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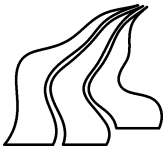

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Numerical solution method for the dbar-equation in the plane

K. Knudsen*, J. Mueller† S. Siltanen‡

Abstract

A fast method for solving $\bar{\partial}$-equations of the form $\bar{\partial}v = T\overline{v}$ is presented, where v and T are complex-valued function of two real variables. The multigrid method of Vainikko [*Direct and inverse problems of mathematical physics*, Int. Soc. Anal. Appl. Comput., 5, Kluwer 2000] is adapted to the problem with a FFT implementation. Convergence with rate $\mathcal{O}(h)$ is proved for the method applied to equations of the form above. One-grid and two-grid versions of the method are implemented and their effectiveness is demonstrated on an application arising in electrical impedance tomography (EIT).

1 Introduction

In this paper we consider the numerical computation of the solution $v : \mathbb{R}^2 \to \mathbb{C}$ to the $\bar{\partial}$-equation

$$\overline{\partial}v(k) = -T(k)\overline{v(k)} \tag{1.1}$$

with the asymptotic condition $\lim_{|k|\to\infty} v(k) = 1$. Here the $\bar{\partial}$-operator is defined by

$$\bar{\partial} = \overline{\partial}_k = \frac{\partial}{\partial \bar{k}} = \frac{1}{2}\left(\frac{\partial}{\partial k_1} + i\frac{\partial}{\partial k_2}\right) \tag{1.2}$$

and the multiplier $T : \mathbb{R}^2 \to \mathbb{C}$ is assumed to have compact support. In the sequel we identify $k \in \mathbb{R}^2$ with $k \in \mathbb{C}$ and use $k = (k_1, k_2)$ and $k = k_1 + ik_2$ interchangeably.

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By convolving in (1.1) with the Green's function $1/\pi k$ for $\bar{\partial}$, it follows that the equation (1.1) together with the asymptotic condition is equivalent to the integral equation

$$v(k) = 1 - \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{T(k')}{k - k'} \overline{v(k')} dk'_1 dk'_2, \qquad k = k'_1 + ik'_2 \in \mathbb{C}, \tag{1.3}$$

or

$$v(k) = 1 - \frac{1}{\pi k} * (T(k)\overline{v(k)}) \tag{1.4}$$

where $*$ denotes convolution. We consider (1.4) as an equation in an appropriate function space on a bounded domain containing the support of T, and to solve this equation numerically we adapt the multigrid method introduced by Vainikko [35]. This method is a fast method based on FFT for solving integral equations with weakly singular kernels.

The equation (1.1) arises in connection with problems in inverse scattering and nonlinear evolution equations. In the context of inverse scattering, the idea behind the $\bar{\partial}$-method is to apply the $\bar{\partial}$-operator to an integral equation that governs the solution of the scattering problem and derive a $\bar{\partial}$-equation that the solution satisfies. This method leads to linear integral equations for reconstructing the eigenfunctions and the potential and also provides necessary conditions which the scattering data must satisfy. The $\bar{\partial}$-method was first used by Beals and Coifman [5] for the quantum inverse scattering problem in 1-D and was extended to 2-D problems in [2] in the context of the Kadomtsev-Petviashvili (KP) equation which has applications in water waves, stratified fluids and plasma physics. See also the references [6, 7, 17, 26, 29] for applications to multidimensional problems. More recently, the $\bar{\partial}$-method has been used in inverse problems in medical imaging including in electrical impedance tomography (EIT) by Nachman [24, 25] and Brown–Uhlmann [10] and in positron-emission tomography (PET) by R. Novikov [27, 28]. Out of the numerous nonlinear evolution equations where equation (1.1) is useful, we single out the Novikov–Veselov equations introduced by Novikov and Veselov [38] and considered by Boiti et al. [9] and Tsai [34] and the Davey–Stewartson equations [14, 1].

The numerical solution of equation (1.1) was considered by Siltanen, Mueller and Isaacson [32] for EIT in the numerical algorithm based on Nachman's uniqueness proof for the 2-D inverse conductivity problem [24]. In [32, 33, 22, 23] equation (1.1) was solved numerically by a 2-D adaptation of the method of product integrals [4]. The numerical solution of equation (1.1) has also been applied to EIT by Knudsen [19] in the numerical algorithm based on the Brown-Uhlmann uniqueness proof for the 2-D inverse conductivity problem

[10]. A fast, direct algorithm for the Lippmann-Schwinger equation in two dimensions is found in [12].

We describe both one- and two-grid implementations of Vainikko's method. Details of this method for the Lippmann-Schwinger equation can be found in [36, 31]. This method has been implemented in [22] for the computation of Faddeev's exponentially growing solutions [16] and in [18] for the numerical computation of the solution to a scattering problem. Furthermore, we will discuss the complexity and accuracy of the method and show that the complexity of the method is $\mathcal{O}(M^2 \log(M))$ for obtaining the solution with accuracy order $\mathcal{O}(h)$, $h = C/M$, on a plane grid of size M^2.

This paper is organized as follows. In section 2 we give the details of the fast algorithm and in section 3 we describe the two–grid extension. Then in section 4 we analyze the accuracy and complexity of the method. Finally in section 5 we provide an example of the application of the method to electrical impedance tomography.

2 The fast algorithm

In this section we describe a fast algorithm for numerical solution of the integral equation (1.3). The method is based on the work of Vainikko [35]. Assume that T satisfies the following:

A1 *Properties of T.* Denote by $C^{0,1}(G)$ the space of Lipschitz continuous functions on $G \subset \mathbb{R}^2$. We assume that T is compactly supported in an open set Ω and that $T \in C^{0,1}(\Omega \setminus \Gamma)$. Here $\Gamma = \cup_{i=1}^{N} \Gamma_i$, where Γ_i are compact, piecewise C^1 curves for which $\Gamma_i \cap \Gamma_j$ is a discrete set whenever $i \neq j$.

With such T the equation (1.3) has a unique solution $v \in C^{0,1}(\mathbb{R}^2)$ [35].

The fast algorithm is based on the following crucial observations:

- If we know $v(k)$ for $k \in \mathrm{supp}(T)$, we can compute v in the whole plane by $v = 1 - (\pi k)^{-1} * (T\bar{v})$.
- Let $\rho > 0$ be such that $\mathrm{supp}(T) \subset B(0, \rho)$, where $B(0, \rho)$ is the open disc with radius ρ and center at the origin. If $k \in \mathrm{supp}(T)$ then the integral in (1.4) does not involve the values of the convolution kernel $(\pi k')^{-1}$ for $|k'| \geq 2\rho$.

