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BRUSHLET CHARACTERIZATION OF THE HARDY SPACE $H_1(\mathbb{R})$ AND THE SPACE BMO

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ABSTRACT. A typical wavelet system constitutes an unconditional basis for various function spaces – Lebesgue, Besov, Triebel-Lizorkin, Hardy, BMO. One of the main reasons is the frequency localization of an element from such a basis. In this paper we study a wavelet-type system, called a brushlet system. In [2] it was noticed that brushlets constitutes unconditional bases for classical function spaces such as the Triebel-Lizorkin and Besov spaces. In this paper we study brushlet expansions of functions in the Hardy space $H_1(\mathbb{R})$ and the space of functions of bounded mean oscillations. We will see that for these spaces we still have a clear similarity between a brushlet and a wavelet expansion.

1. INTRODUCTION

The present paper is a successor to the paper [2] where a wavelet-type system, called a brushlet system, was studied in various function spaces. The basic idea in the construction of the brushlet system is to obtain functions with, roughly, the same frequency content as the elements from a wavelet basis. It was noticed in [2] that under some mild conditions, such a system shares many of the properties of a wavelet system. In the present paper we continue the analysis of a brushlet system. We will see that the analogy between a brushlet and a wavelet system carries into the Hardy space $H_1(\mathbb{R})$ and the space of functions of bounded mean oscillations.

The construction of a brushlet system is based on a local trigonometric basis as introduced in [10] and [3]. A typical atom from such a basis has the form

$$b_I(x)\cos\left[\pi(n+\tfrac{1}{2})\frac{x-\alpha_I}{|I|}\right], \qquad n \in \mathbb{N}_0, \tag{1.1}$$

with $I \subset \mathbb{R}$ an interval, α_I the left endpoint of I, and b_I a smooth bell function with compact support around I. Such atoms thus have perfect localization in time and are well localized in frequency depending on the smoothness properties of b_I. If the interval I is chosen from a fixed segmentation $\mathcal{I}$ of the real line, it is easy to verify that the functions are mutually

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orthogonal in $L_2(\mathbb{R})$ for different $I \in \mathcal{I}$ and different $n \in \mathbb{N}_0$, and that this system of functions span a dense subset of $L_2(\mathbb{R})$.

E. Laeng noticed in [9] that it is possible to map any such basis by the Fourier transform to a new type of orthonormal basis well localized in time and with compact support in the frequency domain. This construction was further developed by F. Meyer and R. Coifman in [11]. They considered bases in $L_2(\mathbb{R})$ constructed using the local trigonometric bases of Wickerhauser (as given in [17]) and called such objects brushlets.

In [2], the analysis of a brushlet system was extended to other spaces than $L_2(\mathbb{R})$. It was noticed that if $\mathcal{I}$ is given by an exponential partition of $\mathbb{R}$, the corresponding brushlet system constitutes an unconditional basis for the Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R})$, $s \in \mathbb{R}$, $1 < p, q < \infty$, and for the Besov spaces $\dot{B}^s_{p,q}(\mathbb{R})$, $s \in \mathbb{R}$, $1 < p, q < \infty$. Furthermore, the norm of these spaces can be calculated from the size of the corresponding brushlet coefficients in a way similar to characterizations given by a wavelet basis. Some results concerning non-linear approximation using brushlet systems were also considered in [2]. It was noticed that a brushlet system constructed using the correct segmentation $\mathcal{I}$ becomes a greedy basis for the Lebesgue spaces $L_p(\mathbb{R})$, $1 < p < \infty$, and that optimal non-linear m-term brushlet approximation can be achieved in these spaces simply by thresholding the brushlet coefficients, provided the function belongs to a given Besov space.

We begin our treatment of brushlet characterizations in Section 2 by introducing the special class of brushlet systems we will consider in this paper. In Section 3 we study brushlet expansions of functions in the Hardy space $H_1(\mathbb{R})$. We will see that it is possible to characterize the $H_1(\mathbb{R})$-norm by the size of the brushlet coefficients. In particular, this means that a brushlet system constitutes an unconditional basis for the Hardy space. Finally, in Section 4 we study brushlet expansions of functions of bounded mean oscillations. Again we obtain a characterization based on the size of the brushlet coefficients.

2. A Brushlet system

In this section we shall define what we will call a brushlet system. The first step is the construction of a segmentation of $\mathbb{R}$. Let $\mathcal{I}$ be a countable covering of $\mathbb{R} \setminus \{0\}$ consisting of pairwise disjoint half-open intervals $I = [\alpha_I, \alpha'_I)$, such that each $I \in \mathcal{I}$ has a unique adjacent interval from $\mathcal{I}$ to the left and to the right of I. Furthermore, assume there exist two constants $1 < \lambda \leq \Lambda < \infty$ such that

$$\lambda \leq \frac{|I|}{|I'|} \leq \Lambda, \tag{2.1}$$

for all adjacent $I, I' \in \mathcal{I}$ with either $\alpha'_{I'} = \alpha_I > 0$ or $\alpha'_I = \alpha_{I'} < 0$, see Figure 1.

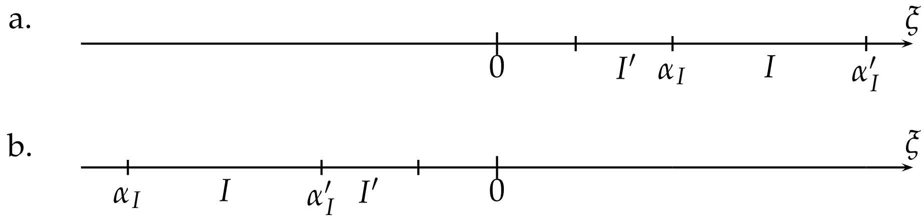

FIGURE 1. Examples of adjacent intervals $I, I' \in \mathcal{I}$. a. $\alpha'_{I'} = \alpha_I > 0$ or b. $\alpha'_I = \alpha_{I'} < 0$.

To each interval $I = [\alpha_I, \alpha'_I) \in \mathcal{I}$ assign a left and right cutoff radius $\varepsilon_I, \varepsilon'_I > 0$, satisfying

$$\begin{cases} \text{(i)} & \varepsilon'_I = \varepsilon_{I'} \text{ whenever } \alpha'_I = \alpha_{I'} \\ \text{(ii)} & \varepsilon_I + \varepsilon'_I \leq |I| \\ \text{(iii)} & \varepsilon_I \geq c|I|, \end{cases} \tag{2.2}$$

for all $I, I' \in \mathcal{I}$, with c independent of I.

Remark 2.1. Notice that if we let

$$\varepsilon_I = (1+\Lambda)^{-1}|I|,$$

then

$$\varepsilon_I + \varepsilon_{I'} = (1+\Lambda)^{-1}(|I| + |I'|) \leq |I|,$$

for any two adjacent intervals $I, I' \in \mathcal{I}$.

The following is a generic example one could keep in mind.

Example 2.2. *Let $\lambda = \Lambda = 2$. Then $\mathcal{I}$ is completely defined by* (2.1) *(up to an absolute scaling factor). Furthermore, if $\varepsilon_I = |I|/3$ and if ε'_I is defined by (i) in* (2.2)*, then (ii) and (iii) are clearly satisfied.*

We are now ready to define a brushlet system. For each $I \in \mathcal{I}$, we would like to construct a smooth bell function localized in a neighborhood of this interval. Take a non-negative ramp function $\rho \in C^r(\mathbb{R})$, for some $r \geq 0$, satisfying

$$\rho(\xi) = \begin{cases} 0 & \text{for } \xi \leq -1, \\ 1 & \text{for } \xi \geq 1, \end{cases} \tag{2.3}$$

and with the property that

$$\rho(\xi)^2 + \rho(-\xi)^2 = 1 \qquad \text{for all } \xi \in \mathbb{R}. \tag{2.4}$$

Define for each interval $I = [\alpha_I, \alpha'_I) \in \mathcal{I}$, a *bell function*

$$b_I(\xi) := \rho\left(\frac{\xi - \alpha_I}{\varepsilon_I}\right)\rho\left(\frac{\alpha'_I - \xi}{\varepsilon'_I}\right). \tag{2.5}$$

Notice that $\text{supp}(b_I) \subset [\alpha_I - \varepsilon_I, \alpha'_I + \varepsilon'_I]$ and $b_I(\xi) = 1$ for $\xi \in [\alpha_I + \varepsilon_I, \alpha'_I - \varepsilon'_I]$. Now the set of local cosine functions

$$\hat{w}_{n,I}(\xi) = \sqrt{\frac{2}{|I|}} b_I(\xi) \cos\left(\pi\left(n + \tfrac{1}{2}\right)\frac{\xi - \alpha_I}{|I|}\right), \quad I \in \mathcal{I}, \quad n \in \mathbb{N}_0, \tag{2.6}$$

constitutes an orthonormal basis for $L_2(\mathbb{R})$, see e.g. [1]. Inspired by [11], we call the collection $\{w_{n,I}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ a *brushlet system*. The brushlets also have an explicit representation in the time domain. Define the set of *central bell functions* $\{g_I\}_{I\in\mathcal{I}}$ by

$$\hat{g}_I(\xi) := \rho\left(\tfrac{|I|}{\varepsilon_I}\xi\right)\rho\left(\tfrac{|I|}{\varepsilon'_I}(1-\xi)\right), \tag{2.7}$$

i.e., $b_I(\xi) = \hat{g}_I(|I|^{-1}(\xi - \alpha_I))$, and let for notational convenience

$$k_{n,I} := \frac{\pi\left(n + \frac{1}{2}\right)}{|I|}, \qquad n \in \mathbb{N}_0,\ I \in \mathcal{I}.$$

Then,

$$w_{n,I}(x) = \sqrt{\frac{|I|}{2}} e^{i\alpha_I x}\left\{g_I\left(|I|(x + k_{n,I})\right) + g_I\left(|I|(x - k_{n,I})\right)\right\}. \tag{2.8}$$

Remark 2.3. Notice that given the setup in Example 2.2, we have

$$\hat{g}_I(\xi) = \hat{g}(\xi) := \rho(3\xi)\rho\left(\tfrac{3}{2}(1-\xi)\right),$$

independent of $I \in \mathcal{I}$.

