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by

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Partitions and domination in a graph

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Abstract

Consider a graph G in which the vertices are partitioned into k subsets. For each subset we want a set of vertices of G that dominate that subset. Note that the vertices doing the domination need not be in the subset itself. We are interested in dominating the *entire* graph G as well as dominating *each* of the k subsets and minimizing the sum of these $k+1$ dominating sets. For trees and for all values of k, we can determine an upper bound on this sum and characterize the trees that achieve it.

1 Motivation

We observe that the connections in a computer network between a workstation and its local file server can be modelled as domination in a graph.

Some files (data and text files for instance) are compatible with all computers on the network while other files (such as binary code files) are only compatible with a particular type of workstation (Mac, Sun, Sparc for example).

The file server ordinarily is shared among the workstations and it must be located either at the workstation it caters to or one communication step away.

We model the computer network by a graph, where each workstation and each file server is represented by a vertex. The requirement that the file server is, or is adjacent to, the workstation it serves corresponds in the model to demanding that some collection of text file servers must dominate every vertex of the graph, since each workstation shall have access to common requirements of data, text, latex, emacs files, internet connection, open windows, dos etc. The workstations are of type $1, 2, \cdots, k$, and for each i, $1 \leq i \leq k$, some collection of file servers must dominate every vertex of the graph of type i.

We wish to minimize the number of file servers. That is, the sum $\gamma+\gamma_1+\gamma_2+\cdots+\gamma_k$ where γ is the number of text file servers needed so that every workstation has access to data, text, internet connection, latex, emacs files, and γ_i is the number needed of specialized file servers for workstations of type i.

2 Definitions

A vertex of valency one is called a *leaf* and the unique neighbour of a leaf is called a *stem*. A path on ℓ vertices is denoted P_ℓ. A *block graph* is a graph in which each block is a complete graph. In particular, a tree is a block graph since each block is a K_2. A *cactus* is a connected graph in which each block is a circuit or a K_2, while an *n-cactus* is a cactus with n circuits. A *packing* is a set of vertices pairwise at distance at least three apart.

D is called a *dominating* set of a graph G if each vertex not in D has a neighbour in D. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set D of G.

Let k be a positive integer and G a graph with at least k vertices. By a *partition* of $V(G)$, denoted by $\pi_k = \{V_1, V_2, \cdots, V_k\}$, we understand k disjoint, nonempty subsets whose union is $V(G)$.

For each $i, 1 \leq i \leq k$, let D_i be a smallest sized subset of $V(G)$ such that each vertex from V_i either belongs to D_i or has a neighbour in D_i. We define $\gamma_i = \gamma_G(V_i) = |D_i|$ and $\gamma_G(\emptyset) = 0$.

3 Theorems

In this paper we restrict our attention to connected graphs. We shall establish upper bounds for

$$\gamma(G, \pi_k) = \gamma(G) + \sum_{i=1}^{k} \gamma_G(V_i).$$

In addition, for all k we also characterize those extremal graphs G and the associated vertex partition π_k which achieve the upper bound of $\gamma(G, \pi_k) = \max\{\gamma(G, \pi) | \pi \text{ is a partition of } V(G) \text{ into } k \text{ sets}\}$.

We first observe that for $k = 1$ we have $V_1 = V(G)$ and if $|V(G)| \geq 2$ we have for a graph with no isolated vertex that $\gamma(G, \pi_1) = \gamma(G) + \gamma(G) \leq |V(G)|$ [6, Prop. 1], [1, Th. 1]. The inequality is sharp, since $\gamma(P_4, \pi_1) = 4$.

For $k = 2$ we formulate the following for a tree T with partition $\pi_2,\ V(T) = V_1 \cup V_2,\ V_1 \cap V_2 = \emptyset,\ V_1 \neq \emptyset,\ V_2 \neq \emptyset$:

Theorem 1 *Let T be a tree with at least two vertices and let $s(T)$ denote the number of stems in T with exactly one leaf attached to it. Then for any partition π_2 of $V(T)$ we have*

$$\gamma(T,\pi_2) = \gamma(T) + \gamma_T(V_1) + \gamma_T(V_2) \le |V(T)| + \frac{s(T)}{2}.$$

Proof: We first observe that the theorem holds for a path, a star, and for a star with some of its edges subdivided.

