

6. Jackson and Bernstein estimates

In this section we establish Jackson and Bernstein type inequalities for functions from the α-modulation spaces, with the error measured in the α-Triebel-Lizorkin spaces. To a large extent we will use the same type of arguments as used by Kyriazis in [15], where similar estimates were established for wavelet systems.

6.1. **Jackson inequality.** We begin with the Jackson inequality, but first we prove a technical lemma that will be of use in obtaining both the Jackson and Bernstein estimate.

Lemma 6.1. *Given* $0<\alpha\le 1$*, let* $\mathbb{I}$ *be a disjoint* α*-covering. For* $\Lambda\subset\mathbb{N}_0\times\mathbb{I}$*, with* $\#\Lambda<\infty$*, let* $I_\Lambda(x)=\max_I\{|I|\colon (n,I)\in\Lambda,\ x\in E_{n,I}\}$*. Then,*

$$\sum_{(n,I)\in\Lambda}|I|^q\chi_{E_{(n,I)}}\le CI_\Lambda(x)^{q+\frac{1-\alpha}{\alpha}},$$

for any $q>-(1-\alpha)/\alpha$*.*

Proof. For $\alpha=1$ the result is well known, see e.g., [3]. Suppose $0<\alpha<1$. For $k\in\mathbb{N}$ let $a_k=k^{1/(1-\alpha)}$ and $\Delta_k=|a_{k+1}-a_k|$, then it is straight forward to show that $\Delta_k\asymp a_k^\alpha$, i.e., $\{[-a_{k+1},-a_k),[-1,1),[a_k,a_{k+1})\}_{k\in\mathbb{N}}$ is an α-covering. Moreover, for any $N\in\mathbb{N}$

$$\sum_{k=1}^N\Delta_k^q\le C\sum_{k=1}^N k^{\frac{q}{r}}\le C'N^{\frac{q}{r}+1},$$

where $r=(1-\alpha)/\alpha$ and C' is independent of N. Hence,

$$\sum_{k=1}^N\Delta_k^q\le C''\Delta_N^{(\frac{q}{r}+1)r}=\Delta_N^{q+r}.$$

For a given $k\in\mathbb{N}$ let $A_k=\{I\in\mathbb{I}\colon I\cap[a_k,a_{k+1})\neq\emptyset\}$. By Lemma 3.7, $\#A_k\le d_A<\infty$ independent of $k\in\mathbb{N}$. Given $x\in\mathbb{R}$, let $N\in\mathbb{N}$ be the smallest positive integer such that $\arg\max_I\{|I|\colon (n,I)\in\Lambda,\ x\in E_{n,I}\}\subset[-a_{N+1},a_{N+1})$. Then,

$$\begin{aligned}\sum_{(n,I)\in\Lambda}|I|^q\chi_{E_{(n,I)}}&\le\sum_{|I|\le I_\Lambda(x)}|I|^q\le\sum_{k=0}^N\sum_{I\in A_k}|I|^q\le C\sum_{k=0}^N\sum_{I\in A_k}\Delta_k^q\\&\le C'\sum_{k=0}^N\Delta_k^q\le C''\Delta_N^{q+\frac{1-\alpha}{\alpha}}\le C'''I_\Lambda(x)^{q+\frac{1-\alpha}{\alpha}},\end{aligned}$$

using that $|I|\asymp\Delta_k$ for $I\in A_k$. □

We have the following Jackson estimates for m-term brushlet approximation to functions in the α-modulation space $M_\tau^{\gamma,\alpha}(L_\tau)$ where the error is measured in the α-Triebel-Lizorkin space $F_t^{\beta,\alpha}(L_p)$.

Proposition 6.2. *Given* $0<\alpha\le 1$*, let* $\mathcal{B}$ *be a brushlet system associated to a disjoint* α*-covering* $\mathbb{I}$*. Let* $1<p<\infty$*,* $1\le t<\infty$*, and* $\beta<\gamma$*. Define r by*

$$r=r(\alpha,p,t):=\begin{cases}0 & \textit{for}\quad t\ge p\\ \frac{1-\alpha}{\alpha} & \textit{for}\quad t<p\end{cases}.$$

and τ *by* $1/\tau=(\gamma-\beta)/\alpha+1/p-r/t$*. Then, for every* $f\in M_\tau^{\gamma,\alpha}(L_\tau)$*,*

$$\sigma_n(f,\mathcal{B})_{F_t^{\beta,\alpha}(L_p)}\le Cn^{-\frac{\gamma-\beta}{\alpha}}\|f\|_{M_\tau^{\gamma,\alpha}(L_\tau)}.\tag{6.1}$$

Remark 6.3. Notice that the factor r is the price for introducing the α-coverings. For $\alpha=1$, we have $r\equiv 0$.