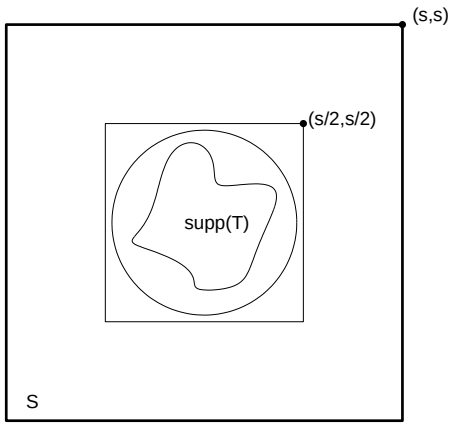

Figure 1: The large square S determines the periodization of the $\bar{\partial}$-equation. The circle and both of the squares are centered at the origin, and the radius of the circle is ρ.

We will consider a periodic equation allowing fast solution and giving the solution to the full-plane equation. Choose $s > 2\rho$ and set $S := (-s, s)^2$; see Figure 1. Consider the following equation for functions that are $2s$-periodic in k_1 and k_2:

$$w(k) = f(k) - \int_{-s}^{s}\int_{-s}^{s} g(k-k')T(k')\overline{w(k')}dk'_1dk'_2, \tag{2.1}$$

or more briefly

$$[I + g * (T\,\cdot^{-})]w = f, \tag{2.2}$$

where $*$ denotes convolution on the torus and f satisfies the following:

A2 *Smoothness of f.* We assume that f is $2s$-periodic in k_1, k_2 and Lipschitz.

Clearly, $f \equiv 1$ satisfies A2. The $2s$-periodic function g appearing in (2.1) and (2.2) is given by $g(k) = (\pi k)^{-1}$ for $k \in S$. (It is also possible to truncate g sharply or smoothly at $|k| = 2\rho$). It is easily checked that

$$v|_{\mathrm{supp}(T)} = w|_{\mathrm{supp}(T)},$$

and that unique solvability of (1.3) is equivalent to that of (2.1) with $f \equiv 1$.

Next we discretize equation (2.2). Choose a positive integer m, denote $M = 2^m$, and set $h = 2s/M$. Define a grid $\mathcal{G}_m \subset \overline{S}$ by

$$\begin{aligned} \mathcal{G}_m &= \{jh \,|\, j \in \mathbb{Z}^2_m\}, \\ \mathbb{Z}^2_m &= \{j = (j_1, j_2) \in \mathbb{Z}^2 \,|\, -2^{m-1} \le j_l < 2^{m-1},\ l = 1, 2\}. \end{aligned} \tag{2.3}$$

Note that the number of points in $\mathcal{G}_m$ is M^2. To each grid point $x \in \mathcal{G}_m$ we attach a *cell*, an open set that contains x. The cells are given by

$$B_{j,h} = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, (j_l - \frac{1}{2})h < x_l < (j_l + \frac{1}{2})h,\ l = 1, 2\} \tag{2.4}$$

for $j \in \mathbb{Z}^2_m$.

Let $\varphi : \mathbb{R}^2 \to \mathbb{C}$ be a $2s$-periodic function that is continuous in a neighborhood of ∂S, and in $S \setminus \Gamma$, where Γ is as in assumption A1. Define the grid approximation $\varphi_h : \mathbb{Z}^2_m \to \mathbb{C}$ of φ by

$$\varphi_h(j) = \begin{cases} \varphi(jh), & \text{if} \quad B_{j,h} \cap \Gamma = \emptyset, \\ h^{-2} \sum_{p=1}^{P_j} \varphi(x^{(p)}_{j,h}) |B^{(p)}_{j,h}| & \text{otherwise,} \end{cases} \tag{2.5}$$

where the sets $B^{(p)}_{j,h}$ for $p = 1, \ldots, P_j$ are the connected components of the open set $B_{j,h} \setminus \Gamma$, the evaluation point $x^{(p)}_{j,h} \in B^{(p)}_{j,h}$ is arbitrary, and $|\cdot|$ denotes the Lebesgue measure. However, the Green's function $(\pi k)^{-1}$ is singular at $k = 0$, and (2.5) cannot be readily used. We set

$$g_h(j) = \begin{cases} g(jh), & \text{for } j \in \mathbb{Z}^2_m \setminus 0, \\ 0, & \text{for } j = 0, \end{cases} \tag{2.6}$$

and since the singularity at $k = 0$ is integrable, the error caused by (2.6) becomes small when the discretization is refined.

We discretize the periodic convolution operator

$$(A\varphi)(k) = (g * \varphi)(k) = \int_{-s}^{s} \int_{-s}^{s} g(k - k')\varphi(k') dk'_1 dk'_2 \tag{2.7}$$

with the formula

$$A_h \varphi_h = \mathcal{F}^{-1}\big(\mathcal{F} g_h \cdot \mathcal{F} \varphi_h\big), \tag{2.8}$$

where $\mathcal{F}$ is the discrete Fourier transform and $\cdot$ denotes component-wise multiplication. The practical value of formula (2.8) is that application of the operator A_h can be implemented using the fast Fourier transform.

The discrete version of (2.2) is now

$$[I + A_h(T_h \cdot {}^{-})]w_h = f_h, \tag{2.9}$$

where $T_h \cdot$ denotes component-wise multiplication by the matrix $[T_h(j)]$. Solvability of this equation (for sufficiently large m) is a consequence of the solvability of (2.1). To avoid explicit representation of the inverse operator

$[I + A_h(T_h \cdot {}^-)]^{-1}$ in the numerical solution of (2.9) we employ an iterative method, such as GMRES [30].

Note that since the operator $[I + A_h(T_h \cdot {}^-)]$ is real-linear but not complex linear, the real and imaginary parts of the complex solution vector w_h must be kept separate when using GMRES. An algorithm which avoids the use of the equivalent linear system of doubled size is found in [15].

3 Two-grid extension of the algorithm

We show how a two-grid method gives extra resolution with reasonable computational expense compared to solving (2.9) iteratively on one grid.

We construct a two grid scheme with one fine grid and one coarse grid. Choose $0 < m^* < m$ and define the fine grid $\mathcal{G}_m$ by (2.3) and its cells by (2.4). We next introduce the coarse grid $\mathcal{G}_{m^*}$ and its *panels* $B^*_{j^*,h^*}$ (we call the cells of the coarse grid panels). The following two requirements must be fulfilled [35, section 5.12]:

- R1 Every point of the coarse grid must belong to the fine grid: $\mathcal{G}_{m^*} \subset \mathcal{G}_m$.
- R2 Every cell $B_{j,h}$ of the fine grid must be contained in some panel B_{j^*,h^*}, and conversely, the closure of any panel must be the union of some collection of closures of fine grid cells.