(1) For a path on ℓ vertices, $P_\ell = x_1x_2\ldots x_\ell,\ \ell \ge 2$, we have for $\ell = 3n,\ n \ge 1$ or $\ell = 3n+2,\ n \ge 0$, that $\gamma(P_\ell) = \lceil \frac{\ell}{3} \rceil$ and $3\lceil \frac{\ell}{3} \rceil \le \ell + 1 = |V(P_\ell)| + \frac{s(P_\ell)}{2}$, so the theorem holds in these cases.

For $\ell = 3n+1,\ n \ge 1$ we define

$$\begin{array}{rcl} D &=& \{x_2, x_5, \ldots, x_{2+3i}, \ldots, x_{3n-1}, x_{3n+1}\} \\ D_1 &=& \{x_2, x_5, \ldots, x_{2+3i}, \ldots x_{3n-1}\} \cup (\{x_{3n+1}\} \cap V_1) \\ D_2 &=& \{x_2, x_5, \ldots, x_{2+3i}, \ldots x_{3n-1}\} \cup (\{x_{3n+1}\} \cap V_2) \end{array}$$

and we obtain $\gamma(P_\ell) = |D| = n+1,\ \gamma_T(V_1) \le |D_1|,\ \gamma_T(V_2) \le |D_2|,\ s(P_\ell) = 2$, and hence $\gamma(P_\ell) + \gamma_{P_\ell}(V_1) + \gamma_{P_\ell}(V_2) \le 3n+2 = |V(P_\ell)| + \frac{s(P_\ell)}{2}$. Thus the theorem holds for a path.

(2) For a star $T = K_{1,p-1}, p \ge 2$, with central vertex x let $D = D_1 = D_2 = \{x\}$. Then $\gamma(T) + \gamma_T(V_1) + \gamma_T(V_2) = 3 \le p + \frac{s(T)}{2}$.

(3) For T a star with some edges subdivided,

$$\begin{array}{rcl} V(T) &=& \{x\} \cup \{w_1, \ldots, w_m\} \cup \{y_1, z_1, y_2, z_2, \ldots, y_n, z_n\}, \\ E(T) &=& (\bigcup_{i=1}^m xw_i) \cup (\bigcup_{i=1}^n \{xy_i, y_iz_i\}), \qquad n \ge 1, m \ge 0, \text{ and } n+m \ge 2 \end{array}$$

we define

$$\begin{array}{rcl} D &=& \left\{ \begin{array}{ll} \{y_1, \ldots, y_n\}, & \text{if } m = 0 \text{ and } n \ge 2 \\ \{x\} \cup \{y_1, \ldots, y_n\} & \text{otherwise} \end{array} \right. \\ D_1 &=& \{x\} \cup (\{z_1, \ldots, z_n\} \cap V_1), \\ D_2 &=& \{x\} \cup (\{z_1, \ldots, z_n\} \cap V_2). \end{array}$$

Then

$$\gamma(T) + \gamma_T(V_1) + \gamma_T(V_2) \le |D| + |D_1| + |D_2| \le |V(T)| + 1 \le |V(T)| + \frac{s(T)}{2},$$

if $s(T) \geq 2$; $s = 0$ cannot occur since $n \geq 1$ and y_1 is a stem with a single leaf; if $s(T) = 1$ then $n = 1, m \geq 2$ and we obtain $|D| + |D_1| + |D_2| \leq |V(T)| \leq |V(T)| + \frac{s(T)}{2}$. So the theorem is proven for case (3).

We proceed by induction on $p = |V(T)|$, the number of vertices of T. The cases $p = 2, 3, 4$ are proven above. Let $p \geq 5$ and assume the theorem is true for trees with at most $p-1$ vertices.

Assume T as a subgraph contains a path $xyzw$ pendant from w, i.e., $d_T(x) = 1,\ d_T(y) = d_T(z) = 2$ and consider the tree $T' = T - \{x, y, z\}$. From $p \geq 5$ it follows that $|V(T')| \geq 2$. If one of $V_1' = V_1 \cap V(T'),\ V_2' = V_2 \cap V(T')$ is empty, it is easy to verify the theorem, since $\gamma(T') + (\gamma_{T'}(V_1') + \gamma_{T'}(V_2')) \leq \frac{p-3}{2} + \frac{p-3}{2}$ and adding y to each of the three dominaton sets in T' we obtain $\gamma(T, \pi_2) \leq p$. Otherwise $\{V_1', V_2'\}$ is a partition of $V(T')$ and by the induction hypothesis

$$\gamma(T') + \gamma_{T'}(V_1') + \gamma_{T'}(V_2') \leq p - 3 + \frac{s(T')}{2} \leq p - 3 + \frac{s(T)}{2}.$$

Adding y to all three domination sets in T' we obtain the desired inequality for T.