From (2.8) we see that when the central bell function g_I is well localized at zero, the brushlet essentially consists of two peaks localized at $\pm k_{n,I}$. By a straight forward calculation, using (ii) in (2.2) and (2.1) it can be verified that

$$|g_I(x)| \leq C(1 + |x|)^{-r}, \tag{2.9}$$

with $r \geq 0$ given by the smoothness of the ramp function. Thus a brushlet $w_{n,I}$ essentially consists of two humps at $\pm k_{n,I}$.

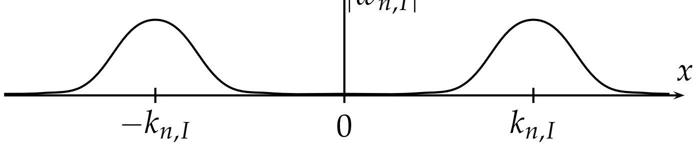

FIGURE 2. A sketch of the way one can think of a typical graph of $|w_{n,I}|$. But one should keep in mind, though, that a brushlet is not compactly supported.

Since $\lambda > 1$, we have for any $\xi \in \text{supp}(b_I)$,

$$|\xi| \leq |I| + |I| \sum_{k=0}^{\infty} \lambda^{-k} = \frac{2\lambda - 1}{\lambda - 1}|I|. \tag{2.10}$$

Likewise, $|\xi| \geq \frac{1}{2\Lambda^2}|I|$, for $\xi \in \operatorname{supp}(b_I)$, such that

$$|\xi| \asymp |I|, \qquad \text{for } \xi \in \operatorname{supp}(b_I), \qquad I \in \mathcal{I}, \tag{2.11}$$

with equivalence depending only on λ and Λ. Here and in the rest of the paper $A \asymp B$ means that there exist two constants $0 < c \leq C < \infty$ such that $cA \leq B \leq CA$.

In [2] it was noticed that we have the following equivalent norms for the homogeneous Besov and Triebel-Lizorkin spaces $\dot{B}^s_{p,q}(\mathbb{R})$ and $\dot{F}^s_{p,q}(\mathbb{R})$, $s \in \mathbb{R}$, $1 < p, q < \infty$, based on the brushlet coefficients $\{\langle f, w_{n,I}\rangle\}_{n\in\mathbb{N}_0, I\in\mathcal{I}}$,

$$|f|_{\dot{B}^s_{p,q}} \asymp \Big(\sum_{I\in\mathcal{I}}\Big(\sum_{n\in\mathbb{N}_0}\big(|I|^{s+\frac{1}{2}-\frac{1}{p}}|\langle f, w_{n,I}\rangle|\big)^p\Big)^{q/p}\Big)^{1/q}$$

and

$$|f|_{\dot{F}^s_{p,q}} \asymp \Big\|\Big(\sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}}\big(|\langle f, w_{n,I}\rangle||I|^{s+\frac{1}{2}}\chi_{E_{(n,I)}}\big)^q\Big)^{1/q}\Big\|_{L_p},$$

where $E_{(n,I)} := \{x \in \mathbb{R} \colon |x - k_{n,I}| < |I|^{-1}\}$. Here χ_Ω denotes the indicator function for $\Omega \subset \mathbb{R}$. We notice that these two expression are very similar to the corresponding expressions given by wavelet expansion coefficients, see e.g. [8].

In this paper the main focus is on the Hardy space $H_1(\mathbb{R})$ and on the space BMO. However, let us conclude this section with a result concerning a general Hardy space $H_p(\mathbb{R})$, $0 < p$. Given a bell function b_I, define an operator $P_I : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ by

$$\widehat{P_I f}(\xi) := b_I(\xi)\big[b_I(\xi)\hat{f}(\xi) + b_I(2\alpha_I - \xi)\hat{f}(2\alpha_I - \xi) - b_I(2\alpha'_I - \xi)\hat{f}(2\alpha'_I - \xi)\big].$$

It can be verified that P_I is an orthogonal projection, mapping $L_2(\mathbb{R})$ onto $\overline{\operatorname{span}}\{w_{n,I} \colon n \in \mathbb{N}_0\}$. Moreover, it was proved in [2] that P_I extends to a bounded operator on $L_p(\mathbb{R})$ for $1 < p < \infty$. This result can be extended to the Hardy spaces $H_p(\mathbb{R})$ for $0 < p \leq 1$.

Proposition 2.4. *Given $p \in (0, 1]$, suppose the ramp function $\rho \in C_r(\mathbb{R})$, for some $r \geq \lceil \frac{1}{p} - \frac{1}{2}\rceil$. Then P_I extends to a bounded operator on $H_p(\mathbb{R})$.*

Proof. By (2.5) and (2.2) we have for any $k = 0, 1, \ldots, r$

$$\begin{aligned}|b_I^{(k)}(\xi)| &\leq \sum_{n=0}^{k}\binom{k}{n}|\varepsilon_I^l|^{-n}|\varepsilon_I^r|^{-(k-n)}\Big|\rho^{(n)}\Big(\frac{\xi-\alpha_I}{\varepsilon_I}\Big)\rho^{(k-n)}\Big(\frac{\xi-\alpha'_I}{\varepsilon'_I}\Big)\Big| \\ &\leq C_k|I|^{-k}\chi_{[\alpha_I-\varepsilon_I, \alpha'_I+\varepsilon'_I]}(\xi).\end{aligned}$$

Thus $\int_{\mathbb{R}} |b_I^{(k)}(\xi)|^2\,d\xi \leq C|I|^{-2k+1}$, for $k = 0, 1, \ldots, r$, that is, b_I satisfies the Hörmander condition of order r. Hence, by [14, Theorem 12.5.1], the operator S_I, $I \in \mathcal{I}$, given by $\widehat{S_I f} := b_I\hat{f}$, is a bounded operator on $H_p(\mathbb{R})$, $0 < p \leq 1$, with norm independent of I. Notice that,

$$P_I f = S_I\big[S_I f + e^{i2\alpha_I}S_I f(-\cdot) - e^{i2\alpha'_I}S_I f(-\cdot)\big],$$

and, since $\|e^{i2\alpha x}f(-x)\|_{H_p} = \|f\|_{H_p}$ the result follows. $\square$

3. Brushlet Expansion in the Hardy Space $H_1(\mathbb{R})$

In this section we are going to study brushlet expansions of functions in the Hardy space $H_1(\mathbb{R})$. This space can be described in various ways (see e.g. [13, 7]). Here we will use the following simple definition:

$$H_1(\mathbb{R}) := \{f \in L_1(\mathbb{R})\colon Hf \in L_1(\mathbb{R})\},$$

where H is the Hilbert transform,

$$Hf(x) = \mathrm{pv}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy.$$

Furthermore,

$$\|f\|_{H_1} := \|f\|_{L_1} + \|Hf\|_{L_1},$$

defines a Banach space norm on $H_1(\mathbb{R})$. Let $\{w_{n,I}\}_{n\in\mathbb{N}_0, I\in\mathcal{I}}$ be a brushlet system as defined above. For notational convenience we define, formally, the square function

$$Wf := \Big(\sum_{n\in\mathbb{N}_0, I\in\mathcal{I}} |\langle f, w_{n,I}\rangle|^2 |I| \chi_{E_{(n,I)}}\Big)^{1/2},$$

where $E_{(n,I)} := \{x \in \mathbb{R}\colon |x - k_{n,I}| < |I|^{-1}\}$. We will prove that we can characterize the Hardy space by the size of this square function, in fact, we will see that $\|f\|_{H_1} \asymp \|Wf\|_{L_1}$ for all $f \in H_1$. Let us start by proving that $Wf \in L_1(\mathbb{R})$ for any $f \in H_1(\mathbb{R})$.

For any $I \in \mathcal{I}$ let h_I be the inverse Fourier transform of b_I, i.e. $h_I(x) = |I|g_I(|I|x)e^{ix\alpha_I}$. Then from (2.8) we have that

$$w_{n,I}(x) = (2|I|)^{-1/2}\Big[e^{-ik_{n,I}\alpha_I}h_I(x+k_{n,I}) + e^{ik_{n,I}\alpha_I}h_I(x-k_{n,I})\Big],$$

and thus

$$\begin{aligned} &|I|^{1/2}\chi_{E_{(n,I)}}(x)|\langle f, w_{n,I}\rangle| \\ &\qquad \le 2^{-1/2}\chi_{E_{(n,I)}}(x)\Big[\big|f * h_I(-k_{n,I})\big| + \big|f * h_I(k_{n,I})\big|\Big] \\ &\qquad \le 2^{\beta-1/2}\big[h^*_{I,\beta}f(-x) + h^*_{I,\beta}f(x)\big], \end{aligned} \tag{3.1}$$

for $\beta \ge 1$, where

$$h^*_{I,\beta}f(x) := \sup_{|y|\le\frac{1}{|I|}} \frac{|f * h_I(x-y)|}{(1+|I||y|)^\beta}.$$

As a consequence of the Fefferman-Stein maximal inequality [4] we have the following result from [8, Lemma 6.3.5].

Lemma 3.1. *Given a band-limited function u on $\mathbb{R}$, define for $\beta > 0$ the maximal function*

$$u_\beta^*(x) := \sup_{y\in\mathbb{R}} \frac{|u(x-y)|}{(1+|y|)^\beta}.$$

If $u_\beta^*(x) < \infty$ for all $x \in \mathbb{R}$, there exists a constant C_β such that

$$u_\beta^*(x) \le C_\beta \Big[\mathcal{M}\big(|u|^{1/\beta}\big)(x)\Big]^\beta, \quad \text{for } x \in \mathbb{R}.$$

Here

$$\mathcal{M}u(x) := \sup_{\{J\colon x\in J\}} |J|^{-1} \int_J |u(y)|dy,$$

is the Hardy-Littlewood maximal operator defined for any locally integrable function u on $\mathbb{R}$. The sup is taken over all intervals $J \subset \mathbb{R}$ that contains x.