Denote $M^* = 2^{m^*}$ and set $h^* = 2s/M^*$; then $0 < h < h^*$. Define the coarse grid $\mathcal{G}_{m^*}$ equivalently to (2.3):

$$\mathcal{G}_{m^*} := \{\, j^*h^* \,|\, j^* \in \mathbb{Z}^2_{m^*} \}. \tag{3.1}$$

Requirement R1 clearly holds: the coarse grid is an equispaced collection of points in $\mathcal{G}_m$. The definition of panels must, however, be different from the definition of fine grid cells: the coarse grid points cannot be centerpoints of square cells without violating R2. We give the following definition as a compromise:

$$B^*_{j^*,h^*} := \mathrm{int}\Big(\bigcup_{j \in J_{j^*}} \overline{B}_{j,h} \Big), \tag{3.2}$$

where int denotes topological interior and

$$J_{j^*} := \{ j \in \mathbb{Z}^2_m \,|\, j^*_l \le j_l < j^*_l + 2^{m-m^*}h,\ l = 1,2 \}.$$

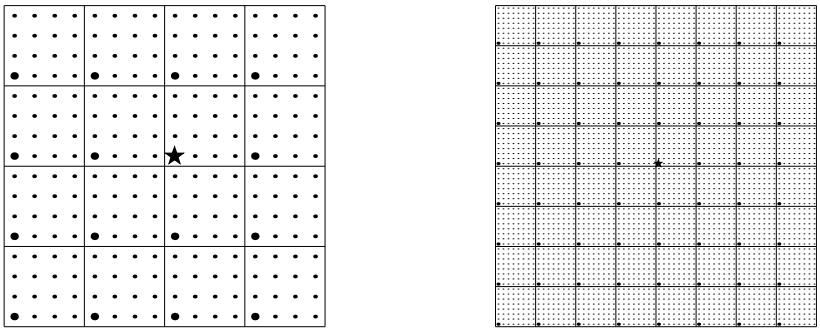

Figure 2: Left: coarse and fine grid of the two-grid method corresponding to $m^* = 2$ and $m = 4$. Small dots denote the points of the fine grid and big dots denote the points of the coarse grid. Star denotes the origin (that belongs to both grids). Lines indicate boundaries of panels. Note that the coarse grid collocation points are regularly distributed but are not at the center of panels. Right: same as left plot but $m^* = 3$ and $m = 6$.

An illustration of the coarse and fine grids is found in figure 2. Definitions (2.5) and (2.6) apply for the coarse grid just by changing j to j^* and h to h^*.

We require operators for transitions from one grid to the other. The operator p_{hh^*} taking functions $\mathbb{Z}^2_{m^*} \to \mathbb{C}$ on the coarse grid to functions $\mathbb{Z}^2_m \to \mathbb{C}$ on the fine grid is defined by piecewise constant interpolation:

$$(p_{hh^*} w_{h^*})(j) = w_{h^*}(j^*) \text{ for unique } j^* \in \mathbb{Z}^2_{m^*} \text{ for which } B_{j,h} \subset B^*_{j^*,h^*}. \quad (3.3)$$

The fine-to-coarse operator p_{h^*h} is simply restriction:

$$(p_{h^*h} w_h)(j^*) = w_h(j) \text{ for unique } j \in \mathbb{Z}^2_m \text{ for which } jh = j^*h^*. \quad (3.4)$$

The two-grid scheme is based on the observation that the solution w_h of (2.9) satisfies

$$w_h = f''_{h,h^*} - \mathcal{T}''_{h,h^*} w_h \quad (3.5)$$

where

$$f''_{h,h^*} = f_h - p_{hh^*}(I_h + A_{h^*}(T_{h^*} \cdot \ ^-))^{-1} p_{h^*h} A_{h^*}(T_{h^*} \cdot \ ^-) f_h.$$

and the operator $\mathcal{T}''_{h,h^*}$ is defined by

$$\mathcal{T}''_{h,h^*} = [I_h - p_{hh^*}(I_{h^*} + A_{h^*}(T_{h^*} \cdot \ ^-))^{-1} p_{h^*h}(I_h + A_h(T_h \cdot \ ^-))] A_h(T_h \cdot \ ^-). \quad (3.6)$$

The point is that when h^* is sufficiently small then $I + \mathcal{T}''_{h,h^*}$ can be inverted by a Neumann series (see Lemma 4.1 below), and hence the solution to (3.5) can be computed by

$$w_h = (I_h + \mathcal{T}''_{h,h^*})^{-1} f''_{h,h^*} = \sum_{n=0}^{\infty} (-\mathcal{T}''_{h,h^*})^n f''_{h,h^*}.$$

This gives the following two-grid scheme for computing an approximate solution to (2.9):

$$\begin{array}{ll} w_h^0 & = p_{hh^*} w_{h^*}, \\ \texttt{For} & n = 0 : (N-1) \\ & r_h^n = w_h^n + A_h(T_h \cdot \overline{w_h^n}) - f_h \\ & w_h^{n+1} = w_h^n - r_h^n - p_{hh^*}(I_{h^*} + A_{h^*}(T_{h^*} \cdot \ ^-))^{-1} p_{h^*h} A_h(T_h \cdot r_h^n) \end{array} \tag{3.7}$$

Here m^* has to be large enough for the inverse of the coarse grid operator $(I_{h^*} + A_{h^*}(T_{h^*} \cdot ^-))$ to exist. This operator is then inverted using the one-grid approximation described in section 2. The iteration number N is related to the accuracy of the method and will be discussed in section 4.

4 Accuracy and complexity of the method

The one-grid method described in section 2 is inspired by method (5.13) in [35], but since the integral operator in (2.2) is conjugate linear (and not linear) and the kernel T is allowed to have discontinuities across curves, the convergence rates in Theorem 5.1 of [35] are not immediate. However, the expected convergence rates follow from a proof analogous to that of Theorem 5.1 of [35]. We will outline the main steps below.

The theory of discrete convergence is used in [35] to analyze the convergence of the discretization method. We also adopt this language for the proofs. The following definitions are taken from [35].

Definition 4.1 *Let E be a Banach space and let E_h be a family of Banach spaces parameterized by $h \geq 0$ (all spaces real or all complex). Let $p_h \in \mathcal{B}(E, E_h)$ be the so-called connection operators satisfying*

$$\|p_h u\|_{E_h} \to \|u\|_E.$$

A family $\{u_h\}_{0<h<H}$ of elements $u_h \in E_h$ is called discretely convergent to an element $u \in E$ if $\|u_h - p_h u\|_{E_h} \to 0$ as $h \to 0$.

Definition 4.2 *A family $\{u_h\}$ of elements $u_h \in E_h$ is called discretely compact if any sequence $\{u_h\}$ formed by the elements of the family with $h_n \to 0$ contains a discretely convergent subsequence.*

Definition 4.3 *A family of linear bounded operators $T_h \in L(E_h, E_h)$ is called discretely convergent to $T \in L(E, E)$ if the following implication holds: discrete convergence of a family of elements $\{u_h\} \in E_h$ to an element $u \in E$ implies discrete convergence of $T_h u$ to Tu.*

Definition 4.4 *The bounded linear operator T_h converges to T discretely compactly if the implication in definition 4.3 holds and*

$$\limsup_{h\to 0} \|u_h\|_{E_h} < \infty \quad \textit{implies} \quad \{T_h u_h\} \quad \textit{is discretely compact.} \tag{4.1}$$

We extend definitions 4.3 and 4.4 to include conjugate linear operators. Note that Lemmas 4.1, 4.2 and Theorem 4.1 of [35] hold for conjugate linear operators as well.