If T contains at most one vertex of degree at least three, then one of the above cases occurs and the theorem holds. So, we may assume that T contains two vertices of degree at least three.

Let $z_1 z_2 \dots z_\ell,\ \ell \geq 2$, be a path such that $d_T(z_1) \geq 3,\ d_T(z_\ell) \geq 3$ and
$d_T(z_i) = 2,\ 2 \leq i \leq \ell - 1$.

$T - z_1 z_2$ consists of two trees, namely T_{z_1} containing z_1 and T_{z_ℓ} containing z_ℓ. Both have at least two vertices.

By the induction hypothesis we have

$$\gamma(T_{z_1}) + \gamma_{T_{z_1}}(V_1 \cap V(T_{z_1})) + \gamma_{T_{z_1}}(V_2 \cap V(T_{z_1})) \leq p' + \frac{s'}{2},$$

$$\gamma(T_{z_\ell}) + \gamma_{T_{z_\ell}}(V_1 \cap V(T_{z_\ell})) + \gamma_{T_{z_\ell}}(V_2 \cap V(T_{z_\ell})) \leq p'' + \frac{s''}{2},$$

where

$p' = |V(T_{z_1})|,\ p'' = |V(T_{z_\ell})|;\ p' + p'' = p = |V(T)|;$
$s' = s(T_{z_1}),\ s'' = s(T_{z_\ell});\ s = s(T).$

If $\ell = 2$ then $s' + s'' \leq s$ and the theorem holds. So we may assume $\ell \geq 3$, and hence $s' + s'' \leq s + 1$.

$\gamma(T, \pi_2)$ satisfies $\gamma(T, \pi_2) \leq p + \left\lfloor \frac{s'}{2} + \frac{s''}{2} \right\rfloor = p + \lfloor \frac{s+1}{2} \rfloor$. If $s(T)$ is even, then $\gamma(T, \pi_2) \leq p + \frac{s}{2}$ and the theorem holds. So we may assume that $s(T)$ is odd. Then $s' + s'' = s + 1$ is even and either s', s'' are both odd or both even. If s', s'' are odd, then $\left\lfloor \frac{s'}{2} \right\rfloor + \left\lfloor \frac{s''}{2} \right\rfloor = \frac{s-1}{2} \leq \frac{s}{2}$ and the theorem holds. If s', s'' are both even, then we use for the induction argument $T - z_{\ell-1}z_\ell$ where the tree containing z_ℓ has an odd number, namely $s'' - 1$, of vertices with exactly one leaf attached. This completes the proof of Theorem 1.

□

Corollary 1 *Theorem 1 also holds for a block graph.*

Proof: From the definition of domination we have that $\gamma(G, \pi_k) \leq \gamma(T, \pi_k)$ where k is a positive integer, G is a connected graph with at least k vertices, π_k is a partition of $V(G)$ and T is a spanning tree for G. Consider a spanning tree by selecting for each interior block a Hamilton path between two of its cut vertices and selecting $t - 1$ leaves in each end block with t vertices. □

The inequality of Theorem 1 has been proven valid for graphs with minimum degree two by Seager [7]; in particular $\gamma(G, \pi_2) \leq |V(G)|$ for such graphs.

The following result shows that the trees which achieve the upper bound for $\gamma(T, \pi_2)$ have the same structure as those which are optimal for $\gamma(T, \pi_1)$, namely that each vertex either is a leaf or has exactly one leaf as a neighbour, and furthermore , once the leaves are removed, the resulting tree also has that structure.