Proof. Let $f \in M_\tau^{\gamma,\alpha}(L_\tau)$. We put $c_{n,I}(f) = \langle f, w_{n,I}\rangle |I|^{\gamma/\alpha-1/\tau+1/2}$. Then $\|f\|_{M_\tau^{\gamma,\alpha}(L_\tau)} \asymp \|\{c_{n,I}(f)\}\|_{\ell_\tau} := M$. Notice also that $|\langle f, w_{n,I}\rangle||I|^{\beta/\alpha+1/2} = |c_{n,I}(f)||I|^{1/p-r/t}$. Define

$$\Lambda_j = \{(n,I) : 2^{-j} < |c_{n,I}(f)| \le 2^{-j+1}\}.$$

Standard estimates show that

$$\#\Lambda_j \le CM^\tau 2^{j\tau} \quad \Rightarrow \quad \sum_{j\le k} \#\Lambda_j \le CM^\tau 2^{k\tau}.$$

Put

$$T_k = \sum_{j\le k} \sum_{(n,I)\in\Lambda_j} \langle f, w_{n,I}\rangle w_{n,I}.$$

Since $\sigma_n(f,\mathcal{B})_{F_t^{\beta,\alpha}(L_p)}$ is decreasing, it suffices to prove (6.1) for a subsequence n_k. Specifically, we will prove that

$$\|f - T_k\|_{F_t^{\beta,\alpha}(L_p)} \le C(M^\tau 2^{k\tau})^{-\frac{\gamma-\beta}{\alpha}} \|f\|_{M_\tau^{\gamma,\alpha}(L_\tau)} = CM^{\tau/p} 2^{-k(1-\tau/p)}.$$

By the above considerations, we have

$$\begin{aligned}
\|f - T_k\|_{F_t^{\beta,\alpha}(L_p)} &= \int_{\mathbb{R}} \Bigg(\sum_{j\ge k+1} \sum_{(n,I)\in\Lambda_j} \big(|\langle f, w_{n,I}\rangle||I|^{\beta/\alpha+1/2} \chi_{E_{(n,I)}}\big)^t \Bigg)^{\frac{1}{t}} dx \\
&= \int_{\mathbb{R}} \Bigg(\sum_{j\ge k+1} \sum_{(n,I)\in\Lambda_j} \big(|c_{n,I}(f)||I|^{1/p-r/t} \chi_{E_{(n,I)}}\big)^t \Bigg)^{\frac{1}{t}} dx \\
&\le C \int_{\mathbb{R}} \Bigg(\sum_{j\ge k+1} \sum_{(n,I)\in\Lambda_j} \big(2^{-j}|I|^{1/p-r/t} \chi_{E_{(n,I)}}\big)^t \Bigg)^{\frac{1}{t}} dx.
\end{aligned}$$

Two distinct cases need to be considered. The first case is $p \le t$, where the Jackson inequality is universally true. We have,

$$\begin{aligned}
\|f - T_k\|^p_{F_t^{\beta,\alpha}(L_p)} &\le \int_{\mathbb{R}} \sum_{j\ge k+1} \sum_{(n,I)\in\Lambda_j} \big(2^{-j}|I|^{1/p} \chi_{E_{(n,I)}}\big)^p dx \\
&\le C \sum_{j\ge k+1} 2^{-jp} \int_{\mathbb{R}} \sum_{(n,I)\in\Lambda_j} \big(|I|^{1/p} \chi_{E_{(n,I)}}\big)^p dx \\
&\le C \sum_{j\ge k+1} 2^{-jp} \#\Lambda_j \qquad \text{since } |E_{n,I}| = 2|I|^{-1} \\
&\le CM^\tau 2^{-k(p-\tau)}.
\end{aligned}$$