For our purpose we let $E = C(S)$, the set of continuous functions on the set S equipped with the supremum norm, and let $E_h = \mathcal{M}_{2^m \times 2^m}$ be the space of $2^m \times 2^m$ matrices equipped with the supremum norm. Further, we define the connection operators $p_h \in \mathcal{B}(E, E_h)$ by $(p_h\phi)(j) = \phi(jh),\ j \in \mathbb{Z}_m^2$.

Theorem 4.1 *Let conditions A1 and A2 hold for equation (2.2). Then there exists $m_0 > 0$ such that for $m > m_0$ the system (2.9) has a unique solution w_h with*

$$\max_{j\in\mathbb{Z}_m^2} |w_h(j) - w(jh)| \le Ch, \tag{4.2}$$

where $h = 2s/2^m$ and w is the unique solution to (2.2).

Proof. Define the operators $\mathcal{T} \in \mathcal{B}(E), \mathcal{T}_h, \mathcal{T}_h'' \in \mathcal{B}(E_h)$ by

$$\begin{aligned}
\mathcal{T}u(k) &= \frac{1}{\pi}\int_S \frac{T(k')}{k-k'}\overline{u(k')}dk_1'dk_2', \quad k \in S \\
\mathcal{T}_h u_h(j) &= \frac{1}{\pi}\sum_{l\in\mathbb{Z}_m^2}\int_{B_{l,h}} \frac{T(k')}{jh-k'}dk_1'dk_2'\overline{u_h(l)}, \\
\mathcal{T}_h'' u_h(j) &= h^2\sum_{l\in\mathbb{Z}_m^2} g_h(j-l)T_h(l)\overline{u_h(l)}.
\end{aligned}$$

Note that these operators are compact. The proof is divided into three steps:

1. Show the existence of $h_0 > 0$ such that for $0 < h < h_0$ the system (2.9) has a unique solution w_h with

$$\max_{j \in \mathbb{Z}_m^2} |w_h(j) - w(jh)| \leq C \|\mathcal{T}_h'' p_h w - p_h \mathcal{T} w\|_{E_h}. \tag{4.3}$$

2. Show that

$$\|\mathcal{T}_h p_h w - p_h \mathcal{T} w\|_{E_h} \leq Ch \tag{4.4}$$

for the solution $w \in E$ to (2.2).

3. Show that

$$\|\mathcal{T}_h - \mathcal{T}_h''\|_{\mathcal{B}(E_h)} \leq Ch. \tag{4.5}$$

It is clear that (4.2) follows from (4.3) – (4.5) by the triangle inequality.

Claim (1) follows from [35, Theorem 4.1], since $\mathcal{T}_h \to \mathcal{T}$ discretely compactly. For a complete proof of this fact we refer to the proof of [35, Lemma 5.2], which is easily adapted to our setting.

To prove (4.4) we use the fact that the solution w to (2.2) is Lipschitz continuous in S. Hence it follows that

$$\begin{aligned}
\|\mathcal{T}_h p_h w - p_h \mathcal{T} w\|_{E_h} &= \max_{j \in \mathbb{Z}_m^2} \left| \sum_{l \in \mathbb{Z}_m^2} \int_{B_{l,h}} \frac{T(k')}{jh - k'} \overline{(p_h w(l) - w(k'))} dk_1' dk_2' \right| \\
&\leq \max_{k \in S} |T(k)| \int_S \frac{1}{|k - k'|} dk_1' dk_2' \max_{\substack{l \in \mathbb{Z}_m^2 \\ z_1, z_2 \in B_{l,h}}} |w(z_1) - w(z_2)| \\
&\leq C \max_{\substack{l \in \mathbb{Z}_m^2, \\ z_1, z_2 \in B_{l,h}}} |z_1 - z_2| \\
&\leq Ch.
\end{aligned}$$

Concerning (4.5) note that

$$\|\mathcal{T}_h - \mathcal{T}_h''\|_{\mathcal{B}(E_h)} = \max_{j \in \mathbb{Z}_m^2} \sum_{l \in \mathbb{Z}_m^2} |\mathcal{T}_{jl,h} - \mathcal{T}_{jl,h}''|,$$

where for $j, l \in \mathbb{Z}_m^2$

$$\begin{aligned}
\mathcal{T}_{jl,h} &= \int_{B_{l,h}} \frac{T(k')}{jh - k'} dk_1' dk_2', \\
\mathcal{T}_{jl,h}'' &= \begin{cases} h^2 \frac{T_h(l)}{jh - lh}, & l \neq j, \\ 0, & l = j. \end{cases}
\end{aligned}$$

If $B_{l,h} \cap \Gamma = \emptyset$ then

$$\begin{aligned}
|\mathcal{T}_{jl,h} - \mathcal{T}''_{jl,h}| &= \left| \int_{B_{l,h}} \left(\frac{T(k')}{jh - k'} - \frac{T_h(l)}{jh - lh} \right) dk'_1 dk'_2 \right| \\
&\leq \int_{B_{l,h}} \left| \frac{T(k')}{jh - k'} - \frac{T_h(l)}{jh - lh} \right| dk'_1 dk'_2 \\
&\leq C \int_{B_{l,h}} |lh - k'| dk'_1 dk'_2 \\
&\leq Ch^3,
\end{aligned}$$

since $T(k')/(jh - k')$ is Lipschitz continuous on the cell $B_{l,h}, j \neq l$ defined in (2.4). If $B_{l,h} \cap \Gamma \neq \emptyset$ then we find (see (2.5) for the notation)

$$\begin{aligned}
|\mathcal{T}_{jl,h} - \mathcal{T}''_{jl,h}| &= \left| \int_{B_{l,h}} \left(\frac{T(k')}{jh - k'} - \frac{T_h(l)}{jh - lh} \right) dk'_1 dk'_2 \right| \\
&= \left| \sum_{p=1}^{P_j} \int_{B^{(p)}_{j,h}} \left(\frac{T(k')}{jh - k'} - \frac{T(x^{(p)}_{j,h})}{j - l} \right) dk'_1 dk'_2 \right| \\
&= \sum_{p=1}^{P_j} \int_{B^{(p)}_{j,h}} \left(\left| \frac{T(k')}{jh - k'} - \frac{T(x^{(p)}_{j,h})}{jh - x^{(p)}_{j,h}} \right| + \left| \frac{T(x^{(p)}_{j,h})}{jh - x^{(p)}_{j,h}} - \frac{T(x^{(p)}_{j,h})}{jh - lh} \right| \right) dk'_1 dk'_2 \\
&\leq \sum_{p=1}^{P_j} \int_{B^{(p)}_{j,h}} \left(C|k' - x^{(p)}_{j,h}| + |T(x^{(p)}_{j,h})| \left| \frac{1}{jh - x^{(p)}_{j,h}} - \frac{1}{jh - lh} \right| \right) dk'_1 dk'_2 \\
&\leq C \sum_{p=1}^{P_j} \int_{B^{(p)}_{j,h}} (|k' - x^{(p)}_{j,h}| + |x^{(p)}_{j,h} - lh|) dk'_1 dk'_2 \\
&\leq Ch^3,
\end{aligned}$$