Theorem 2 *Let T be a tree with p vertices, $p \geq 3$. For any partition $\pi_2 = \{V_1, V_2\}$ of $V(T)$ we have*

$$\gamma(T, \pi_2) = \gamma(T) + \gamma_T(V_1) + \gamma_T(V_2) \leq \frac{5p}{4}. \tag{1}$$

$\gamma(T, \pi_2) = \frac{5p}{4}$ *occurs if and only if*

(i) $p \equiv 0$ *(mod 4),*

(ii) every vertex is a leaf or has precisely one leaf attached to it,

(iii) every vertex of degree three or more has precisely one P_2 attached.

(iv) If w is a vertex of degree 3 or more and z is its unique leaf, and wxy is its unique attached P_2, then y and z must belong to one class V_i, while x must belong to the other class V_{3-i} where $i = 1$ or $i = 2$ and w can be in either of the two classes.

Proof: In any tree on p vertices, $p \geq 3$, the number $s = s(T)$ of stems with precisely one leaf attached obviously is at most $\frac{p}{2}$. Together with Theorem 1 that implies the inequality. Equality in (1) is attained for a tree with the structure described in (i)-(iv), when V_1, say, contains all leaves, V_2 contains all degree two vertices and vertices of degree ≥ 3 are distributed arbitrarily between V_1 and V_2.

Conversely, if equality holds, then $p = 4\alpha$, $\alpha \geq 1$, since $\frac{5p}{4}$ is an integer. It remains to prove that equality in (1) implies (ii) and (iii). We shall do that by induction on p.

A tree on four vertices is $K_{1,3}$ or P_4. The path P_4 has the structure of (ii) and (iii), the star $K_{1,3}$ does not, but equality cannot occur for $K_{1,3}$ since $\gamma(K_{1,3}, \pi_2) = 3$ for any partition.

Let $p > 4$, and assume equality in (1) implies (ii) and (iii) for trees with fewer than p vertices. Assume T and V_1, V_2 satisfies

$$\gamma(T) + \gamma_T(V_1) + \gamma_T(V_2) = \frac{5p}{4}.$$

We shall prove (ii) and (iii). Obviously $s(T) \leq \frac{p}{2}$ and together with Theorem 1 that implies $\gamma(T, \pi_2) = \frac{5p}{4} \leq p + \frac{s(T)}{2} \leq \frac{5p}{4}$. Hence $s(T) = \frac{p}{2}$ and by definition of s every vertex of T either is a leaf or has exactly one leaf attached to it. That proves (ii).

Let $x_1^{(1)} x_2^{(1)} x_3 x_4 \ldots x_\ell$ be a longest path in T. It is easy to see that $s = \frac{p}{2}$ implies that $T = P_4$, for which (iii) is true, or $\ell \geq 5$, as we assume henceforth. From (ii) it follows that $\deg_T(x_2^{(1)}) = 2$, and to x_3 is attached a single leaf w. Further, there are attached m P_2's to x_3, one of which is $x_1^{(1)} x_2^{(1)} x_3$, namely

$$x_1^{(1)} x_2^{(1)};\ x_1^{(2)} x_2^{(2)};\ \ldots;\ x_1^{(m)} x_2^{(m)}, \qquad m \geq 1.$$

$T - x_3 x_4$ has precisely two components. Let T' be the component containing x_4. Define $V_1' = V_1 \cap V(T')$, $V_2' = V_2 \cap V(T')$. Even if one of V_1', V_2' is empty we have (using the definition $\gamma_{T'}(\emptyset) = 0$)

$$\gamma(T') + \gamma_{T'}(V_1') + \gamma_{T'}(V_2') \leq \frac{5}{4}|V(T')| = \frac{5}{4}(4\alpha - 2m - 2) \qquad (2)$$

Let D', D_1', D_2' be the sets in $V(T')$ which dominate $V(T'), V_1', V_2'$, respectively. Then the three sets

$$\begin{array}{lcl} D & = & D' \cup \{w, x_2^{(1)}, x_2^{(2)}, x_2^{(3)}, \ldots, x_2^{(m)}\} \\ D_1 & = & D_1' \cup \{x_3\} \cup \left(\{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(m)}\} \cap V_1\right) \\ D_2 & = & D_2' \cup \{x_3\} \cup \left(\{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(m)}\} \cap V_2\right) \end{array} \quad \text{in } T \text{ dominate,}$$

respectively, $V(T), V_1$ and V_2. We infer that

$$5\alpha = \gamma(T) + \gamma_T(V_1) + \gamma_T(V_2) \leq \frac{5}{4}(4\alpha - 2m - 2) + (m+1) + 2 + m = 5\alpha + \frac{1-m}{2},$$

so that $m = 1$. Thus $|V(T')| = 4\alpha - 4$ and T', V_1', V_2' achieve equality in (2). Hence by the induction hypothesis T' has the structure described in (i)-(iii), implying that T has the structure described in (i)-(iii). Finally, it is easy to verify that the partition must satisfy (iv). □