Next, we consider $p > t$. Given $\delta > 0$ such that $p(t-\delta) > t\tau$, we have,

$$\begin{aligned}
\|f - T_k\|^p_{F_t^{\beta,\alpha}(L_p)} &\le \int_{\mathbb{R}} \Bigg(\sum_{j\ge k+1} \sum_{(n,I)\in\Lambda_j} 2^{-jt} |I|^{\frac{t}{p}-r} \chi_{E_{(n,I)}} \Bigg)^{\frac{p}{t}} dx \\
&= \int_{\mathbb{R}} \Bigg(\sum_{j\ge k+1} 2^{-j\delta} \sum_{(n,I)\in\Lambda_j} 2^{-j(t-\delta)} |I|^{\frac{t}{p}-r} \chi_{E_{(n,I)}} \Bigg)^{\frac{p}{t}} dx
\end{aligned}$$

$$\leq \int_{\mathbb{R}} \Bigg(\sum_{j\geq k+1} 2^{-j\delta\left(\frac{p}{p-t}\right)} \Bigg)^{\frac{p}{t}\left(\frac{p-t}{p}\right)}$$
$$\Bigg(\sum_{j\geq k+1} \Bigg(\sum_{(n,I)\in\Lambda_j} 2^{-j(t-\delta)} |I|^{\frac{t}{p}-r} \chi_{E_{(n,I)}} \Bigg)^{\frac{p}{t}} \Bigg) dx$$
$$\leq 2^{-\frac{k\delta p}{t}} \sum_{j\geq k+1} 2^{-\frac{jp(t-\delta)}{t}} \int_{\mathbb{R}} \Bigg(\sum_{(n,I)\in\Lambda_j} |I|^{\frac{t}{p}-r} \chi_{E_{(n,I)}} \Bigg)^{\frac{p}{t}} dx,$$

where we used Hölder's inequality. Now, by Lemma 6.1,

$$\int_{\mathbb{R}} \Bigg(\sum_{(n,I)\in\Lambda_j} |I|^{\frac{t}{p}-\frac{1-\alpha}{\alpha}} \chi_{E_{(n,I)}} \Bigg)^{\frac{p}{t}} dx, \leq \int_{\mathbb{R}} \left(I_{\Lambda_j}(x)^{\frac{t}{p}} \right)^{\frac{p}{t}} dx$$
$$= \int_{\mathbb{R}} I_{\Lambda_j}(x)\, dx \leq \#\Lambda_j.$$

Thus,

$$\begin{aligned} \|f - T_k\|^p_{F_t^{\beta,\alpha}(L_p)} &\leq 2^{-k\delta p/t} \sum_{j\geq k+1} 2^{-jp(t-\delta)/t} \#\Lambda_j \\ &\leq 2^{-k\delta p/t} M^\tau \sum_{j\geq k+1} 2^{-j(p(t-\delta)/t-\tau)} \\ &\leq M^\tau 2^{-k(\delta p/t + p(t-\delta)/t-\tau)} \\ &\leq M^\tau 2^{-k(p-\tau)}. \end{aligned}$$

□

6.2. **Bernstein inequality.** We can also establish a Bernstein-type inequality for the α-Triebel-Lizorkin and α-modulation spaces. The first result concerns *m*-term brushlet approximation to functions in the α-modulation space $M_\tau^{\gamma,\alpha}(L_\tau)$, where the error is measure in $M_t^{\beta,\alpha}(L_p)$.

Proposition 6.4. *Given* $0 < \alpha \leq 1$*, let* $\mathcal{B}$ *be a brushlet system associated to a disjoint* α*-covering* $\mathbb{I}$*. Let* $1 < p, t < \infty$ *and* $\beta < \gamma$*. Suppose*

$$\frac{1}{\tau} - \frac{1}{p} = \frac{1}{r} - \frac{1}{t} = \frac{\gamma-\beta}{\alpha},$$

then for every $S \in \Sigma_n(\mathcal{B})$

$$\|S\|_{M_r^{\gamma,\alpha}(L_\tau)} \leq C n^{\frac{\gamma-\beta}{\alpha}} \|S\|_{M_t^{\beta,\alpha}(L_p)}.$$