using the Lipschitz continuity of $T(k')/(jh - k')$ in the first term and the Lipschitz continuity of $1/k$ in the second term. Moreover,

$$|\mathcal{T}_{jj,h}| = \left| \int_{B_{j,h}} \frac{T(k')}{jh - k'} dk'_1 dk'_2 \right| \leq \max_{k \in B_{j,h}} |T(k)| \int_{|k'| < \sqrt{2}h} \frac{1}{|k'|} dk'_1 dk'_2 \leq Ch.$$

Hence

$$\begin{aligned}
\|\mathcal{T}_h - \mathcal{T}''_h\|_{\mathcal{B}(E_h)} &= \max_{j \in \mathbb{Z}^2_m} (|\mathcal{T}_{jj,h}| + \sum_{l \neq j} |\mathcal{T}_{jl,h} - \mathcal{T}''_{jl,h}|) \\
&\leq Ch + CM^2 h^3 \\
&\leq Ch.
\end{aligned}$$

□

Recall the definition of the operator $\mathcal{T}''_{h,h^*}$ from (3.6) and note that

$$\mathcal{T}''_{h,h^*} = (I_h - p_{hh^*}p_{h^*h})\mathcal{T}''_h + p_{hh^*}(I_{h^*} + \mathcal{T}''_{h^*})^{-1}(p_{h^*h}\mathcal{T}''_h - \mathcal{T}''_{h^*}p_{h^*h})\mathcal{T}''_h.$$

Define $\mathcal{T}_{h,h^*}$ by

$$\begin{aligned}\mathcal{T}_{h,h^*} &= [I_h - p_{hh^*}(I_{h^*} + \mathcal{T}_{h^*})^{-1}p_{h^*h}(I_h + \mathcal{T}_h)]\mathcal{T}_h \\ &= (I_h - p_{hh^*}p_{h^*h})\mathcal{T}_h + p_{hh^*}(I_{h^*} + \mathcal{T}_{h^*})^{-1}(p_{h^*h}\mathcal{T}_h - \mathcal{T}_{h^*}p_{h^*h})\mathcal{T}_h\end{aligned}$$

We first prove a lemma needed to establish the convergence rate of the two-grid method.

Lemma 4.1 $\|\mathcal{T}_{h,h^*}\|_{\mathcal{B}(E_h,E_{h^*})} \le Ch^*$.

Proof. For $w_h \in E_h$, define the piecewise constant function $\hat{w}_h \in E$ by

$$\hat{w}_h(k) = \sum_{j\in\mathbb{Z}_m^2} w_h(j)\xi_{j,h}(k), \tag{4.6}$$

where

$$\xi_{j,h}(k) = \begin{cases} 1, & \text{if} \quad k \in B_{j,h}, \\ 0, & \text{otherwise.}\end{cases} \tag{4.7}$$

Then $\|\hat{w}_h\|_E = \|w_h\|_{E_h}$ and

$$\mathcal{T}_h w_h(j) = \frac{1}{\pi}\int_S \frac{T(k')}{jh - k'}\overline{\hat{w}_h(k')}dk'_1dk'_2.$$

Note that

$$\begin{aligned}|\mathcal{T}_h u_h(s_1) - \mathcal{T}_h u_h(s_2)| &= \frac{1}{\pi}\left|\int_S \frac{T(k')}{s_1 - k'}\overline{u_h(k')}dk'_1dk'_2 - \int_S \frac{T(k')}{s_2 - k'}\overline{u_h(k')}dk'_1dk'_2\right| \\ &\le \|u_h\|_E\frac{1}{\pi}\int_S\left|\frac{T(k')}{s_1 - k'} - \frac{T(k')}{s_2 - k'}\right|dk'_1dk'_2 \\ &\le C\|u_h\|_E|s_1 - s_2|\end{aligned}$$

So in particular we have

$$|\mathcal{T}_h\hat{w}_h(lh) - \mathcal{T}_h\hat{w}_h(jh)| \le C\|w_h\|_E|lh - jh|. \tag{4.8}$$

Thus,

$$\begin{aligned}|\mathcal{T}_h\hat{w}_h(lh) - \mathcal{T}_h\hat{w}_h(jh)| &= \frac{1}{\pi}\left|\int_S \frac{T(k')}{lh-k'}\overline{\hat{w}_h(k')}dk_1'dk_2' - \int_S \frac{T(k')}{jh-k'}\overline{\hat{w}_h(k')}dk_1'dk_2'\right| \\ &\le C\|w_h\|_E|lh-jh| \\ &\le C\|w_h\|_E h\end{aligned}$$

So we have

$$|\mathcal{T}_h w_h(lh) - \mathcal{T}_h w_h(jh)| \le C\|w_h\|_E h. \tag{4.9}$$

By (3.4)

$$|(I_h - p_{hh^*}p_{h^*h})\mathcal{T}_h w_h(j)| = |\mathcal{T}_h w_h(j) - p_{hh^*}\mathcal{T}_h w_h(j)|$$

where j is the unique $j \in \mathbb{Z}_m^2$ such that $jh = j^*h^*$. By (3.3)

$$|\mathcal{T}_h w_h(j) - p_{hh^*}\mathcal{T}_h w_h(j)| = |\mathcal{T}_h w_h(j) - \mathcal{T}_h w_h(j^*)|$$

where j^* is the unique $j^* \in \mathbb{Z}_{m^*}^2$ such that $B_{j,h} \subset B_{j,h}^*$. Then by (4.9)

$$|(I_h - p_{hh^*}p_{h^*h})\mathcal{T}_h w_h(j)| \le C\|w_h\|_E h^*$$

since j and j^* are in the same panel. Thus,

$$\|(I_h - p_{hh^*}p_{h^*h})\mathcal{T}_h\|_{\mathcal{B}(E_h)} \le Ch^*. \tag{4.10}$$

Now let v_h denote $\mathcal{T}_h w_h$.