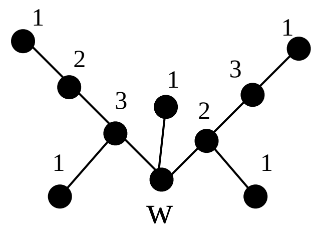

Figure 1: Note that the bound given in equation 3 of Theorem 3 is sharp for $p \equiv 0 \pmod{10}$. Simply take copies of a graph with the indicated partition and join the vertices labelled w.

Theorem 3 *Let p be an integer, $p \geq 3$. Let T be a tree on p vertices, such that $T \notin \{P_4, P_7\}$ and let $\pi_3 = \{V_1, V_2, V_3\}$ be a partition of $V(T)$. Then*

$$\gamma(T, \pi_3) = \gamma(T) + \gamma_T(V_1) + \gamma_T(V_2) + \gamma_T(V_3) \leq \frac{7p}{5}. \qquad (3)$$

$\gamma(T, \pi_3) = \frac{7p}{5}$ occurs if and only if $T = P_5, T = P_{10}$ or T is composed of graphs from fig. 1 with vertex partition π_3 as indicated on fig. 1.

If $\gamma(T, \pi_3) = \frac{7p}{5}$ then $p = 5$ or $p \equiv 0 \pmod{10}$.

Proof:

A path $P_4 = x_1x_2x_3x_4$ with partition $\pi_3 = \{\{x_1, x_4\}, \{x_2\}, \{x_3\}\}$ has $\gamma(P_4, \pi_3) = 6 > \frac{7}{5} \cdot 4$ and a path $P_7 = x_1x_2x_3x_4x_5x_6x_7$ with partition $\pi_3 = \{\{x_1, x_4, x_7\}, \{x_2, x_5\}, \{x_3, x_6\}\}$ has $\gamma(P_7, \pi_3) = 10 > \frac{7}{5} \cdot 7$, so P_4 and P_7 do not satisfy inequality (3).

We use induction on p. The inequality is easily verified for P_3, for the star on four vertices, and for the three trees on five vertices. We note that for $p = 5$ equality in (3) can only occur for $T = P_5$. Let $p > 5$, assume the result is true for trees on at most $p - 1$ vertices, and let T be a tree on p vertices such that $T \neq P_7$.

Assume that some vertex y in T has two leaves x_1, x_2 attached to it. We shall then prove that (3) holds with strict inequality. Define

$T' = T - x_1$. If $T' = P_7$, we can verify the result. Otherwise, let $V_i' = V_i \cap V(T'),\ 1 \leq i \leq 3$; $\pi_3' = \{V_1', V_2', V_3'\}$. We then have

$$\gamma(T', \pi_3') = \gamma(T') + \gamma_{T'}(V_1') + \gamma_{T'}(V_2') + \gamma_{T'}(V_3') \leq \frac{7}{5}(p-1).$$

If $\pi_3' = \{V_1', V_2', V_3'\}$ is a partition of $V(T')$ this inequality follows from the induction hypothesis and we can obtain that $\gamma(T, \pi_3) < \frac{7p}{5}$. If V_1', V_2', V_3' is not a partition of $V(T')$ then one of the sets, say V_3', is empty. That implies $V_1 = V_1', V_2 = V_2', V_3 = \{x_1\}$, and by Theorem 2 there exist subsets D', D_1', D_2' of $V(T')$ which dominate respectively $V(T'), V_1'$, and V_2', and which satisfy $|D'| + |D_1'| + |D_2'| \leq \frac{5}{4}(p-1)$. Since in T' there is a leaf attached to y, we may assume $y \in D'$, so in T the sets $D = D', D_1 = D_1', D_2 = D_2', D_3 = \{x_1\}$ dominate respectively $V(T), V_1, V_2$ and V_3 and thus we have the desired inequality $\gamma(\pi_3, T) \leq \frac{5}{4}(p-1) + 1 < \frac{5}{4}p < \frac{7}{5}p$. We observe from the arguments above that an extremal graph for inequality (3) cannot have a stem with more than one leaf.