Proof. Let $S = \sum_{(n,I)\in\Lambda} c_{n,I} w_{n,I}$, with $\#\Lambda = n$, and define for every $I \in \mathbb{I}$, $\Lambda_I := \{n \in \mathbb{N}_0 \colon (n,I) \in \Lambda\}$. Then since $p > \tau$ and $t > r$ we can use Hölders inequality,

$$\|S\|^r_{M_r^{\gamma,\alpha}(L_\tau)} \leq C \sum_{I\in\mathbb{I}} \Bigg(\sum_{n\in\Lambda_I} (|I|^{\frac{\gamma}{\alpha}+\frac{1}{2}-\frac{1}{\tau}} c_{(n,I)})^\tau \Bigg)^{\frac{r}{\tau}}$$

$$
\begin{aligned}
&= C\sum_{I\in\mathbb{I}}\Big(\sum_{n\in\Lambda_I}(|I|^{\frac{\beta}{\alpha}+\frac12-\frac1p}c_{(n,I)})^\tau\Big)^{\frac{r}{\tau}}\\
&\le C\sum_{I\in\mathbb{I}}(\#\Lambda_I)^{r(\frac1\tau-\frac1p)}\Big(\sum_{n\in\Lambda_I}(|I|^{\frac{\beta}{\alpha}+\frac12-\frac1p}c_{(n,I)})^p\Big)^{\frac{r}{p}} \quad (p>\tau)\\
&\le C\Big(\sum_{I\in\mathbb{I}}\#\Lambda_I\Big)^{r(\frac1\tau-\frac1p)}\Big(\sum_{I\in\mathbb{I}}\Big(\sum_{n\in\Lambda_I}(|I|^{\frac{\beta}{\alpha}+\frac12-\frac1p}c_{(n,I)})^p\Big)^{\frac{t}{p}}\Big)^{\frac{r}{t}} \quad (t>r)\\
&= (\#\Lambda)^{r(\frac1\tau-\frac1p)}\|S\|_{M_t^{\beta,\alpha}(L_p)} = n^{r\frac{\gamma-\beta}{\alpha}}\|S\|_{M_t^{\beta,\alpha}(L_p)}.
\end{aligned}
$$

□

Recall that $F_q^{s,\alpha}(L_p)\hookrightarrow M^{s,\alpha}_{\max\{p,q\}}(L_p)$, which gives us

Corollary 6.5. *Given* $0<\alpha\le 1$*, let* $\mathcal{B}$ *be a brushlet system associated to a disjoint* α*-covering* $\mathbb{I}$*. Let* $1<p\le t<\infty$ *and* $\beta<\gamma$*. Suppose*

$$\frac1\tau-\frac1p=\frac1r-\frac1t=\frac{\gamma-\beta}{\alpha},$$

then for every $S\in\Sigma_n(\mathcal{B})$

$$\|S\|_{M_r^{\gamma,\alpha}(L_\tau)}\le Cn^{\frac{\gamma-\beta}{\alpha}}\|S\|_{F_t^{\beta,\alpha}(L_p)}.$$

For a generel $p,t\in(1,\infty)$ we cannot hope for as good a Bernstein inequality as in the previous corollary. However, we have

Proposition 6.6. *Given* $0<\alpha\le 1$*, let* $\mathcal{B}$ *be a brushlet system associated to a disjoint* α*-covering* $\mathbb{I}$*. Let* $1<p<\infty$*,* $1\le t<\infty$*, and* $\beta<\gamma$*. Define* τ *by* $\frac1\tau=\frac{\gamma-\beta}{\alpha}+\frac1p$*. Then, for every* $S\in\Sigma_n(\mathcal{B})$

$$\|S\|_{M_\tau^{\gamma-\frac{1-\alpha}{\tau},\alpha}(L_\tau)}\le Cn^{\frac{\gamma-\beta}{\alpha}}\|S\|_{F_t^{\beta,\alpha}(L_p)}.$$

Proof. Suppose $S=\sum_{(n,I)\in\Lambda}c_{n,I}w_{n,I}$, with $\#\Lambda=n$. Then,