Since we are considering brushlets, we are especially interested in functions with compact support in the frequency domain. In this case we can use the following Nikol'skij-Plancherel-Polya type inequality (see e.g. [15, p. 19] or Remark 4.2)

Theorem 3.2 (Nikol'skij-Plancherel-Polya inequality)**.** *Let $u \in L_p(\mathbb{R})$ for some $p \in [1,\infty]$, and suppose there exists an $0 < R < \infty$ such that*

$$\operatorname{supp}(\hat{u}) \subseteq \{\xi \in \mathbb{R}\colon |\xi| \le R\}.$$

Then there exists a constant C_p depending only on p such that

$$\sum_{k\in\mathbb{Z}} |u(x_k)|^p \le C_p R \|u\|_{L_p}^p, \tag{3.2}$$

for any set $\{x_k\}_{k\in\mathbb{Z}}$ with $x_k \in [Rk, R(k+1)]$.

Due to the Nikol'skij-Plancherel-Polya inequality (and the fact that $\ell_p \hookrightarrow \ell_\infty$ for $1 \le p < \infty$) we have[1]

$$|u_\beta^*(x)| \le \sup_{x\in\mathbb{R}} |u(x)| \le \Big(\sum_{k\in\mathbb{Z}} |u(x_k)|^p\Big)^{1/p} \le C\|u\|_{L_p} < \infty,$$

uniformly in $\beta > 0$, if u is a band-limited function in $L_p(\mathbb{R})$ for some $p \in [1,\infty]$. Using Lemma 3.1 we can estimate the function $h_{I,\beta}^*(x)$ by a maximal inequality.

Corollary 3.3. *Given a function f such that $h_I * f \in L_p(\mathbb{R})$ for some $p \in [1,\infty]$ and all $I \in \mathcal{I}$. Then for any $\beta > 0$ there exists a constant C_β such that*

$$h_{I,\beta}^* f(x) \le C_\beta \Big[\mathcal{M}\big(|h_I * f|^{1/\beta}\big)(x)\Big]^\beta, \quad \text{for } x \in \mathbb{R}.$$

[1]By $X \hookrightarrow Y$ we mean that the two (quasi)normed spaces X and Y satisfy $X \subset Y$ and there is a constant $C < \infty$ such that $\|\cdot\|_X \le C\|\cdot\|_Y$.

The proof is a straightforward generalization of the one given in [8, pp. 271–272], but will be given here for completeness.

Proof. Fix $I \in \mathcal{I}$ and $\beta > 0$. Let $u_I(x) := h_I * f(|I|^{-1}x)$, such that $u_I \in L_p(\mathbb{R})$. Since $\hat{h}_I = b_I$, the Nikol'skij-Plancherel-Polya inequality (Theorem 3.2) implies that $u^*_{I,\beta}(x) \le C\|u_I\|_{L_p} < \infty$ with C independent of $x \in \mathbb{R}$, $\beta > 0$ and $I \in \mathcal{I}$. Thus, by Lemma 3.1,

$$u^*_{I,\beta}(x) \le C_\beta \Big[\mathcal{M}\big(|u_I|^{1/\beta}\big)(x)\Big]^\beta,$$

for all $x \in \mathbb{R}$. Now, on one hand

$$\begin{aligned} u^*_{I,\beta}(t) &= \sup_{y\in\mathbb{R}} \frac{|h_I * f(|I|^{-1}t - |I|^{-1}y)|}{(1+|y|)^\beta} \\ &= \sup_{y\in\mathbb{R}} \frac{|h_I * f(|I|^{-1}t - y)|}{(1+|I||y|)^\beta} \ge h^*_{I,\beta}(|I|^{-1}t), \end{aligned}$$

and on the other hand

$$\begin{aligned} \mathcal{M}\big(|u_I|^{1/\beta}\big)(t) &= \sup_{\{Q:\, t\in Q\}} |Q|^{-1} \int_Q |h_I * f(|I|^{-1}y)|^{1/\beta} dy \\ &= \sup_{\{Q:\, |I|^{-1}t\in Q\}} |Q|^{-1} \int_Q |h_I * f(y)|^{1/\beta} dy \\ &= \mathcal{M}\big(|h_I * f|^{1/\beta}\big)(|I|^{-1}t), \end{aligned}$$

proving the corollary for $x = |I|^{-1}t$. □

We now turn to the question whether $h_I * f \in L_p(\mathbb{R})$ for some $p \in [1,\infty]$. We are particularly interested in the situation where $f \in H_1(\mathbb{R})$. Let us recall the following result from [7, p. 494].

Theorem 3.4 ([7]). *Given a Banach space X, let $T : L_{p_0}(\mathbb{R}) \to L_{p_0}(\mathbb{R}, X)$ be a bounded linear operator for some $p_0 \in [1,\infty]$. Assume there exists an X-valued measurable function K satisfying*

$$\sup_{y\in\mathbb{R}} \int_{|x|>2|y|} \|K(x-y) - K(x)\|_X dx < \infty,$$

such that for any bounded and compactly supported function $f \in L_{p_0}(\mathbb{R})$,

$$Tf(x) = \int_{\mathbb{R}} K(x-y) f(y) dy, \qquad \text{for } x \notin \operatorname{supp}(f).$$

Then T extends to a bounded operator from $L_p(\mathbb{R})$ to $L_p(\mathbb{R}, X)$, $1 < p < \infty$, and from $H_1(\mathbb{R})$ to $L_1(\mathbb{R}, X)$.

We will use the theorem for $X = \ell_2(\mathcal{I})$.

Theorem 3.5. *Let $\{w_{n,I}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ be a brushlet system based on a ramp function $\rho \in C^r(\mathbb{R})$ with $r \geq 2$. Then*

$$\Big\| \Big[\sum_{I\in\mathcal{I}} |h_I * f|^2 \Big]^{1/2} \Big\|_{L_1} \leq C\|f\|_{H_1},$$

for all $f \in H_1(\mathbb{R})$.

Proof. As we will see below, we need the two inequalities

$$|g_I(x)| \leq C(1+|x|)^{-1-\gamma} \qquad \text{for all } x \in \mathbb{R} \tag{3.3}$$

and

$$\int_{\mathbb{R}} |g_I(x+h) - g_I(x)| dx \leq C|h|^\gamma \qquad \text{for all } h \in \mathbb{R}, \tag{3.4}$$

for some constants $C < \infty$ and $0 < \gamma < 1$ independent of $I \in \mathcal{I}$. Clearly, (3.3) is satisfied for any $0 < \gamma \leq 1$ according to (2.9). Moreover, since $\rho \in C^r(\mathbb{R})$ for some $r \geq 2$ and $|\operatorname{supp}(\hat{g}_I)| \leq 2$, we have

$$\Big| \tfrac{d^r}{d\xi^r}[\xi \hat{g}_I(\xi)] \Big| \leq C < \infty \qquad \text{for all } \xi \in \mathbb{R}.$$

Thus,

$$|g_I'(x)| \leq C(1+|x|)^{-r} \tag{3.5}$$

and (3.4) is satisfied for any $0 < \gamma \leq 1$.

With these observations we can now prove the theorem. Recall that

$$\sum_{I\in\mathcal{I}} |b_I(\xi)|^2 = 1, \qquad \text{for all } \xi \in \mathbb{R},$$

which together with Plancherel's theorem gives

$$\Big\| \Big[\sum_{I\in\mathcal{I}} |h_I * f|^2 \Big]^{1/2} \Big\|_{L_2} = \Big(\sum_{I\in\mathcal{I}} \|b_I \hat{f}\|_{L_2}^2 \Big)^{1/2} = \|f\|_{L_2},$$

for all $f \in L_2(\mathbb{R})$. Thus, the operator $T : L_2(\mathbb{R}) \to L_2(\mathbb{R}, \ell_2(\mathcal{I}))$, given by $Tf = \{h_I * f\}_{I\in\mathcal{I}}$ is bounded. Now, according to Theorem 3.4 and since $\ell_1 \hookrightarrow \ell_2$, the theorem is proved if we can show that

$$\sup_{y\in\mathbb{R}} \sum_{I\in\mathcal{I}} \int_{|x|>2|y|} |h_I(x-y) - h_I(x)| dx < \infty.$$

Fix $y \neq 0$ and divide the sum over $I \in \mathcal{I}$ into two sums with $|I| > |y|^{-1}$ and $|I| \leq |y|^{-1}$ respectively. Since

$$h_I(x) = |I| g_I(|I|x) e^{i\alpha_I x},$$

we get, using (3.3)

$$\sum_{|I|>|y|^{-1}} \int_{|x|>2|y|} |h_I(x-y)-h_I(x)|dx$$
$$\leq 2 \sum_{|I|>|y|^{-1}} |I| \int_{|x|>|y|} |g_I(|I|x)|dx$$
$$\leq 2C \sum_{|I|>|y|^{-1}} |I|^{-\gamma} \int_{|x|>|y|} |x|^{-1-\gamma}dx$$
$$= \frac{4C}{\gamma}|y|^{-\gamma} \sum_{|I|>|y|^{-1}} |I|^{-\gamma}$$
$$\leq \frac{8C}{\gamma} \sum_{k=0}^{\infty} \lambda^{-k\gamma} < \infty.$$

For the other sum we first notice that

$$|h_I(x-y)-h_I(x)|$$
$$\leq |I||g_I(|I|(x-y))-g_I(|I|x)| + |I||g_I(|I|x)||1-e^{i\alpha_I y}|.$$

Since $|\alpha_I| \leq A|I|$ according to (2.11), we have

$$|1-e^{i\alpha_I y}| \leq C\min\{|\alpha_I y|, 1\} \leq C'|y||I|,$$

yielding

$$\sum_{|I|\leq|y|^{-1}} |I| \int_{|x|>2|y|} |g_I(|I|x)||1-e^{i\alpha_I y}|dx$$
$$\leq C \sum_{|I|\leq|y|^{-1}} |I|^{1-\gamma}|y| \int_{|x|>2|y|} |x|^{-1-\gamma}dx$$
$$= \frac{2C}{\gamma}|y|^{1-\gamma} \sum_{|I|\leq|y|^{-1}} |I|^{1-\gamma} < \infty.$$

Likewise, using (3.4)

$$\sum_{|I|\leq|y|^{-1}} |I| \int_{|x|>2|y|} |g_I(|I|(x-y))-g_I(|I|x)|dx \leq C|y|^{\gamma} \sum_{|I|\leq|y|^{-1}} |I|^{\gamma} < \infty.$$

□

Now combining Corollary 3.3 and Theorem 3.5 with the inequality (3.1) we can give a lower bound on the H_1-norm using brushlet coefficients.