We may now assume that no vertex in T has more than one leaf attached to it. Consider a longest path $x_1^{(1)} x_2^{(1)}\ x_3^{(1)} x_4\ \ldots x_\ell$ in T. Then $x_1^{(1)}$ is the only leaf attached to $x_2^{(1)}$. Let there be precisely m paths on two vertices $x_1^{(1)} x_2^{(1)};\ x_1^{(2)} x_2^{(2)};\ \ldots;\ x_1^{(m)} x_2^{(m)},\quad m \geq 1$, attached to $x_3^{(1)}$ and at most one leaf. Assume first that no leaf is attached to $x_3^{(1)}$. Let T' be the component of $T - x_3^{(1)}$ which contains x_4 and let $V_i' = V_i \cap V(T'),\quad 1 \leq i \leq 3$; $\pi_3' = \{V_1', V_2', V_3'\}$. If $|V(T')| \leq 4$ or $T' = P_7$ we can verify case by case that inequality (3) holds for T. Hence we assume $5 \leq |V(T')| < p$, $T' \neq P_7$, and by the induction hypothesis, or by Theorem 2 as explained below, we obtain

$$\gamma(T') + \gamma_{T'}(V_1') + \gamma_{T'}(V_2') + \gamma_{T'}(V_3') \leq \frac{7}{5}|V(T')|. \qquad (4)$$

If $V_i' \neq \emptyset, 1 \leq i \leq 3$, then π_3' is a partition of $V(T')$ and by the induction hypothesis (4) holds. If precisely one of the sets V_i' is empty, say $V_3' = \emptyset$, we recall that by definition $\gamma_{T'}(\emptyset) = 0$ and we obtain from Theorem 2 that $\gamma(T') + \gamma_{T'}(V_1') + \gamma_{T'}(V_2') \leq \frac{5}{4}|V(T')| < \frac{7}{5}|V(T')|$, so (4) holds. Finally, if two of the sets V_i' are empty, say

$V_2' = V_3' = \emptyset$, then $\gamma(T') + \gamma_{T'}(V_1') \leq |V(T')| < \frac{7}{5}|V(T')|$. So in each case T' and V_1', V_2', V_3' satisfy (4). We note that if one or two of the sets V_1', V_2', V_3' is empty, then T' is not extremal.

We can dominate the rest of T, i.e., $T - T'$, with $m + m + 3$ more vertices and thus

$$\gamma(T, \pi_3) \leq \frac{7}{5}(p - 2m - 1) + 2m + 3.$$

The expression above is $< \frac{7}{5}p$ for $m \geq 3$, equals $\frac{7}{5}p$ for $m = 2$, and for $m = 1$ we, in fact, only needed 4 more vertices, so in that case as well

$$\gamma(T, \pi_3) \leq \frac{7}{5}(p - 3) + 4 < \frac{7}{5}p.$$

Thus, inequality (3) holds if no leaf is attached to $x_3^{(1)}$. Furthermore, the inequality must be strict as can be verified by checking any optimal graph for T' (by the induction hypothesis T' equals P_5, P_{10} or is composed of graphs from fig. 1) joined to the central vertex of P_5 (i.e., $T - T'$ where $m = 2$).

We assume next that a leaf and m paths on two vertices, $m \geq 1$, are attached to $x_3^{(1)}$. For $0 \leq |V(T')| \leq 4$ and for $T' = P_7$ we can verify Theorem 3 case by case, so assume $|V(T')| \geq 5$ and $T' \neq P_7$.

As above, dominations in T' (= the component of $T - x_3^{(1)}$ containing x_4) can be done with at most $\frac{7}{5}|V(T')| = \frac{7}{5}(p-2m-2)$ vertices. The rest, $T - T'$, can be dominated with $2m + 4$ further vertices, namely $m + 1$ for overall domination, plus $x_3^{(1)}$ once for each of the three partition classes, plus m vertices at distance two from $x_3^{(1)}$. We have $\frac{7}{5}(p - 2m - 2) + 2m + 4 < \frac{7}{5}p$ for $m \geq 2$, so we may assume $m = 1$, i.e., $\deg_T(x_3^{(1)}) = 3$.