$$
\begin{aligned}
\|S\|^\tau_{M_\tau^{\gamma-(1-\alpha)/\tau,\alpha}(L_\tau)} &\le C\sum_{(n,I)\in\Lambda}(|I|^{\frac{\gamma}{\alpha}+\frac12-\frac{1}{\tau\alpha}}c_{(n,I)})^\tau\\
&= 2C\int_{\mathbb{R}}\sum_{(n,I)\in\Lambda}|I|^{\frac{\tau(\gamma-\beta)}{\alpha}-\frac{1-\alpha}{\alpha}}(|I|^{\frac{\beta}{\alpha}+\frac12}c_{(n,I)})^\tau\chi_{E_{(n,I)}}(x)\,dx\\
&\le 2C\int_{\mathbb{R}}S_t^{\beta,\alpha}(S,x)^\tau\sum_{(n,I)\in\Lambda}|I|^{\frac{\tau(\gamma-\beta)}{\alpha}-\frac{1-\alpha}{\alpha}}\chi_{E_{(n,I)}}(x)\,dx\\
&\le 2C\|S_t^{\beta,\alpha}(S,\cdot)\|^\tau_{L_p}\Big(\int_{\mathbb{R}}\Big(\sum_{(n,I)\in\Lambda}|I|^{\frac{\tau(\gamma-\beta)}{\alpha}-\frac{1-\alpha}{\alpha}}\chi_{E_{(n,I)}}(x)\Big)^{\frac{p}{p-\tau}}dx\Big)^{\frac{p-\tau}{p}}.
\end{aligned}
$$

By Lemma 6.1 we have

$$\int_{\mathbb{R}}\Big(\sum_{(n,I)\in\Lambda}|I|^{\frac{\tau(\gamma-\beta)}{\alpha}-\frac{1-\alpha}{\alpha}}\chi_{E_{(n,I)}}(x)\Big)^{\frac{p}{p-\tau}}dx \le C\int_{\mathbb{R}} I_\Lambda(x)^{\frac{\tau(\gamma-\beta)}{\alpha}\cdot\frac{p}{p-\tau}}dx$$
$$= C\int_{\mathbb{R}} I_\Lambda(x)\,dx \le C\#\Lambda.$$

Thus,

$$\|S\|^\tau_{M_\tau^{\gamma-(1-\alpha)/\tau,\alpha}(L_\tau)} \le C'(\#\Lambda)^{\frac{p-\tau}{p}}\|S_t^{\beta,\alpha}(S,\cdot)\|^\tau_{L_p} = C' n^{\tau\frac{\gamma-\beta}{\alpha}}\|S\|^\tau_{F_t^{\beta,\alpha}(L_p)}.$$

□

7. Some interpolation results

In this section we state and prove interpolation results for the α-modulation and α-Triebel-Lizorkin spaces. The results for the α-modulation spaces were previously proved by Gröbner in [11], but the results on the α-Triebel-Lizorkin spaces are new. First, we have the following proposition, where we use the complex method of interpolation, see e.g. [2], which seems to be the most efficient method for the spaces considered here.

Proposition 7.1. *Suppose* $1 \le p_1, p_2, q_1, q_2 \le \infty$ *and* $\alpha \in [0,1]$. *Define* $1/q = (1-\theta)/q_1 + \theta/q_2$, $1/p = (1-\theta)/p_1 + \theta/p_2$ *and* $s = (1-\theta)s_1 + \theta s_2$. *Then*

$$\big(M_{q_1}^{s_1,\alpha}(L_{p_1}), M_{q_2}^{s_2,\alpha}(L_{p_2})\big)_{[\theta]} = M_q^{s,\alpha}(L_p) \tag{7.1}$$

$$\big(F_{q_1}^{s_1,\alpha}(L_{p_1}), F_{q_2}^{s_2,\alpha}(L_{p_2})\big)_{[\theta]} = F_q^{s,\alpha}(L_p) \tag{7.2}$$

Proof. We prove (7.1) first. We have noticed in Remark 3.10 that $M_{q_i}^{s_i,\alpha}(L_{p_i})$ is a retract of the weighted sequence space $\ell_{q_i}\big(\mathbb{I}, |I|^{\frac{s_i}{\alpha}+\frac{1}{2}-\frac{1}{p_i}}, \ell_{p_i}(\mathbb{N}_0)\big)$, through the operators $\mathcal{J}$ and $\mathcal{P}$ also defined in Remark 3.10. Hence, we have bounded operators,

$$M_{q_1}^{s_1,\alpha}(L_{p_1}) \xrightarrow{\mathcal{J}} \ell_{q_1}\big(\mathbb{I}, |I|^{\frac{s_1}{\alpha}+\frac{1}{2}-\frac{1}{p_1}}, \ell_{p_1}(\mathbb{N}_0)\big) \xrightarrow{\mathcal{P}} M_{q_1}^{s_1,\alpha}(L_{p_1})$$
$$M_{q_2}^{s_2,\alpha}(L_{p_2}) \xrightarrow{\mathcal{J}} \ell_{q_2}\big(\mathbb{I}, |I|^{\frac{s_2}{\alpha}+\frac{1}{2}-\frac{1}{p_2}}, \ell_{p_2}(\mathbb{N}_0)\big) \xrightarrow{\mathcal{P}} M_{q_2}^{s_2,\alpha}(L_{p_2}).$$