Theorem 3.6. *Let $\{w_{n,I}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ be a brushlet system with associated ramp function $\rho \in C^r(\mathbb{R})$ for some $r \geq 2$. Then $Wf \in L_1(\mathbb{R})$ for any $f \in H_1(\mathbb{R})$, in fact, $\|Wf\|_{L_1} \leq C\|f\|_{H_1}$ for a constant C depending only on the brushlet system.*

Proof. Suppose $f \in H_1(\mathbb{R})$. By (3.1) it suffices to prove that

$$\Big\| \Big(\sum_{I\in\mathcal{I}} |h^*_{I,\beta} f|^2 \Big)^{1/2} \Big\|_{L_1} \leq C \|f\|_{H_1}$$

for some $\beta > 1$. By Theorem 3.5, $\{h_I * f\}_{I\in\mathcal{I}} \in L_1(\mathbb{R}, \ell_2(\mathcal{I}))$ for all $I \in \mathcal{I}$. Thus, by Corollary 3.3 we have for $\beta > 1$

$$\begin{aligned} \Big\| \Big[\sum_{I\in\mathcal{I}} |h^*_{I,\beta} f|^2 \Big]^{1/2} \Big\|_{L_1} &\leq C_\beta \Big\| \Big[\sum_{I\in\mathcal{I}} \{\mathcal{M}(|h_I * f|^{1/\beta})\}^{2\beta} \Big]^{\frac{1}{2}} \Big\|_{L_1} \\ &= C_\beta \Big\| \Big[\sum_{I\in\mathcal{I}} \{\mathcal{M}(|h_I * f|^{1/\beta})\}^{2\beta} \Big]^{\frac{1}{2\beta}} \Big\|^\beta_{L_\beta}. \end{aligned}$$

Now, the Fefferman-Stein maximal inequality [4] implies that

$$\begin{aligned} \Big\| \Big[\sum_{I\in\mathcal{I}} \{\mathcal{M}(|h_I * f|^{1/\beta})\}^{2\beta} \Big]^{\frac{1}{2\beta}} \Big\|^\beta_{L_\beta} &\leq C_\beta \Big\| \Big[\sum_{I\in\mathcal{I}} |h_I * f|^2 \Big]^{\frac{1}{2\beta}} \Big\|^\beta_{L_\beta} \\ &= C_\beta \Big\| \Big[\sum_{I\in\mathcal{I}} |h_I * f|^2 \Big]^{\frac{1}{2}} \Big\|_{L_1} \leq C' \|f\|_{H_1}, \end{aligned}$$

where the last inequality follows from Theorem 3.5. □

We can also prove an inverse to Theorem 3.6. That is to say, there exists a constant $C > 0$ such that $\|f\|_{H_1} \leq C \|Wf\|_{L_1}$. We will prove that the Hardy space consists of functions which can be expanded as a sum of molecules satisfying certain properties. Normally, a molecule is a function localized around a single point $p_0 \in \mathbb{R}$, but since a brushlet is real-valued in the frequency domain and thus essentially consists of two humps in direct space, we need a slightly different definition of a molecule. Given a point $p_0 \in \mathbb{R}$, let

$$d(x, p_0) := \min\{|x + p_0|, |x - p_0|\},$$

denote the distance from $x \in \mathbb{R}$ to the set $\{-p_0, p_0\}$. Our definition of a molecule is motivated by the following lemma.

Lemma 3.7. *Assume $M \in L_2(\mathbb{R})$ is integrable and satisfies*

$$\|d(\cdot, p_0) M\|_{L_2} < \infty.$$

If furthermore, $\operatorname{supp}(\hat{M}) \subset \mathbb{R}^+$ *or* $\operatorname{supp}(\hat{M}) \subset \mathbb{R}^-$, *then $M \in H_1(\mathbb{R})$. In fact,*

$$\|M\|_{H_1} \leq 8 \|M\|^{1/2}_{L_2} \|d(\cdot, p_0) M\|^{1/2}_{L_2}.$$

Proof. Given $a > 0$, we have

$$\begin{aligned} \|M\|_{L_1} &\leq \Big(\int_{d(x,p_0)\leq a} + \int_{d(x,p_0)>a} \Big) |M(x)| dx \\ &\leq 2\sqrt{a} \|M\|_{L_2} + \frac{2}{\sqrt{a}} \|d(\cdot, p_0) M\|_{L_2}. \end{aligned}$$

Thus, taking $a = \|M\|_{L_2}^{-1}\|d(\cdot,p_0)M\|_{L_2}$, we obtain

$$\|M\|_{L_1}^2 \leq 16\|M\|_{L_2}\|d(\cdot,p_0)M\|_{L_2}.$$

Recall that

$$\widehat{Hf}(\xi) = \begin{cases} -i\hat{f}(\xi) & \text{for a.e. } \xi > 0 \\ i\hat{f}(\xi) & \text{for a.e. } \xi < 0 \end{cases},$$

for all $f \in L_2(\mathbb{R})$, [7, p. 196]. Now, since $\hat{M}$ has support solely on either $\mathbb{R}^+$ or $\mathbb{R}^-$, we have $HM = \pm iM$ in $L_2(\mathbb{R})$. Hence, $\|HM\|_{L_1} = \|M\|_{L_1}$, and the lemma follows. □

We define a *molecule* as a function M satisfying the conditions in Lemma 3.7. We can now prove the converse result of Theorem 3.6.

Theorem 3.8. *Let $\{w_{n,I}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ be a brushlet system. Suppose the central bell functions, g_I, satisfy* (2.9) *for some $r > 2$. Then there exists a constant $0 < C < \infty$ such that*

$$\|f\|_{H_1} \leq C\Big\|\Big(\sum_{n\in\mathbb{N}_0,I\in\mathcal{I}} |\langle f, w_{n,I}\rangle|^2|I|\chi_{E_{(n,I)}}\Big)^{1/2}\Big\|_{L_1}$$

for all $f \in H_1(\mathbb{R})$.

Proof. Let for $m \in \mathbb{Z}$

$$\Omega_m = \{x \in \mathbb{R}\colon Wf(x) > 2^m\}. \tag{3.6}$$

Clearly, $\Omega_m \supseteq \Omega_{m+1}$ and since

$$\begin{aligned}\int_{\mathbb{R}} Wf(x)dx &= \sum_{m\in\mathbb{Z}} \int_{\Omega_m\setminus\Omega_{m+1}} Wf(x)dx \geq \sum_{m\in\mathbb{Z}} 2^m|\Omega_m\setminus\Omega_{m+1}| \\ &= \sum_{m\in\mathbb{Z}} 2^m(|\Omega_m| - |\Omega_{m+1}|) = 1/2\sum_{m\in\mathbb{Z}} 2^m|\Omega_m|,\end{aligned}$$

we have

$$\sum_{m\in\mathbb{Z}} 2^m|\Omega_m| \leq 2\|Wf\|_{L_1}. \tag{3.7}$$

Define

$$B_m := \{(n,I) \in \mathbb{Z}\times\mathcal{I}\colon |E_{(n,I)}\cap\Omega_m| \geq |I|^{-1},\ |E_{(n,I)}\cap\Omega_{m+1}| < |I|^{-1}\}.$$

Notice that for each $(n,I) \in \mathbb{Z}\times\mathcal{I}$ there exists at most one m such that $(n,I) \in B_m$. Let

$$B_m^+ := \{(n,I) \in B_m\colon I \subset \mathbb{R}^+\} \quad \text{and} \quad B_m^- := \{(n,I) \in B_m\colon I \subset \mathbb{R}^-\}.$$

For each nonempty $B_m^\pm$, take (one of) the smallest interval(s) $I_{m,1}^\pm$ with associated integer $n_{m,1}^\pm$ such that $(n_{m,1}^\pm, I_{m,1}^\pm) \in B_m^\pm$. Denote

$$B_{m,1}^\pm = \{(n,I) \in B_m^\pm\colon E_{(n,I)} \cap E_{(n_{m,1}^\pm,I_{m,1}^\pm)} \neq \emptyset\},$$

and $\tilde{B}_m^{\pm} = B_m^{\pm} \backslash B_{m,1}^{\pm}$. Take (one of) the smallest interval(s) $I_{m,2}^{\pm}$ with associated integer $n_{m,2}^{\pm}$ such that $(n_{m,2}^{\pm}, I_{m,2}^{\pm}) \in \tilde{B}_m^{\pm}$ and let

$$B_{m,2}^{\pm} = \{(n,I) \in \tilde{B}_m^{\pm} \colon E_{(n,I)} \cap E_{(n_{m,2}^{\pm}, I_{m,2}^{\pm})} \neq \emptyset\}.$$

Continue this process until we have a partition $B_m^{\pm} = \cup_{i \in \mathbb{N}} B_{m,i}^{\pm}$.