$$\begin{aligned}((p_{h^*h}\mathcal{T}_h - \mathcal{T}_{h^*}p_{h^*h})v_h)(j^*) &= \mathcal{T}_h v_h(j^*) - \mathcal{T}_{h^*}p_{h^*h}v_h(j^*) \\ &= \frac{1}{\pi}\sum_{l\in\mathbb{Z}_m^2}\int_{B_{j,h}} \frac{T(k')}{j^*h-k'}dk_1'dk_2'v_h(l) \\ &\quad - \frac{1}{\pi}\sum_{i^*\in\mathbb{Z}_{m^*}^2}\int B_{j^*,h^*}^* \frac{T(k')}{j^*h^*-k'}dk_1'dk_2'v_h(i^*) \\ &= \frac{1}{\pi}\sum_{l\in\mathbb{Z}_m^2}\int_{B_{j,h}} \frac{T(k')}{j^*h-k'}dk_1'dk_2'(v_h(l) - v_h(l^*))\end{aligned}$$

where l^* is defined by (3.4). Thus,

$$\begin{aligned}|((p_{h^*h}\mathcal{T}_h - \mathcal{T}_{h^*}p_{h^*h})v_h)(j^*)| &\le \left(\sup_{k\in S}\int_S \left|\frac{T(k')}{k-k'}\right|dk_1'dk_2'\right)C\|w_h\|_E h^* \\ &\le C\|w_h\|_E h^*.\end{aligned}$$

This proves the lemma. □

Concerning the complexity of the one-grid method we note that with an FFT implementation one application of the discrete convolution operator (2.8) is done in $\mathcal{O}(M^2 \log M)$ arithmetical operations. Hence to solve (2.9) using an iterative solver with a fixed upper bound on the numbers of iterations also requires $\mathcal{O}(M^2 \log M)$ arithmetical operations.

The following theorem gives the convergence rate of the two-grid method.

Theorem 4.2 *Let conditions A1 and A2 hold for equation (2.2). Then for sufficiently small $h^* > 0$ and $0 < h < h^*$, the function w_h^N in (3.7) satisfies*

$$\max_{j \in \mathbb{Z}_m^2} |w_h^N(j) - w_h(j)| \leq \max_{j \in \mathbb{Z}_m^2} |w_h^0(j) - w_h(j)| (Ch^*)^N, \tag{4.11}$$

where $N = 0, 1, 2, \ldots,$ and w_h is the unique solution to (2.9).

Proof. By lemma 4.2 [35], for sufficiently small h, the operator $I_h + \mathcal{T}_h$ is invertible and the inverse is uniformly bounded. Thus, there exist $h_1 > 0$ and c_1 constant such that

$$\|(I_h + \mathcal{T}_h)^{-1}\|_{\mathcal{B}(E_h)} \leq c_1, \quad 0 < h < h_1.$$

By (4.5),

$$\|\mathcal{T}_h'' - \mathcal{T}_h\|_{\mathcal{B}(E_h)} \leq Ch < Ch^*. \tag{4.12}$$

By the Lipschitz continuity of w,

$$\|p_{hh^*}(I_{h^*} + \mathcal{T}_{h^*})^{-1} p_{h^*h}(I_h + \mathcal{T}_h)\|_{\mathcal{B}(E_h)} \leq Ch^*,$$

and so

$$\|\mathcal{T}_h - \mathcal{T}_{h,h^*}\|_{\mathcal{B}(E_h)} \leq \|\mathcal{T}_h\|_{\mathcal{B}(E_h)} \|p_{hh^*}(I_{h^*} + \mathcal{T}_{h^*})^{-1} p_{h^*h}(I_h + \mathcal{T}_h)\|_{\mathcal{B}(E_h)} \leq Ch^*. \tag{4.13}$$

Similarly,

$$\|\mathcal{T}_{h,h^*}'' - \mathcal{T}_h''\|_{\mathcal{B}(E_h)} \leq \|\mathcal{T}_h''\|_{\mathcal{B}(E_h)} \|p_{hh^*}(I_{h^*} + \mathcal{T}_{h^*}'')^{-1} p_{h^*h}(I_h + \mathcal{T}_h'')\|_{\mathcal{B}(E_h)} \leq Ch^*. \tag{4.14}$$

Then it follows from (4.12), (4.13) and (4.14) that

$$\|\mathcal{T}_{h,h^*}'' - \mathcal{T}_{h,h^*}\|_{\mathcal{B}(E_h)} \leq Ch^*. \tag{4.15}$$

Thus, (4.15) and lemma 4.1 imply

$$\|\mathcal{T}_{h,h^*}''\|_{\mathcal{B}(E_h)} \leq \|\mathcal{T}_{h,h^*}'' - \mathcal{T}_{h,h^*}\|_{\mathcal{B}(E_h)} + \|\mathcal{T}_{h,h^*}\|_{\mathcal{B}(E_h)} \leq Ch^*.$$

If $Ch^* < 1$, then $\|\mathcal{T}''_{h,h^*}\| < 1$ and (3.5) is uniquely solvable. Thus, the iteration scheme (3.7) converges with rate of convergence

$$\|w_h^n - w_h\|_{E_h} \leq \|w_h^0 - w_h\|_{E_h}(Ch^*)^n, \quad n = 0, 1, 2, \ldots$$

where C is independent of h and h^*. This proves the theorem. □

We analyze the work necessary to attain accuracy $\mathcal{O}(h)$ for the two-grid method as for the one-grid method, following the analysis in [35]. The choice of the initial guess in (3.7) gives

$$\max_{j \in \mathbb{Z}_m^2} |w_h^0(j) - w_h(j)| \leq Ch^*,$$

so by Theorem 4.2 we have

$$\max_{j \in \mathbb{Z}_m^2} |w_h^N(j) - w_h(j)| \leq h^q \quad q \geq 1,$$

for

$$N \geq \frac{q|\log h|}{|\log h^* + \log C|} - 1.$$

Choose $h^* \sim h^\tau$ where $0 < \tau < 1$. Then the desired accuracy will be reached asymptotically in $N = [q/\tau] - 1$ steps where $[z]$ denotes the smallest integer larger than z. Thus, choosing $h^* \sim h^{1/3}$, one can obtain accuracy $\mathcal{O}(h)$ in three iterations ($N = 3$).

Note that the complexity of the two-grid method is also $\mathcal{O}(M^2 \log(M))$, since the algorithm (3.7) involves the application of the discrete convolution operator (2.8) on the fine grid.

5 Application to the inverse conductivity problem

The inverse conductivity problem is the mathematical problem behind a recent method for medical imaging called Electrical Impedance Tomography (EIT). The problem is to recover a bounded and strictly positive conductivity γ in a body Ω from static electric measurements on the boundary of the body. See, for example, [13] for further information about EIT. We consider here the two-dimensional problem, i.e. $\Omega \subset \mathbb{R}^2$. If we apply a voltage potential $f \in H^{1/2}(\partial\Omega)$ on the boundary of Ω, a voltage distribution $u \in H^1(\Omega)$ is induced in Ω described uniquely as the solution to the conductivity equation

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \qquad u = f \text{ on } \partial\Omega. \tag{5.1}$$

The voltage potential in Ω gives rise to a current flux through the boundary given by

$$\gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

which can be measured at the boundary. All possible boundary measurements are encoded in the map

$$\Lambda_\gamma : f \mapsto \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

the so-called Dirichlet-to-Neumann map or voltage-to-current map that maps any voltage distribution on the boundary to the resulting current flux.