Using the complex method of interpolation, we thus obtain

$$\begin{aligned}&\big(M_{q_1}^{s_1,\alpha}(L_{p_1}), M_{q_2}^{s_2,\alpha}(L_{p_2})\big)_{[\theta]}\\ &\quad\xrightarrow{\mathcal{J}} \Big(\ell_{q_1}\big(\mathbb{I}, |I|^{\frac{s_1}{\alpha}+\frac{1}{2}-\frac{1}{p_1}}, \ell_{p_1}(\mathbb{N}_0)\big), \ell_{q_2}\big(\mathbb{I}, |I|^{\frac{s_2}{\alpha}+\frac{1}{2}-\frac{1}{p_2}}, \ell_{p_2}(\mathbb{N}_0)\big)\Big)_{[\theta]}\\ &\quad\xrightarrow{\mathcal{P}} \big(M_{q_1}^{s_1,\alpha}(L_{p_1}), M_{q_2}^{s_2,\alpha}(L_{p_2})\big)_{[\theta]}\end{aligned}$$

Standard results (see e.g. [22, Sec. 1.18.1]) show that

$$\Big(\ell_{q_1}\big(\mathbb{I},|I|^{\frac{s_1}{\alpha}+\frac{1}{2}-\frac{1}{p_1}},\ell_{p_1}(\mathbb{N}_0)\big),\ell_{q_2}\big(\mathbb{I},|I|^{\frac{s_2}{\alpha}+\frac{1}{2}-\frac{1}{p_2}},\ell_{p_2}(\mathbb{N}_0)\big)\Big)_{[\theta]}$$
$$=\big(\ell_q\big(\mathbb{I},|I|^{\frac{s}{\alpha}+\frac{1}{2}-\frac{1}{p}},\ell_p(\mathbb{N}_0)\big),$$

and (7.1) follows. We turn to (7.2). The proof of (7.2) is quite similar to the proof of (7.2) so we only give an outline of it. From the definition of the α-modulation spaces we see that $F^{s_i,\alpha}_{q_i}(L_{p_i})$ is a retract of $L_{p_i}(\ell_{q_i}(\mathbb{N}_0\times\mathbb{I},|I|^{s_i/\alpha+1/2}))$, $i=1,2$. We interpolate the operators inducing the retracts, and it follows that $\big(F^{s_1,\alpha}_{q_1}(L_{p_1}),F^{s_2,\alpha}_{q_2}(L_{p_2})\big)_{[\theta]}$ is a retract of $\big(L_{p_1}(\ell_{q_1}),L_{p_2}(\ell_{q_2})\big)_{[\theta]}$. However, from [22, Sec. 1.18.4] we have

$$\big(L_{p_1}(\ell_{q_1}(\mathbb{N}_0\times\mathbb{I},|I|^{s_1/\alpha+1/2})),L_{p_2}(\ell_{q_2}(\mathbb{N}_0\times\mathbb{I},|I|^{s_2/\alpha+1/2}))\big)_{[\theta]}$$
$$=L_p(\ell_q(\mathbb{N}_0\times\mathbb{I},|I|^{s/\alpha+1/2})),$$

and (7.2) follows. □

7.1. **Additional Jackson and Bernstein estimates.** It is well known that the main tool in the characterization of $\mathcal{A}^\gamma_q(X,\mathcal{B})$ comes from the link between approximation theory and interpolation theory (see e.g. [6, Theorem 9.1, Chapter 7]). Let Y be a Banach space with semi-(quasi)norm $|\cdot|_Y$ continuously embedded in X. Given $r>0$, the Jackson inequality

$$\sigma_m(f,\mathcal{B})_X\le Cm^{-r}|f|_Y,\qquad \forall f\in Y:\forall m\in\mathbb{N},\tag{7.3}$$

and the Bernstein inequality

$$|S|_Y\le C'm^r\|S\|_X,\qquad \forall S\in\Sigma_m(\mathcal{B})\tag{7.4}$$

with constants C and C' independent of f, S and m, imply, respectively, the continuous embeddings

$$(X,Y)_{s/r,q}\hookrightarrow\mathcal{A}^s_q(X,\mathcal{B})$$

and

$$\mathcal{A}^s_q(X,\mathcal{B})\hookrightarrow(X,Y)_{s/r,q}$$

for all $0<s<r$ and $q\in(0,\infty]$.