Now, let

$$\alpha_{m,i}^{\pm} = |I_{m,i}^{\pm}|^{-1/2} \Big(\sum_{(n,I) \in B_{m,i}^{\pm}} |\langle f, w_{|n|,I} \rangle|^2 \Big)^{1/2},$$

and define

$$M_{m,i}^{\pm}(x) := \begin{cases} \frac{1}{\alpha_{m,i}^{\pm}} \sum_{(n,I) \in B_{m,i}^{\pm}} \langle f, w_{|n|,I} \rangle w_{|n|,I} & \text{if } \alpha_{m,i}^{\pm} \neq 0, \\ 0 & \text{if } \alpha_{m,i}^{\pm} = 0. \end{cases}$$

Then we obtain the decomposition $f = \sum_{m \in \mathbb{Z}, i \in \mathbb{N}} \alpha_{m,i}^+ M_{m,i}^+ + \alpha_{m,i}^- M_{m,i}^-$. We want to show that

1. $M_{m,i}^{\pm}$, $m \in \mathbb{Z}$, $i \in \mathbb{N}$, are molecules with H_1-norm bounded by a constant independent of m and i.
2. $\sum_{m \in \mathbb{Z}, i \in \mathbb{N}} |\alpha_{m,i}^+| + |\alpha_{m,i}^-| \leq C \|Wf\|_{L_1}$ for some constant depending only on the brushlet system.

This will imply

$$\|f\|_{H_1} \leq \sum_{\substack{m \in \mathbb{Z} \\ i \in \mathbb{N}}} |\alpha_{m,i}^+| \|M_{m,i}^+\|_{H_1} + |\alpha_{m,i}^-| \|M_{m,i}^-\|_{H_1} \leq C' \|Wf\|_{L_1}.$$

For notational convenience we suppress the index $\pm$ in the following.

1. Fix $m \in \mathbb{Z}$ and $i \in \mathbb{N}$ such that $B_{m,i} \neq \emptyset$. Since the brushlets are orthonormal, we have

$$\|M_{m,i}\|_{L_2} = \frac{1}{|\alpha_{m,i}|} \Big(\sum_{I \in B_{m,i}} |\langle f, w_{|n|,I} \rangle|^2 \Big)^{1/2} = |I_{m,i}|^{1/2}.$$

Let $\kappa_{m,i} := k_{n_{m,i}, I_{m,i}} = \pi(n_{m,i} + \frac{1}{2}) |I_{m,i}|^{-1}$ and $R_{m,i} = \{x \in \mathbb{R} \colon d(x, \kappa_{m,i}) > 4|I_{m,i}|^{-1}\}$. Then

$$\|M_{m,i} d(\cdot, \kappa_{m,i})\|_{L_2}^2 = \Big(\int_{\mathbb{R} \backslash R_{m,i}} + \int_{R_{m,i}} \Big) |M_{m,i}(x) d(x, \kappa_{m,i})|^2 \, dx := A_1 + A_2.$$

The first integral is easily estimated

$$A_1 \leq 16 |I_{m,i}|^{-2} \|M_{m,i}\|_{L_2}^2 = 16 |I_{m,i}|^{-1}.$$

For A_2 we have, using Cauchy-Schwarz inequality on the sum,

$$A_2 \leq |\alpha_{m,i}|^{-2} \int_{R_{m,i}} \Big(\sum_{(n,I)\in B_{m,i}} |\langle f, w_{|n|,I}\rangle||w_{|n|,I}(x) d(x,\kappa_{m,i})| \Big)^2 dx$$

$$\leq |\alpha_{m,i}|^{-2} \Big(\sum_{(n,I)\in B_{m,i}} |\langle f, w_{|n|,I}\rangle|^2 \Big) \sum_{(n,I)\in B_{m,i}} \int_{R_{m,i}} |w_{|n|,I}(x) d(x,\kappa_{m,i})|^2 dx$$

(3.8)

$$= |I_{m,i}| \sum_{(n,I)\in B_{m,i}} \int_{R_{m,i}} |w_{|n|,I}(x) d(x,\kappa_{m,i})|^2 dx.$$

From (2.8) we get that

$$\Big[\int_{R_{m,i}} |w_{n,I}(x) d(x,\kappa_{m,i})|^2 dx \Big]^{1/2}$$

$$\leq \Big[\frac{|I|}{2} \int_{R_{m,i}} |g_I(|I|(x-k_{n,I})) d(x,\kappa_{m,i})|^2 dx \Big]^{1/2} + \Big[\frac{|I|}{2} \int_{R_{m,i}} |g_I(|I|(x+k_{n,I})) d(x,\kappa_{m,i})|^2 dx \Big]^{1/2}$$

$$\leq \Big[\frac{|I|}{2} \int_{|x-\kappa_{m,i}|>4|I_{m,i}|^{-1}} |g_I(|I|(x-k_{n,I}))|^2 |x-\kappa_{m,i}|^2 dx \Big]^{1/2}$$

$$+ \Big[\frac{|I|}{2} \int_{|x+\kappa_{m,i}|>4|I_{m,i}|^{-1}} |g_I(|I|(x+k_{n,I}))|^2 |x+\kappa_{m,i}|^2 dx \Big]^{1/2}. \tag{3.9}$$

Let $\kappa'_{m,i} := |I|(\kappa_{m,i} - k_{n,I})$. Then by a change of variables we obtain

$$\frac{|I|}{2} \int_{|x-\kappa_{m,i}|>4|I_{m,i}|^{-1}} |g_I(|I|(x-k_{n,I}))|^2 |x-\kappa_{m,i}|^2 dx = \frac{1}{|I|^2 2} \int_{|x'-\kappa'_{m,i}|>4\frac{|I|}{|I_{m,i}|}} |g_I(x')|^2 |x'-\kappa'_{m,i}|^2 dx'.$$

Notice that $|\kappa_{m,i} - k_{n,I}| \leq |I_{m,i}|^{-1} + |I|^{-1} \leq 2|I_{m,i}|^{-1}$ for any $(n,I) \in B_{m,i}$. Thus

$$\int_{|x'-\kappa'_{m,i}|>4\frac{|I|}{|I_{m,i}|}} |g_I(x')|^2 |x'-\kappa'_{m,i}|^2 dx' \leq \int_{|x'-\kappa'_{m,i}|>2|\kappa'_{m,i}|} |g_I(x')|^2 |x'-\kappa'_{m,i}|^2 dx' \leq C < \infty,$$

where C is independent of $\kappa'_{m,i}$ by (2.9). Similar estimates can be made for the last term in (3.9), such that

$$\int_{R_{m,i}} |w_{n,I}(x) d(x,\kappa_{m,i})|^2 dx \leq C|I|^{-2}.$$

Notice that, given an $m \in \mathbb{Z}$ and two intervals $I, I' \in \mathcal{I}$ with $|I| > |I'|$, there are at most $\lfloor |I|/|I'| \rfloor + 1$ intervals $E_{(n,I)}$, $n \in \mathbb{Z}$, such that $E_{(n,I)} \cap E_{(m,I')} \neq \emptyset$. Hence,

$$\sum_{(n,I)\in B_{m,i}} |I|^{-2} \leq |I_{m,i}|^{-2} \sum_{I\in\mathcal{I}:\, |I|\geq|I_{m,i}|} \left(\frac{|I_{m,i}|}{|I|}\right)^2 \left(\frac{|I|}{|I_{m,i}|}+1\right)$$
$$\leq |I_{m,i}|^{-2} \sum_{k=0}^{\infty} \lambda^{-k} + \lambda^{-2k} \leq c|I_{m,i}|^{-2}.$$

Now, according to (3.8) we obtain $A_2 \leq cC|I_{m,i}|^{-1}$, and putting the inequalities together, we get

$$\|M_{m,i}\|_{L_2} \|M_{m,i} d(\cdot, \kappa_{m,i})\|_{L_2} \leq C' |I_{m,i}|^{1/2} |I_{m,i}|^{-1/2} = C',$$

where C' is independent of m and i. Moreover, since either $\mathrm{supp}(\hat{w}_{n,I}) \subset \mathbb{R}^+$ or $\mathrm{supp}(\hat{w}_{n,I}) \subset \mathbb{R}^-$ for any $n \in \mathbb{N}_0$ and $I \in \mathcal{I}$, $M_{m,i}$ is a molecule. Furthermore, $\|M_{m,i}\|_{H_1} \leq C$ independent of m and i, according to Lemma 3.7.

2. Notice that for a given $(n, I) \in B_m$ we have

$$|E_{(n,I)} \backslash \Omega_{m+1}| = |E_{(n,I)}| - |E_{(n,I)} \cap \Omega_{m+1}| > 2|I|^{-1} - |I|^{-1} = |I|^{-1}.$$

Let $E_i := \cup_{(n,I)\in B_{m,i}} E_{(n,I)}$, then in particular

$$|E_i| \leq 3|E_{(n_{m,i}, I_{m,i})}| = 6|I_{m,i}|^{-1}.$$

Now,

$$\sum_{(n,I)\in B_{m,i}} |\langle f, w_{|n|,I}\rangle|^2 < \sum_{(n,I)\in B_{m,i}} |\langle f, w_{|n|,I}\rangle|^2 |I| |E_{(n,I)} \backslash \Omega_{m+1}|$$
$$= \int_{E_i \backslash \Omega_{m+1}} \sum_{(n,I)\in B_{m,i}} |\langle f, w_{|n|,I}\rangle|^2 |I| \chi_{E_{(n,I)}}(x) dx$$
$$\leq 2 \int_{E_i \backslash \Omega_{m+1}} |Wf(x)|^2 dx \leq 12 |I_{m,i}|^{-1} 2^{2(m+1)}$$

by (3.6), which implies that $|\alpha_{m,i}| \leq 2\sqrt{6}|I_{m,i}|^{-1} 2^m$. Since $(n_{m,i}, I_{m,i}) \in B_m$, we have $|I_{m,i}|^{-1} \leq |E_{(n_{m,i}, I_{m,i})} \cap \Omega_m|$, and since the $E_{(n_{m,i}, I_{m,i})}$, $i \in \mathbb{N}$, are disjoint by construction, we obtain

$$\sum_{m\in\mathbb{Z},\, i\in\mathbb{N}} |\alpha_{m,i}| \leq C \sum_{m\in\mathbb{Z}} 2^m \sum_{i\in\mathbb{N}} |E_{(n_{m,i}, I_{m,i})} \cap \Omega_m|$$
$$\leq C \sum_{m\in\mathbb{Z}} 2^m |\Omega_m| \leq C' \|Wf\|_{L_1}$$

by (3.7). □

Finally, combining Theorem 3.6 and Theorem 3.8 we get a characterization of the Hardy space $H_1(\mathbb{R})$ using a brushlet system.