The inverse conductivity problem as considered by Calderón [11] consists of two questions. First, does the Dirichlet-to-Neumann map Λ_γ determine the conductivity γ uniquely, and then if so, how can the conductivity be reconstructed?

For sufficiently regular conductivities having essentially two derivatives Nachman [24] gave a uniqueness proof and a reconstruction algorithm, and for less regular conductivities with only one derivative Brown-Uhlmann [10] generalized the uniqueness result. Both methods are based on solving $\overline{\partial}$-equations of the type (1.1). In this section we will outline a reconstruction method based on the latter approach and show how the implementations described above can be used in this context.

Let u be a solution to the conductivity equation (5.1) and identify $(z_1, z_2) \in \Omega$ with the complex number $z = z_1 + iz_2$. Then $(v, w) = \gamma^{1/2}(\partial_z u, \overline{\partial}_z u)$ solves the system

$$\begin{aligned} \overline{\partial}_z v &= qw, \\ \partial_z w &= \overline{q} v, \end{aligned} \tag{5.2}$$

where

$$q = -\gamma^{-1/2} \partial_z \gamma^{1/2}. \tag{5.3}$$

If we assume that $\gamma = 1$ near the boundary of Ω then q can be extended smoothly to $\mathbb{R}^2$ by setting $q = 0$ in $\mathbb{R}^2 \setminus \Omega$. The idea in the $\overline{\partial}$-method of inverse scattering theory is then to look for a special exponentially growing solution $\Psi(z, k) = (\Psi_1, \Psi_2)(z, k)$ in $\mathbb{R}^2$ to the system (5.2) with

$$\Psi(z, k) = m(z, k) e^{izk}, \qquad \lim_{|z|\to\infty} m(z, k) = \lim_{|z|\to\infty} (m_1, m_2) = (1, 0). \tag{5.4}$$

To construct m let $e(z,k) = \exp(i(zk + \overline{zk})) = \exp(i2\mathrm{Re}(zk))$ and define

$$m_\pm(z,k) = m_1(z,k) \pm \overline{m_2(z,k)}e(z,-k). \tag{5.5}$$

This function satisfies by (5.2) and (5.4) the $\overline{\partial}$-equation

$$\overline{\partial}_z m_\pm(z,k) = \pm q(z)e(z,-k)\overline{m_\pm(z,k)}, \qquad \lim_{|z|\to\infty} m_\pm(z,k) = 1, \tag{5.6}$$

from which $m_\pm$ can be recovered uniquely both theoretically and numerically. This also gives $m(z,k)$ by (5.5).

Associated with q is then the function

$$\mathcal{S}(k) = -\frac{i}{\pi}\int_\Omega e(z,k)\overline{q}(z)m_1(z,k)dz_1dz_2, \tag{5.7}$$

the so-called non-physical scattering transform of the potential q.

The usefulness of introducing the scattering transform in the solution of the inverse problem is two-fold. First, the scattering transform can be computed from boundary data [19, 21], and second, the conductivity can be computed from $\mathcal{S}$. This gives a reconstruction procedure consisting of the two steps

$$\Lambda_\gamma \xrightarrow{1} \mathcal{S} \xrightarrow{2} \gamma.$$

Here we will only consider the numerical implementation of the second step. See references [19, 20, 32, 33, 22] for complete implementations.

To compute γ from $\mathcal{S}$ consider the $\bar{\partial}$-equation in the parametric variable k,

$$\overline{\partial}_k \tilde{m}^+(z,k) = \overline{\mathcal{S}(-k)}e(z,-k)\overline{\tilde{m}^+(z,k)}, \quad \lim_{|k|\to\infty} \tilde{m}^+(z,k) = 1. \tag{5.8}$$

This equation has the unique solution $\tilde{m}^+(z,k)$, which is highly related to $m(z,k)$ (see [19, 21]). Moreover, from this solution we can compute

$$\gamma(z) = (\mathrm{Re}(\tilde{m}^+(z,0)))^2. \tag{5.9}$$

Hence by knowing $\mathcal{S}$, we can solve (5.8) and then obtain the γ from (5.9).

In the next subsections we will use the implementations described in section 2 and 3 to compute the scattering transform for a particular potential and reconstruct the conductivity from the scattering transform. First we will by solving (5.6) and using (5.7) compute the scattering transform of a particular potential defined from a conductivity by (5.3). Then we consider the inverse problem and compute the conductivity from the scattering transform by solving (5.8) and using (5.9).

5.1 Test example

Our test example is a numerical chest phantom consisting of a heart and two lungs, where in one lung there is an abnormality. To simulate a cross-section of the chest during expiration, the conductivity of the phantom lungs was taken to be 1 mS/cm, the conductivity of the phantom heart was taken to be the approximate longitudinal conductivity of heart muscle, 6 mS/cm, and the background conductivity was chosen to be 3 mS/cm. The conductivity of the abnormality was taken to be 4 mS/cm to simulate a tumor. These conductivities were then scaled by dividing by 3 mS/cm so that the background conductivity is 1, the conductivity of the phantom heart is 2, the conductivity of the phantom lungs is .33, and the conductivity of the phantom tumor is 1.33. The phantom organs and tumor are constructed so that the conductivity is a smooth function on the unit circle; see the top row of Figure 3. For this conductivity we have computed the corresponding potential q by (5.3) using numerical differentiation.

We now compute the scattering transform $\mathcal{S}(k)$ on a 60×60 uniform k-mesh in the square $[-20, 20]^2$. First we solve the equation (5.6) for each k in the grid. Since the potential q is supported inside $[-1, 1]^2$ the equivalent periodic integral equation is

$$m_{\pm}(z,k) = 1 \pm \int_{-2}^{2}\int_{-2}^{2} \frac{q(z')e(z',k)}{z-z'}\overline{m_{\pm}(z',k)}dz_1dz_2,$$

which can be solved numerically by the one- and two-grid implementations described in section 2 and 3. In the one-grid implementation we have chosen the grid-size $m = 7$, and in the two-grid implementation we have chosen $m^* = 2$ and $m = 7$. Next, to compute $\mathcal{S}$ we use the known function $e(z,k)\overline{q(z)}$ and the computed function $m_1(z,k) = m_+(z,k) + m_-(z,k)$ to evaluate the integral (5.7) by the trapezoid rule. Since (5.6) was solved both using the one- and two-grid implementations, we get two discrete approximations $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{S}$. The relative l^2-difference of the two functions is approximately 1%.