Now, using the Jackson and Bernstein inequalities from Propositions 6.2 and 6.6 and from Corollary 6.5 we get the following embeddings.

Proposition 7.2. *Given $0<\alpha\le 1$, let $\mathcal{B}$ be a brushlet system associated to a disjoint α-covering $\mathbb{I}$. Let $1<p<\infty$, $1\le t<\infty$, and $\beta<\gamma$. Define τ by $1/\tau-1/p=1/\eta-1/t=(\gamma-\beta)/\alpha$. Suppose $t\ge p$. Then we have the Jackson embedding*

$$\Big(F^{\beta,\alpha}_t(L_p),M^{\gamma,\alpha}_\tau(L_\tau)\Big)_{\frac{s}{\gamma-\beta},q}\hookrightarrow\mathcal{A}^{s/\alpha}_q\big(F^{\beta,\alpha}_t(L_p),\mathcal{B}\big),\qquad s<\gamma-\beta.$$

Conversely, we have the Bernstein estimate

$$\mathcal{A}^{s/\alpha}_q\big(F^{\beta,\alpha}_t(L_p),\mathcal{B}\big)\hookrightarrow\Big(F^{\beta,\alpha}_t(L_p),M^{\gamma,\alpha}_\eta(L_\tau)\Big)_{\frac{s}{\gamma-\beta},q},\qquad s<\gamma-\beta.$$

For $t < p$ we have the weaker embeddings:

$$\left(F_t^{\beta,\alpha}(L_p), M_{\tau'}^{\gamma,\alpha}(L_{\tau'})\right)_{\frac{s}{\gamma-\beta},q} \hookrightarrow \mathcal{A}_q^{s/\alpha}\left(F_t^{\beta,\alpha}(L_p), \mathcal{B}\right) \tag{7.5}$$

and

$$\mathcal{A}_q^{s/\alpha}\left(F_t^{\beta,\alpha}(L_p), \mathcal{B}\right) \hookrightarrow \left(F_t^{\beta,\alpha}(L_p), M_\tau^{\gamma',\alpha}(L_\tau)\right)_{\frac{s}{\gamma-\beta},q} \tag{7.6}$$

for $s < \gamma - \beta$, where $\frac{1}{\tau'} = \frac{1}{\tau} - \frac{1-\alpha}{t\alpha}$ and $\gamma' = \gamma - (1-\alpha)/\tau$.

Remark 7.3. Notice that we are very close to having a complete characterization of the approximation space $\mathcal{A}_q^{s/\alpha}\left(F_t^{\beta,\alpha}(L_p), \mathcal{B}\right)$ for $t \geq p$. For $t < p$ there is a gap between the two sandwiching spaces. This gap grows as $\alpha \to 0$, see Figure 1.

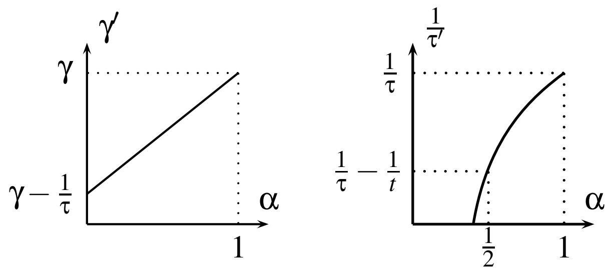


FIGURE 1. The left figure illustrates the decay of the smoothness parameter γ' in (7.6) as $\alpha \to 0$, while the right figure shows how the Lebesgue parameter $1/\tau'$ in (7.5) decays.

We see that the only case where the above result gives a complete characterization is when $\alpha = 1$, that is the case where the α spaces reduce to classical Besov and Triebel-Lizorkin spaces, respectively. We have the following result which concludes the paper.