Proposition 3.9. *Let $\{w_{n,I}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ be a brushlet system with associated ramp function $\rho \in C^r(\mathbb{R})$, for some $r > 2$. Then $\{w_{n,I}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ constitutes an unconditional basis for $H_1(\mathbb{R})$, and we have the following characterization for all $f \in H_1(\mathbb{R})$*

$$\|f\|_{H_1} \asymp \Big\| \Big(\sum_{\substack{I\in\mathcal{I}\\ n\in\mathbb{N}_0}} |\langle f, w_{n,I}\rangle|^2 |I| \chi_{E_{(n,I)}} \Big)^{1/2} \Big\|_{L_1}, \tag{3.10}$$

with the equivalence depending only on the brushlet system.

Proof. We only need to verify that the brushlet system is dense in $H_1(\mathbb{R})$. However, this follows immediately from the fact that $f \to Wf$ is a continuous operation from $H_1(\mathbb{R})$ to $L_1(\mathbb{R})$ by Theorem 3.6. □

4. Brushlet Characterization of BMO

In the previous section we obtained a brushlet characterization of functions in the Hardy space $H_1(\mathbb{R})$. This characterization bear some resemblance to the brushlet characterization of the Lebesgue spaces $L_p(\mathbb{R})$, $1 < p < \infty$, thus in some sense the Hardy space can be seen as a substitute for $L_1(\mathbb{R})$. Since the dual space of $H_1(\mathbb{R})$ is the space of functions with bounded mean oscillation, it is natural to investigate if these functions have a brushlet characterization too.

Given a function $f \in L_{1,loc}(\mathbb{R}^d)$, let f_J denote its mean value over the interval $J \subset \mathbb{R}$, that is to say $f_J = |J|^{-1}\int_J f(x)dx$. f is said to be of *bounded mean oscillation* (BMO) if

$$\|f\|_{\mathrm{BMO}} := \sup_J |J|^{-1} \int_J |f(x) - f_J|\, dx < \infty,$$

where the sup is taken over all intervals $J \subset \mathbb{R}$. The class of functions of bounded mean oscillations, modulo constants, is a Banach space with the norm $\|\cdot\|_{\mathrm{BMO}}$ defined above. The source of inspiration of characterizing BMO using a brushlet system is [12, p. 154]: there exists a characterization of BMO using compactly supported wavelets

Since the brushlets do not have compact support, we have to use a slightly different approach than the one used in [12]. In order to obtain a lower bound on the BMO-norm we need the following local version of the Nikol'skij-Plancherel-Polya inequality.

Lemma 4.1 (local Nikol'skij-Plancherel-Polya)**.** *For any finite interval $I \subset \mathbb{R}$ and function u satisfying* $\mathrm{supp}(\hat{u}) \subset \mathring{I} \subset \mathbb{R}$, *there exists an absolute constant C such that*

$$\sup_{x\in E_{(n,I)}} |u(x)|^2 \le C|I| \sum_{l\in\mathbb{Z}} (1+|l|)^{-2} \int_{E_{(n+l,I)}} |u(y)|^2\, dy, \tag{4.1}$$

where $E_{(n,I)} = \{x \in \mathbb{R} \colon |x - k_{n,I}| < |I|^{-1}\}$.

The proof of this lemma is a modification of the arguments given in [6, p. 782] for dyadic partitions of $\mathbb{R}$ instead of $\{E_{(n,I)}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$.

Proof. By the Paley-Wiener Theorem, there exists a constant $\gamma \in \mathbb{N}$ such that $|u(x)| \le C(1+|x|)^\gamma$ for all $x \in \mathbb{R}$, hence,

$$u_\delta(x) := u(x)\,\mathrm{sinc}^{2\gamma+2}(\delta x) \in L_2(\mathbb{R}),$$

for all $\delta > 0$. Furthermore, since $\mathrm{supp}(\widehat{\mathrm{sinc}(\delta\cdot)}) \subseteq [-\delta,\delta]$, we have that $\mathrm{supp}(\hat{u}_\delta) \subset I$ for δ sufficiently small.

Take η from the space of Schwartz functions $\mathcal{S}$ such that $\mathrm{supp}(\hat{\eta}) \subset \left[-\frac{1}{2},\frac{3}{2}\right]$ and $\hat{\eta} \equiv 1$ on $[0,1]$. Define η_I by $\hat{\eta}_I(\xi) = \hat{\eta}(|I|^{-1}(\xi-\alpha_I))$, where α_I is the left endpoint of the interval I. Notice that for a fixed $x \in \mathbb{R}$, $\hat{\eta}(\xi)e^{ix\xi} = \sum_{k\in\mathbb{Z}} c_k e^{i\pi|I|^{-1}k\xi}$, where

$$\begin{aligned} c_k &= (2|I|)^{-1}\int_{\alpha_I-|I|/2}^{\alpha_I+3|I|/2} \left[\hat{\eta}(\xi)e^{ix\xi}\right]e^{-i\pi|I|^{-1}k\xi}\,d\xi \\ &= (2|I|)^{-1}\int_{\mathbb{R}} \hat{\eta}(\xi)e^{i(x-\pi|I|^{-1}k)\xi}\,d\xi \\ &= \sqrt{\tfrac{\pi}{2}}|I|^{-1}\eta(x-\pi|I|^{-1}k). \end{aligned}$$

Thus,

$$\begin{aligned} u_\delta(x) = \eta_I * u_\delta(x) &= \tfrac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{u}_\delta(\xi)\hat{\eta}_I(\xi)e^{ix\xi}\,d\xi \\ &= \tfrac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{u}_\delta(\xi)\Big[\sqrt{\tfrac{\pi}{2}}|I|^{-1}\sum_{k\in\mathbb{Z}}\eta_I\big(x-\pi|I|^{-1}k\big)e^{i\pi|I|^{-1}k\xi}\Big]\,d\xi \\ &= \sqrt{\tfrac{\pi}{2}}|I|^{-1}\sum_{k\in\mathbb{Z}} u_\delta\big(\pi|I|^{-1}k\big)\eta_I\big(x-\pi|I|^{-1}k\big). \end{aligned}$$

Since $\eta \in \mathcal{S}$, letting $\delta \to 0$, the dominated convergence theorem yields

$$u(x+y) = \eta_I * u^y(x) = \sqrt{\tfrac{\pi}{2}}|I|^{-1}\sum_{k\in\mathbb{Z}} u(\pi|I|^{-1}k+y)\eta_I(x-\pi|I|^{-1}k),$$

where $u^y(x) := u(x+y)$. Especially, since $|E_{n,I}| = 2|I|^{-1}$, we have for any $x,y \in E_{(n,I)}$,

$$\begin{aligned} |u(x)| &\le \sup_{|z|\le 2|I|^{-1}} |u(z+y)| \\ &\le \sqrt{\tfrac{\pi}{2}}|I|^{-1}\sum_{k\in\mathbb{Z}} |u(\pi|I|^{-1}k+y)| \sup_{|z|\le 2|I|^{-1}} |\eta_I(z-\pi|I|^{-1}k)|. \end{aligned}$$

But,

$$\sup_{|z|\le 2|I|^{-1}} |\eta_I(z-\pi|I|^{-1}k)| = \sup_{|z|\le 2} |I||\eta(z-\pi k)| \le C|I|(1+|k|)^{-2},$$

since $\eta \in \mathcal{S}$. Thus, using Cauchy-Schwarz inequality, we obtain

$$\sup_{x \in E_{(n,I)}} |u(x)|^2 \le C \sum_{k \in \mathbb{Z}} |u(\pi|I|^{-1}k + y)|^2 (1+|k|)^{-2}.$$

Finally, integrating with respect to y over $E_{(n,I)}$ yields

$$2|I|^{-1} \sup_{x \in E_{(n,I)}} |u(x)|^2 \le C \sum_{k \in \mathbb{Z}} (1+|k|)^{-2} \int_{E_{(n+k,I)}} |u(y)|^2 \, dy.$$

□

Remark 4.2. Notice that the proof of Lemma 4.1 can easily be generalized to give

$$\sup_{x \in E_{(n,I)}} |g(x)|^p \le C|I| \sum_{l \in \mathbb{Z}} (1+|l|)^{-2} \int_{E_{(n+l,I)}} |g(y)|^p \, dy, \quad 1 \le p < \infty. \tag{4.2}$$

Moreover, if we sum over n on both sides of (4.2) we get an inequality almost identical to (3.2) in Theorem 3.2.

Using the local Nikol'skij-Plancherel-Polya inequality, we can now state and prove a lower bound for the BMO-norm based on a brushlet expansion.

Given two intervals $I, J \in \mathbb{R}$, we write $I \prec J$ if

$$|I| < |J| \qquad \text{and} \qquad I \cap J \neq \emptyset.$$

Theorem 4.3. *Let $\{w_{n,I}\}_{n \in \mathbb{N}_0, I \in \mathcal{I}}$ be a brushlet system with associated ramp function $\rho \in C^r(\mathbb{R})$, for some $r \ge 2$. Then there exists a constant C such that*

$$\sup_{J \subset \mathbb{R}} \Big(|J|^{-1} \sum_{\substack{n \in \mathbb{N}_0, I \in \mathcal{I} \\ E_{(n,I)} \prec J}} |\langle f, w_{n,I} \rangle|^2 \Big)^{1/2} \le C \|f\|_{\mathrm{BMO}}, \tag{4.3}$$

for any $f \in$ BMO, where the sup is taken over all intervals $J \neq \emptyset$.