Consider x_4. To x_4 there may, apart from the component of $T - x_4$ containing x_5, be attached at most one leaf, q paths on two vertices, $q \geq 0$, and m paths on four vertices: $x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, z^{(i)}$, $1 \leq i \leq m$, with each $x_3^{(i)}$ adjacent to x_4. We shall treat the remainder of the proof in two cases.

Case 1. Assume no leaf is attached to x_4. In $T - x_4x_5$ dominations in the component containing x_5 can (argue case by case for ≤ 4 vertices and for P_7, so assume that the component has ≥ 5 vertices and is not P_7) be done with $\leq \frac{7}{5}(p-4m-2q-1)$ vertices while the component containing x_4 can be dominated by $(2m+q)+3+(3m+q)$ more vertices, and

$$5m+2q+3 \leq \tfrac{7}{5}(4m+2q+1)$$

$$8 \leq 3m+4q$$

holds, unless

(i) $m=2,\ q=0$,

(ii) $m=1,\ q=0$,

(iii) $m=1,\ q=1$.

In each case we can find dominations of T with $< \frac{7}{5}p$ vertices. For the component of $T-x_4x_5$ containing x_4 we can in case (i) use 4 vertices for overall domination, use another 6 vertices by placing $x_3^{(1)}$ and $x_3^{(2)}$ in all three domination sets and further use 2 vertices to dominate two leaves at distance 3 from x_4. Adding we obtain $12 < \frac{7}{5} \cdot 9$ as desired. For case (ii) we can dominate with $2+3+1=6$ vertices, and $6 < \frac{7}{5} \cdot 5$. For case (iii) we can dominate with 9 vertices and $9 < \frac{7}{5} \cdot 7$. Thus, in case 1 inequality (3) holds strictly.
<u>Case 2.</u> Assume a leaf is attached to x_4. If the component T_{x_5} of $T - x_4x_5$ containing x_5 has 2, 3 or 4 vertices or is P_7 we can verify Theorem 3 case by case. Assume T_{x_5} has ≥ 5 vertices and is not P_7. Let $\pi_3 = \{V_1, V_2, V_3\}$ be a partition of $V(T)$, and let $\pi'_3 = \{V'_1, V'_2, V'_3\}$, where $V'_i = V_i \cap V(T_{x_5})$, $1 \leq i \leq 3$. Whether π'_3 is a partition or not of $V(T_{x_5})$, we have

$$\gamma(T_{x_5}) + \gamma_{T_{x_5}}(V'_1) + \gamma_{T_{x_5}}(V'_2) + \gamma_{T_{x_5}}(V'_3) \leq \frac{7}{5} \cdot |V(T_{x_5})|,$$

and we remember for later use that equality can only occur if T_{x_5} is P_5, P_{10} or composed of graphs from fig. 1.

Denote by T_{x_4} the component of $T - x_4x_5$ containing x_4. Recall that T_{x_4} consists of one leaf, q paths on two vertices, $q \geq 0$, and m

paths on four vertices $x_1^{(i)}x_2^{(i)}x_3^{(i)}z^{(i)}$ with $x_3^{(i)}$ joined to x_4 for each i, $1 \leq i \leq m$. We can do the required dominations in T_{x_4} with $5m + 2q + 4$ vertices. As $|V(T_{x_4})| = 4m + 2q + 2$ we want

$$5m + 2q + 4 \leq \tfrac{7}{5}(4m + 2q + 2)$$

or

$$6 \leq 3m + 4q.$$

This holds for $m \geq 2$. We obtain equality for $m = 2, q = 0$. That case leads to an extremal graph precisely when T_{x_5} is extremal; and we can verify that T is extremal only if T_{x_5}, and consequently also T, is composed of graphs from fig. 1.