Corollary 7.4. *With the same hypotheses as in Proposition 7.2 we have*

$$\left(F_t^{\beta,1}(L_p), M_\tau^{\gamma,1}(L_\tau)\right)_{\frac{s}{\gamma-\beta},q} = \mathcal{A}_q^{s}\left(F_t^{\beta,1}(L_p), \mathcal{B}\right), \qquad s < \gamma - \beta. \tag{7.7}$$

REFERENCES

[1] P. Auscher, G. Weiss, and M. V. Wickerhauser. Local sine and cosine bases of Coifman and Meyer and the construction of smooth wavelets. In C. K. Chui, editor, *Wavelets*, pages 237–256. Academic Press, Boston, MA, 1992.

[2] J. Bergh and J. Löfström. *Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften, No. 223*. Springer-Verlag, Berlin, 1976.

[3] L. Borup and M. Nielsen. Approximation with brushlet systems. To appear in J. Approx. Theory, 2003.

[4] R. R. Coifman and Y. Meyer. Remarques sur l'analyse de Fourier à fenêtre. *C. R. Acad. Sci. Paris Sér. I Math.*, 312(3):259–261, 1991.

[5] R. A. DeVore, B. Jawerth, and V. Popov. Compression of wavelet decompositions. *Amer. J. Math.*, 114(4):737–785, 1992.

[6] R. A. DeVore and G. G. Lorentz. *Constructive approximation*. Springer-Verlag, Berlin, 1993.

[7] H. G. Feichtinger. Banach spaces of distributions defined by decomposition methods. II. *Math. Nachr.*, 132:207–237, 1987.

[8] H. G. Feichtinger and P. Gröbner. Banach spaces of distributions defined by decomposition methods. I. *Math. Nachr.*, 123:97–120, 1985.

[9] M. Fornasier. *Constructive methods for numerical applications in signal processing and homogenization problems*. PhD thesis, University of Padova, 2002.

[10] R. Gribonval and M. Nielsen. Some remarks on non-linear approximation with Schauder bases. *East J. Approx.*, 7(3):267–285, 2001.

[11] P. Gröbner. *Banachräume glatter Funktionen und Zerlegungsmethoden*. PhD thesis, University of Vienna, 1992.

[12] K. Gröchenig and S. Samarah. Nonlinear approximation with local Fourier bases. *Constr. Approx.*, 16(3):317–331, 2000.

[13] E. Hernández and G. Weiss. *A first course on wavelets*. CRC Press, Boca Raton, FL, 1996.

[14] S. V. Konyagin and V. N. Temlyakov. A remark on greedy approximation in Banach spaces. *East J. Approx.*, 5(3):365–379, 1999.

[15] G. Kyriazis. Non-linear approximation and interpolation spaces. *J. Approx. Theory*, 113(1):110–126, 2001.

[16] E. Laeng. Une base orthonormale de $L^2(\mathbf{R})$ dont les éléments sont bien localisés dans l'espace de phase et leurs supports adaptés à toute partition symétrique de l'espace des fréquences. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(11):677–680, 1990.

[17] H. Malvar. Lapped transforms for efficient transform /subband coding. *IEEE Trans. Acoustics Speech Signal and Processing*, 38:969–978, 1990.

[18] F. G. Meyer and R. R. Coifman. Brushlets: a tool for directional image analysis and image compression. *Appl. Comput. Harmon. Anal.*, 4(2):147–187, 1997.

[19] Y. Meyer. *Wavelets and operators*. Cambridge University Press, Cambridge, 1992.

[20] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Princeton University Press, Princeton, NJ, 1993.

[21] H. Triebel. *Theory of function spaces*. Birkhäuser Verlag, Basel, 1983.

[22] H. Triebel. *Interpolation theory, function spaces, differential operators*. Johann Ambrosius Barth, Heidelberg, second edition, 1995.

[23] M. V. Wickerhauser. Smooth localized orthonormal bases. *C. R. Acad. Sci. Paris Sér. I Math.*, 316(5):423–427, 1993.

DEPARTMENT OF MATHEMATICAL SCIENCES, AALBORG UNIVERSITY, FREDRIK BAJERS VEJ 7G, DK-9220 AALBORG EAST, DENMARK

E-mail address: `lasse@math.auc.dk and mnielsen@math.auc.dk`