Proof. Since translation is a Banach space isometry of BMO to itself, it suffices to show (4.3) for $J = [0, a]$ for some $a > 0$. Recall that

$$\langle f, w_{n,I} \rangle = (2|I|)^{-1/2} \big[e^{ik_{n,I}\alpha_I} f * h_I(-k_{n,I}) + e^{-ik_{n,I}\alpha_I} f * h_I(k_{n,I}) \big],$$

where $\hat{h}_I = b_I$. Since $\operatorname{supp}(b_I) \subset \tilde{I} := [\alpha_I - \varepsilon_I, \alpha'_I + \varepsilon'_I]$, Lemma 4.1 yields

$$\begin{aligned} |I|^{-1} |f * h_I(k_{n,I})|^2 &\le C \sum_{l \in \mathbb{Z}} (1+|l-n|)^{-2} \int_{E_{(l,\tilde{I})}} |f * h_I(x)|^2 dx \\ &\le C \sum_{k \in \mathbb{Z}} \sum_{\substack{l \in \mathbb{Z} \\ E_{(l,I)} \prec E_{(k,I')}}} (1+|l-n|)^{-2} \int_{E_{(l,\tilde{I})}} |f * h_I(x)|^2 dx, \end{aligned}$$

where $I' \in \mathcal{I}$ is a fixed interval such that $|I'| = 2\kappa|J|^{-1}$, for some constant κ, $\Lambda^{-1} < \kappa < \Lambda$. For a fixed $k \in \mathbb{Z}$ and for $l \in \mathbb{Z}$ such that $E_{(l,I)} \prec E_{(k,I')}$ we notice that

$$\sum_{\substack{n\in\mathbb{Z}\\ E_{(n,I)}\prec J}} (1+|l-n|)^{-2} \le C'(1+|k|)^{-2}, \tag{4.4}$$

for some constant C' depending only on Λ. This can be seen as follows: Clearly,

$$\sum_{\substack{n\in\mathbb{Z}\\ E_{(n,I)}\prec J}} (1+|l-n|)^{-2} < \sum_{n'\in\mathbb{Z}} (1+|n'|)^{-2} = \tfrac{\pi^2}{3}.$$

Moreover, notice that $\{n \in \mathbb{Z}\colon E_{(n,I)} \prec J\} \subset \{0,1,\dots,\lfloor\kappa|I|/|I'|\rfloor\}$, and $\frac{|I|}{|I'|}(|k|-1) \le |l|$. Thus for $|k| \ge 2(\kappa+1)$ we have

$$|l-n| \ge |l| - |n| \ge \tfrac{|I|}{|I'|}(|k|-1-\kappa) \ge 1/2\tfrac{|I|}{|I'|}|k|,$$

yielding

$$\sum_{n=0}^{\lfloor\kappa|I|/|I'|\rfloor} |l-n|^{-2} \le \kappa\tfrac{|I|}{|I'|}\big(\tfrac{|I|}{2|I'|}|k|\big)^{-2} \le 4\Lambda|k|^{-2},$$

and (4.4) follows.

Now, using (4.4) we get

$$\begin{aligned}
&\sum_{|I|>2|J|^{-1}} \sum_{\substack{n\in\mathbb{Z}\\ E_{(n,I)}\prec J}} |I|^{-1}|f*h_I(k_{n,I})|^2\\
&\qquad\le C' \sum_{|I|>|I'|/\kappa} \sum_{k\in\mathbb{Z}} \sum_{\substack{l\in\mathbb{Z}\\ E_{(l,I)}\prec E_{(k,I')}}} (1+|k|)^{-2} \int_{E_{(l,\tilde I)}} |f*h_I(x)|^2 dx\\
&\qquad\le C' \sum_{|I|>|I'|/\kappa} \sum_{k\in\mathbb{Z}} (1+|k|)^{-2} \int_{\tilde E_{(k,I')}} |f*h_I(x)|^2 dx\\
&\qquad= C' \sum_{k\in\mathbb{Z}} (1+|k|)^{-2} \sum_{|I|>|I'|/\kappa} \int_{\tilde E_{(k,I')}} |f*h_I(x)|^2 dx,
\end{aligned}$$

where

$$\tilde E_{(k,I')} = \bigcup_{l\in\mathbb{Z}\colon E_{(l,I)}\prec E_{(k,I')}} E_{(l,\tilde I)}.$$

Notice that

$$|I'|^{-1} \le |\tilde E_{(k,I')}| \le 4|I'|^{-1},$$

independent of $k \in \mathbb{Z}$, since $|I'| \le |I|$, and $|I| \le |\tilde I| \le 2|I|$. The theorem will be proved if we can show that

$$\sum_{|I|>|I'|/\kappa} \int_{\tilde E_{(k,I')}} |f*h_I(x)|^2\, dx \le C|I'|^{-1}\|f\|_{\mathrm{BMO}}, \tag{4.5}$$

with a constant C independent of k.

For a fixed $E = \tilde{E}_{(k,I')}$, write[2]

$$f = f_{3E} + (f - f_{3E})\chi_{3E} + (f - f_{3E})\chi_{\mathbb{R}\setminus 3E} = f_1 + f_2 + f_3,$$

where f_{3E} is the mean value of f over $3E$. f_1 contributes nothing to (4.5) since $b_I(0) = 0$ for all $I \in \mathcal{I}$. For f_2 we have,

$$\begin{aligned}\sum_{|I|>|I'|/\kappa} \int_E |f_2 * h_I(x)|^2\,dx &\le \int_{\mathbb{R}} \sum_{|I|>|I'|/\kappa} |b_I(\xi)|^2 |\hat{f}_2(\xi)|^2\,d\xi \\ &\le \|f_2\|_{L_2}^2 \le C|I'|^{-1}\|f\|_{\mathrm{BMO}}^2.\end{aligned}$$

Recall that since $h_I(x) = |I| g_I(|I|x) e^{ix\alpha_I}$ we have $|h_I(x)| \le C|I|(1 + |I||x|)^{-r}$, $r \ge 2$, by (2.9). Hence, for $x \in E$ we have the pointwise estimate

$$\begin{aligned}|f_3 * h_I(x)| &\le C \int_{\mathbb{R}\setminus 3E} |I||f(y) - f_{3E}|(1 + |I||x-y|)^{-2}\,dy \\ &\le C|I|^{-1} \int_{\mathbb{R}\setminus 3E} |f(y) - f_{3E}||x-y|^{-2}\,dy.\end{aligned}$$

Let x_E be the center of E, $E_0 := E - x_E$ and $\tilde{f}(y) := f(y + x_E)$. Then $\tilde{f}_{3E_0} = f_{3E}$, and since for any $x \in E_0$ and $y \in \mathbb{R} \setminus 3E_0$,

$$|x-y|^2 \ge c(|I'|^{-2} + |y|^2),$$

for an absolute constant $0 < c < 1$, we have

$$|f_3 * h_I(x)| \le C'|I|^{-1} \int_{\mathbb{R}} \frac{|\tilde{f}(y) - \tilde{f}_{3E_0}|}{|I'|^{-2} + |y|^2}\,dy \le C'' \tfrac{|I'|}{|I|} \|\tilde{f}\|_{\mathrm{BMO}},$$

where we have used an estimate from [5, p. 142] in the last inequality. The result now follows, since the BMO-norm is translation invariant. □

We can also prove an inverse inequality to (4.3). The idea is to expand the functions $w_{n,I}$ into a sequence of special functions satisfying the three conditions in (4.6) below. The following two lemmas and proofs are modifications of the Lemmas 3.3 and 3.4 in [16]. We denote by Lip1, the set of absolutely continuous functions f satisfying

$$|f|_{\mathrm{Lip1}} := \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x-y|} < \infty.$$

Lemma 4.4. *For a positive integer j, let $\{\beta^j_{n,I}\}_{n\in\mathbb{N}_0, I\in\mathcal{I}} \subset \mathrm{Lip1}(\mathbb{R})$ be a sequence of nontrivial functions, satisfying*

$$\begin{cases} \mathrm{supp}(\beta^j_{n,I}) \subset 2^j E_{(n,I)}, \\ \int_{\mathbb{R}} \beta^j_{n,I}(x)\,dx = 0, \\ |\beta^j_{n,I}|_{\mathrm{Lip1}} \le 2^{-j}|I|. \end{cases} \tag{4.6}$$

[2] For a constant $a > 0$ and an interval $I \subset \mathbb{R}$, aI denotes an interval concentric with I, satisfying $|aI| = a|I|$.

Then there exists a constant $C > 0$, such that

$$\Big\| \sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} c_{n,I}\beta^j_{n,I}\Big\|^2_{L_2} \le C2^{2j} \sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} |c_{n,I}|^2|I|^{-1}. \tag{4.7}$$

Proof. Given $n, m \in \mathbb{N}_0$ and $I, J \in \mathcal{I}$, satisfying $|I| \le |J|$. By (4.6) there exists $x_0 \in 2^jE_{(m,J)}$ such that $\beta^j_{m,J}(x_0) = 0$. Thus,

$$\begin{aligned}\Big|\int_{\mathbb{R}} \beta^j_{n,I}(x)\overline{\beta^j_{m,J}(x)}\,dx\Big| &\le \int_{\mathbb{R}} |\beta^j_{n,I}(x) - \beta^j_{n,I}(x_0)||\beta^j_{m,J}(x) - \beta^j_{m,J}(x_0)|\,dx\\ &\le 2^{-j}|I|2^{-j}|J| \int_{2^jE_{(m,J)}} |x - x_0|^2\,dx\\ &\le 2^{-2j}|I||J|2^{3j}|E_{(m,J)}|^3 = 2^{j+3}|I||J|^{-2}.\end{aligned}$$

For $k \in \mathbb{Z}$, $n \in \mathbb{N}_0$ and $I \in \mathcal{I}$, let

$$s_k(n, I) = \{(m, J) \in \mathbb{N}_0 \times \mathcal{I} \colon 2^k|I| \le |J| \le 2^{k+1}|I|,\ 2^jE_{(m,J)} \cap 2^jE_{(n,I)} \ne \emptyset\}.$$