Having computed the scattering transform we now consider the second step in the reconstruction algorithm for the inverse conductivity problem, the computation of γ from $\mathcal{S}$. First, we want to solve (5.8) for $\tilde{m}^+(z,k)$. Note that since $\mathcal{S}$ is not compactly supported, the equation (5.8) is not readily in the form (1.1). Hence $\mathcal{S}$ has to be truncated before the numerical implementations can be used and this cut-off will introduce a systematic error. However, since $\mathcal{S}$ is a rapidly decaying function, the error can be neglected. In our computations we have chosen to cut-off the scattering transform at $|k| = 20$, which reflects the support of the pre-computed approximations $\mathcal{S}_1$

and $\mathcal{S}_2$. To compute the solution to (5.8) we therefore consider the periodic integral equation

$$\tilde{m}^+(z,k) = 1 + \int_{-40}^{40}\int_{-40}^{40} \frac{\overline{\mathcal{S}(-k')}e(z,-k')}{k-k'}\overline{\tilde{m}^+(z,k)}dk_1'dk_2',$$

which is solved using the one- and the two-grid implementations. In the one-grid method we choose the parameter $m = 7$ and use the approximation $\mathcal{S}_1$, and in the two-grid method we choose $m = 7$, $m^* = 2$ and use the approximation $\mathcal{S}_2$. In both cases the conductivity is computed on a uniform 60×60 z-mesh in the square $[-1,1]^2$. Next, to compute the conductivity we use (5.9), i.e. we evaluate the computed solutions to (5.8) at $k = 0$. This gives two reconstructions based on the one- and two-grid methods. The reconstructions are displayed in Figure 3. The relative l^2-difference in the two reconstructions is 2%, and the relative l^2-error in either of the reconstructions when compared to the true conductivity is approximately 3%.

We now compare the speed and the memory usage of the one-grid and two-grid method. For different values of the discretization level (given by the number m, where the size of each cell is h^2 for $h = 2^{1-m}$) we have measured the speed and the memory requirements for computing $m_+(z,k)$ for fixed $k = 1$. Table 5.1 contains the results for both methods.

		Memory		Speed	
m	m^*	One-grid	Two-grid	One-grid	Two-grid
6	2	101	100	0.7	0.7
7	2	115	110	3.3	1.3
8	2	170	140	18	6.3
9	3	390	290	110	39
10	3	1260	860	450	168

Table 1: Speed and memory usage for computing the function $m_+(z,1)$ by the one-grid and two-grid methods, respectively. The quantity m^* applies only to the two-grid computation. Memory usage is given in megabytes and speed in seconds.

It is clear that the two-grid is superior both concerning memory usage and speed.

Note that for both methods the speed decreases by approximately a factor 4, when going from a grid of size 2^m to a grid of size 2^{m+1}. This is expected since the the complexity is $\mathcal{O}(M^2\log(M)) = \mathcal{O}(m2^{2m})$ for $M = 2^m$.

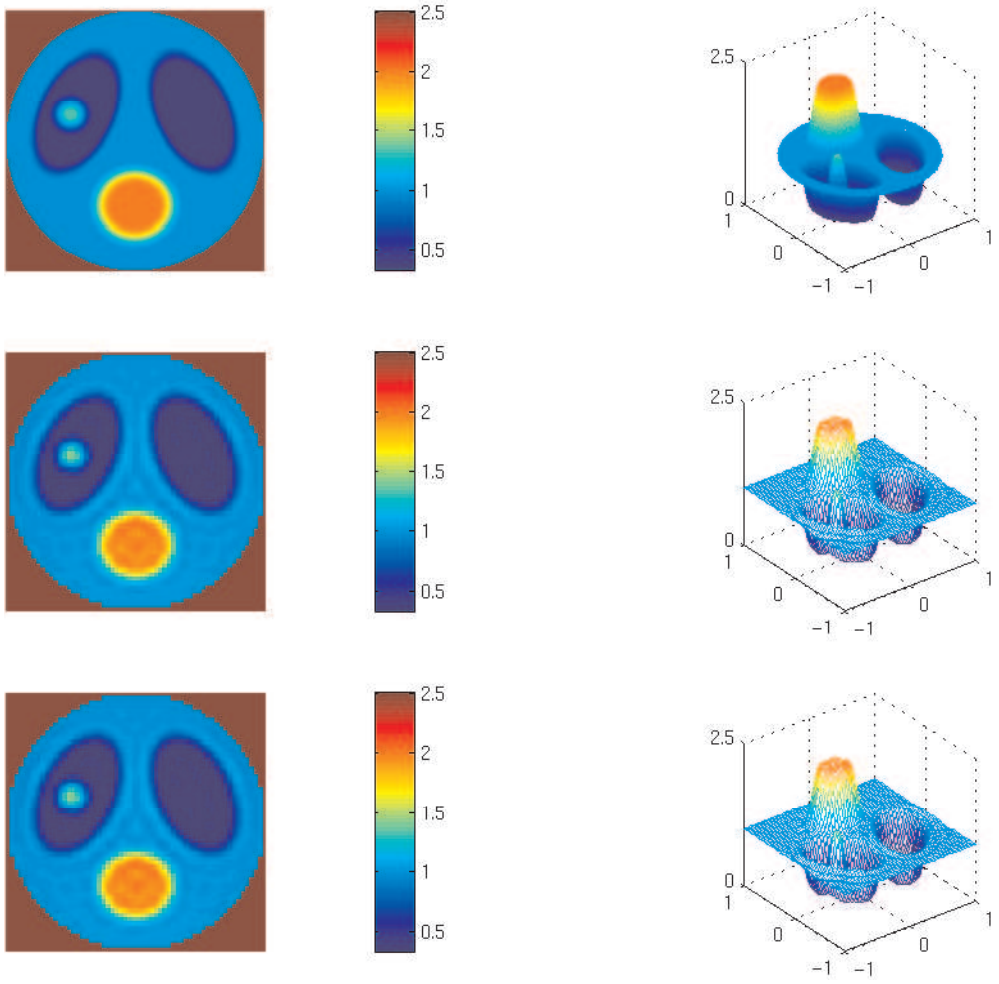

Figure 3: Top row: original conductivity. Middle row: conductivity γ_1 reconstructed by the one-grid method. Bottom row: conductivity γ_2 reconstructed by the two-grid method.

6 Conclusion

We have presented a method for computing the solution to $\bar{\partial}$-equations of the form $\bar{\partial}w = T(k)\overline{w(k)}$. One-grid and two-grid versions of the multigrid method of Vainikko [36, 31] are applied with a FFT implementation. We prove that the accuracy of the method is order h. We show that $\bar{\partial}$-equations in the form above appear naturally in the context of the inverse conductivity problem, which is the underlying mathematical model in electrical impedance tomography (EIT). With the use of the implementations we solved first the forward problem of computing the scattering transform from a known conductivity, and second the inverse problem of computing the conductivity from the scattering transform. Both problems were based on solving a $\overline{\partial}$-equation, and the results show that we have a fast and reliable implementation of a solver for such equations.

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