For $m = 1, q \geq 1$ we have $6 < 3m + 4q$, so inequality (3) holds strictly. That, in fact, is also the case for $m = 1, q = 0$ since we combine $\gamma(\pi_3', T') \leq \frac{7}{5}(p-6)$ with 8 further vertices for dominations in T_{x_4} to obtain $\gamma(\pi_3, T) \leq \frac{7}{5}(p-6) + 8 < \frac{7}{5}(p-6) + \frac{7}{5} \cdot 6 = \frac{7}{5}p$. This finally proves Theorem 3. □

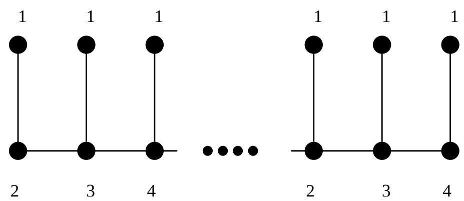

Figure 2: Theorem 4 is best possible. This is demonstrated by this tree with $p \equiv 0 \pmod 6$.

Theorem 4 *Let T be a tree on p vertices, $p \geq 4$, where the vertices are partitioned into k subsets, $k \geq 4$. Then $\gamma(T) + \gamma_1 + \gamma_2 + \ldots + \gamma_k \leq \frac{3}{2}p$.*

Equality, $\gamma(T, \pi_k) = \frac{3}{2}p$, occurs if and only if p is even and the following three conditions hold:

(1) every vertex is either a leaf or has precisely one leaf attached,

(2) the maximum degree of any vertex of T is strictly less than k.

(3) Each $V_i, 1 \leq i \leq k$, is a packing in T.

Proof:

For any tree T on p vertices at most $p/2$ vertices are required in a dominating set (see [1]). For the p vertices in the k subsets, at worst we could let each vertex dominate itself. This establishes the upper bound.

We first consider the "only if" direction for equality. Let k be at least 4 and let T be a tree with partition $\pi_k : V(T) = V_1 \cup V_2 \cup \ldots \cup V_k$. If $\gamma(T, \pi_k) = 3p/2$, then every vertex of T is either a leaf or has exactly one leaf attached. Otherwise, fewer than $p/2$ vertices would be required to dominate T itself, resulting in the total number for T together with the subsets being fewer than $3p/2$. So (1) holds. If k is at least 4 and T, π_k achieves $3p/2$, then the maximum degree of T must be strictly less than k. Assume not. Let v be a vertex of degree at least k. But then the vertices of the k subsets $V_1, V_2, \ldots, V_k$ could be dominated using fewer than p vertices as follows. Using the vertex v once for each of the k subsets, i.e. k times, the closed neighbourhood of v would be dominated, leaving fewer than $p - k$ vertices. Using each one of these to dominate itself results in a total of fewer than p. Since at most $p/2$ are needed to dominate the tree T itself we have a contradiction. This proves (1) and (2). Recall that a packing is a set of vertices with the property that no pair of them are adjacent nor share a common neighbour.

If for some i, V_i contains two vertices which are adjacent or have a common neighbour we have $\gamma(V_i) < |V_i|$ and therefore $\gamma(T, \pi_k) < 3p/2$ proving (3). Thus "the only if" direction follows.

Next consider a tree T which satisfies conditions (1), (2) and (3).

The first condition guarantees that T itself requires $p/2$ vertices to dominate. $V(T)$ is by (3) partitioned into k disjoint packings V_i, so that p vertices will be needed to dominate the vertices of the k subsets. Thus a total of $3p/2$ is forced. This establishes the "if" direction. □

Remarks.

Again, theorem 4 is best possible as illustrated in Figure 2.

For the if direction we need not assume (3) in the sense, that it is proven in [2] that when $n \geq k$ and the maximum degree of T is less than k, we can partition its vertices into at most k disjoint packings.

We conclude with the following observation.

Theorem 5 *If G is a n-cactus and V_1, V_2 is a partition of its vertices then $\gamma(G) + \gamma_G(V_1) + \gamma_G(V_2) \leq p + \frac{s}{2} + \frac{n}{2}$, where s is the number of vertices having exactly one leaf attached to it.*

Proof: If G is just a cycle, the theorem is easy to verify. For G a unicyclic graph delete an edge on a cycle adjacent to a vertex of degree 3 or more. Apply Theorem 1 and note that the resulting tree has at most one leaf that G did not have. The result follows.

Assume the result holds for all n-cactus graphs and consider a $(n+1)$-cactus. Select an endblock that is a circuit and delete an edge incident with a degree 3 vertex. By induction the theorem follows.
□

For general graphs we observe that any spanning tree will give an upper bound. Domination and partitioned domination is also treated in [4, 5] and [8].

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