Notice that $\#s_k(n, I) \le C2^j(1 + 2^k)$ independent of n and I. Now,

$$\begin{aligned}&\int_{\mathbb{R}}\Big| \sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} c_{n,I}\beta^j_{n,I}(x)\Big|^2 dx\\ &\qquad\le C' \sum_{k\in\mathbb{N}_0} \sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} |c_{n,I}| \sum_{(m,J)\in s_k(n,I)} |c_{m,J}|\Big|\int_{\mathbb{R}} \beta^j_{n,I}(x)\overline{\beta^j_{m,J}(x)}\,dx\Big|\\ &\qquad\le C'' \sum_{k\in\mathbb{N}_0} \sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} |c_{n,I}| \sum_{(m,J)\in s_k(n,I)} |c_{m,J}|2^j|I||J|^{-2}\\ &\qquad\le C''2^j \sum_{k\in\mathbb{N}_0} 2^{-2k}\Big(\sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} |c_{n,I}|^2|I|^{-1}\Big)^{1/2}\\ &\qquad\qquad\cdot\Big(\sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} \Big(\sum_{(m,J)\in s_k(n,I)} |c_{m,J}|\Big)^2|I|^{-1}\Big)^{1/2},\end{aligned} \tag{4.8}$$

where we have used that $|I||J|^{-2} \asymp 2^{-2k}|I|^{-1}$ in the last inequality. The last term can be estimated by

$$\begin{aligned}&\Big(\sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} \Big(\sum_{(m,J)\in s_k(n,I)} |c_{m,J}|\Big)^2|I|^{-1}\Big)^{1/2}\\ &\qquad\le \Big(\sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} (\#s_k(n, I))\Big(\sum_{(m,J)\in s_k(n,I)} |c_{m,J}|^2\Big)|I|^{-1}\Big)^{1/2}\\ &\qquad\le C\Big(\sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} 2^{j+k} \sum_{(m,J)\in s_k(n,I)} |c_{m,J}|^2|I|^{-1}\Big)^{1/2} \quad \text{since } k \in \mathbb{N}_0\end{aligned}$$

$$\le C\Big(2^{j+2k+1}\sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}}\sum_{(m,J)\in s_k(n,I)}|c_{m,J}|^2|J|^{-1}\Big)^{1/2}$$

$$= C\Big(2^{j+2k+1}\sum_{\substack{m\in\mathbb{N}_0\\ J\in\mathcal{I}}}(\#s_{-k-1}(m,J))|c_{m,J}|^2|J|^{-1}\Big)^{1/2}$$

$$\le C'2^{j+k}\Big(\sum_{\substack{m\in\mathbb{N}_0\\ J\in\mathcal{I}}}|c_{m,J}|^2|J|^{-1}\Big)^{1/2}.$$

Thus, using this estimate in (4.8) implies (4.7). □

Lemma 4.5. *Suppose $\{\beta^j_{n,I}\}_{n\in\mathbb{N}_0,I\in\mathcal{I}}$ satisfies* (4.6) *for some $j\in\mathbb{N}$. Then there exists a constant $C>0$, such that $\|f\|_{\text{BMO}}\le C2^jM$ for*

$$f=\sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}}c_{n,I}\beta^j_{n,I},$$

where

$$M=\sup_J\Big(|J|^{-1}\sum_{E_{(n,I)}\prec J}|c_{n,I}|^2|I|^{-1}\Big)^{1/2},$$

the sup being taken over all intervals $J\subset\mathbb{R}$.

Proof. For a fixed interval $J\subset\mathbb{R}$ let

$$f_1=\sum_{\substack{n\in\mathbb{N}_0,I\in\mathcal{I}\\ |I|>2^j|J|^{-1}}}c_{n,I}\beta^j_{n,I},\qquad f_2=f-f_1.$$

Clearly, $|c_{n,I}|\le M$ for any $n\in\mathbb{N}_0$, $I\in\mathcal{I}$. Hence for $x,y\in J$ we have

$$|f_2(x)-f_2(y)|\le\sum_{\substack{n\in\mathbb{N}_0,I\in\mathcal{I}\\ |I|\le 2^j|J|^{-1}}}|c_{n,I}||\beta^j_{n,I}(x)-\beta^j_{n,I}(y)|$$

$$\le 2^{-j}M|J|\sum_{\substack{n\in\mathbb{N}_0,I\in\mathcal{I}\\ |I|\le 2^j|J|^{-1},\,2^jE_{(n,I)}\cap J\neq\emptyset}}|I|.$$

Since for each fixed $I\in\mathcal{I}$ there is at most 2^j intervals $E_{(n,I)}$ such that $2^jE_{(n,I)}\cap J\neq\emptyset$, we get

$$|f_2(x)-f_2(y)|\le 2^{-j}M|J|2^j2^j|J|^{-1}\sum_{k=0}^{\infty}\lambda^{-k}\le C2^jM.$$

Moreover, by Lemma 4.4 we have

$$\|f_1\|^2_{L_2(J)}\le C2^{2j}\sum_{\substack{n\in\mathbb{N}_0,I\in\mathcal{I}\\ |I|>2^j|J|^{-1},\,2^jE_{(n,I)}\cap J\neq\emptyset}}|c_{n,I}|^2|I|^{-1}\le C2^{2j}|J|M^2.$$

Notice that by (4.6) and the definition of f_2 there exists $y \in J$ such that $f_J = f_2(y)$, and since the above estimates give

$$\int_J |f(x) - f_2(y)|^2\,dx \le \int_J |f_1(x) + f_2(x) - f_2(y)|^2\,dx \\ \le C2^{2j}|J|M^2,$$

the result now follows. □

We can now proof an upper bound of the BMO-norm given by a brushlet expansion.

Theorem 4.6. *Let $\{w_{n,I}\}_{n\in\mathbb{N}_0,I\in\mathcal{I}}$ be a brushlet system with associated ramp function $\rho \in C^r(\mathbb{R})$, for some $r \ge 2$. Suppose $\{c_{n,I}\}_{n\in\mathbb{N}_0,I\in\mathcal{I}}$ is a sequence of complex numbers satisfying*

$$M = \sup_J \Big(|J|^{-1} \sum_{\substack{n\in\mathbb{N}_0,I\in\mathcal{I}\\ E_{(n,I)}\prec J}} |c_{n,I}|^2\Big)^{1/2} < \infty,$$

where the sup is taken over all intervals $J \subset \mathbb{R}$. Then there exists a constant $C > 0$, such that

$$\Big\|\sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} c_{n,I}w_{n,I}\Big\|_{\mathrm{BMO}} \le CM.$$

In order to prove the theorem we need the following Lemma from [16].

Lemma 4.7 ([16]). *Given an interval $J \subset \mathbb{R}$ with center x_J and a function $p \in C^1(\mathbb{R})$ such that*

$$|p(x)| \le C|J|^2(|J| + |x - x_J|)^{-2}, \qquad \big|\tfrac{d}{dx}p(x)\big| \le C|J|^2(|J| + |x - x_J|)^{-3},$$

and $\int_{\mathbb{R}} p(x)\,dx = 0$. Then there exists a sequence of functions $\{\beta^j\}_{j\in\mathbb{N}_0} \subset C^1(\mathbb{R})$ satisfying

$$\begin{cases} \mathrm{supp}(\beta^j) \subset 2^j J, \\ \int_{\mathbb{R}} \beta^j(x)\,dx = 0, \\ |\beta^j|_{\mathrm{Lip1}} \le C2^{-j}|J|^{-1}, \end{cases}$$

and such that

$$p = \sum_{j\in\mathbb{N}_0} 2^{-2j}\beta^j.$$

Proof of Theorem 4.6. Given a brushlet $w_{n,I}$, define two functions $w^+_{n,I}$ and $w^-_{n,I}$ by

$$w^\pm_{n,I}(x) := \sqrt{\frac{|I|}{2}} e^{i\alpha_I x} g_I(|I|(x \mp k_{n,I})),$$

such that $w_{n,I} = w^+_{n,I} + w^-_{n,I}$. By (2.9) and (3.5) there exists a constant $C < \infty$, such that

$$|I|^{-1/2}|w^{\pm}_{n,I}(x)| \le C(1+|I||x \mp k_{n,I}|)^{-2},$$
$$|I|^{-1/2}\left|\tfrac{d}{dx}w^{\pm}_{n,I}(x)\right| \le C|I|(1+|I||x \mp k_{n,I}|)^{-3},$$

and $\int_{\mathbb{R}} w^{\pm}_{n,I}(x)\,dx = 0$, for any $n \in \mathbb{N}_0$ and $I \in \mathcal{I}$. Thus, by Lemma 4.7, there exists a sequence of functions $\{\beta^j_{n,I}\}_{j\in\mathbb{N}_0}$ satisfying (4.6) and such that

$$|I|^{-1/2}w^{\pm}_{n,I} = \sum_{j\in\mathbb{N}_0} 2^{-2j}\beta^j_{n,I}.$$

Now, using Lemma 4.5 we obtain

$$\Big\|\sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} c_{n,I}w^{\pm}_{n,I}\Big\|_{\mathrm{BMO}} \le \sum_{j\in\mathbb{N}_0} 2^{-2j}\Big\|\sum_{\substack{n\in\mathbb{N}_0\\ I\in\mathcal{I}}} c_{n,I}|I|^{1/2}\beta^j_{n,I}\Big\|_{\mathrm{BMO}} \le C\sum_{j\in\mathbb{N}_0} 2^{-j}M \le 2CM.$$

□

Combining Theorem 4.3 and Theorem 4.6, we obtain a characterization of the space BMO using a brushlet system. This concludes the paper.

Proposition 4.8. *Let $\{w_{n,I}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ be a brushlet system with associated ramp function $\rho \in C^r(\mathbb{R})$, for some $r > 2$. Then $\{w_{n,I}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ constitutes an unconditional basis for the space* BMO, *and we have the following characterization for all $f \in$* BMO,

$$\|f\|_{\mathrm{BMO}} \asymp \sup_{J\subset\mathbb{R}}\Big(|J|^{-1}\sum_{\substack{n\in\mathbb{N}_0, I\in\mathcal{I}\\ E_{(n,I)}\prec J}} |\langle f, w_{n,I}\rangle|^2\Big)^{1/2},$$

with the equivalence depending only on the brushlet system.

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