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by

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Abstract

Let a and b be integers $4 \leq a \leq b$. We give simple, sufficient conditions for graphs to contain an even $[a, b]$-factor. The conditions are on the order and on the minimum degree, or on the edge-connectivity of the graph.

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1 Introduction

We denote by G a graph of order $n = |V(G)|$. For a vertex x in $V(G)$ let $d_G(x)$ denote its degree. By $\delta = \delta(G) = \min\{d_G(x) | x \in V(G)\}$ we denote the minimum degree in G. Let X, Y be an ordered pair of disjoint subsets of $V(G)$, and f, g be mappings from $V(G)$ into $\mathbb{N}$. By $e(X, Y)$ we denote the number of edges with one endvertex in X and the other in Y. By $h(X, Y)$, we denote the number of odd components in $G - (X \cup Y)$. A component C of $G - (X \cup Y)$ is called odd if $e(C, Y) + \sum_{c \in V(C)} f(c)$ is an odd number. An even factor of G is a spanning subgraph all of whose degrees are even. If $g(x) \leq f(x)$ for all x in $V(G)$, by a $[g, f]$-factor we understand a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$, for all $x \in V(G)$.

Theorem 1 *(***Lovász' parity $[g, f]$-factor theorem***[13],[3]).*

Let G be a graph and let g, f be maps from $V(G)$ into the nonnegative integers such that for each $v \in V(G)$, $g(v) \leq f(v)$ and $g(v) \equiv f(v) \pmod 2$. Then G contains a $[g, f]$-factor F such that $d_F(v) \equiv f(v) \pmod 2$, for each $v \in V(G)$, if and only if, for every ordered pair X, Y of disjoint subsets of $V(G)$

$$(*) \qquad h(X,Y) - \sum_{x \in X} f(x) + \sum_{y \in Y} g(y) - \sum_{y \in Y} d_G(y) + e(X,Y) \leq 0.$$

Tutte's f-factor theorem is surveyed in [1]. Let us recall other results on $[a, b]$-factors. In [7], Kano and Saito proved that, for any nonnegative integers k, r, s, t satisfying $k \leq r, 1 \leq r, ks \leq rt$, every graph with degrees in the interval $[r, r + s]$ has a $[k, k + t]$-factor. Berge, Las Vergnas, and independently Amahashi and Kano, proved for any integer $b \geq 2$, that a graph has a $[1, b]$-factor if and only if $b|N(X)| \geq |X|$ for all independent vertex sets X of the graph. Kano proved a sufficient condition for a graph to have an $[a, b]$-factor giving a condition on the sizes $|N(X)|$ for subsets X of $V(G)$ [8]. Cui and Kano generalized Tutte's 1-factor theorem. They consider a map $f : V(G) \rightarrow \{1, 3, 5, \ldots\}$ and call F an odd $[1, f]$-factor of G if F is a factor of G with $d_F(v)$ odd and $d_F(v) \in [1, f(v)]$ for all vertices v in G. They prove that G has an odd $[1, f]$-factor if and only if $G - X$ has at most $\sum_{x \in X} f(x)$ components of odd cardinality for any subset $X \subseteq V(G)$ [5]. Then,Topp and Vestergaard restrict the number of subsets to be considered above, and, as a consequence, proved that a graph of even order n in which no vertex v is the center of an induced $K_{1,nf(v)+1}$ -star has an odd $[1, f]$-factor [15]. In [9, 10], Kouider and Maheo prove the existence of connected [a,b] factors in graphs of high degree. For even factors with degrees between 2 and b we establish a sufficent condition in [11].

Theorem 2 *Let $b \geq 2$ be an even integer and let G be a 2-edge connected graph with n vertices and with minimum degree $\delta(G) \geq \min\{3, \frac{2n}{b+2}\}$. Then G contains an even $[2, b]$-factor.*

We shall now generalize this to even factors with degrees between a and b, where a is an even integer ≥ 4.

2 Results

Let a, b, $a \leq b$, be even, positive integers. In the inequality (*), we substitute $e(X, Y)$ by $|X||Y|$, and derive a sufficient condition for existence of an even $[a, b]$-factor in G:

$$(**) \qquad h(X,Y) - b|X| + a|Y| - \delta|Y| + |X||Y| \leq 0.$$

We shall prove the following results.

Theorem 3 *Let a, b be two even integers satisfying $4 \le a \le b$. Let G be a 2-edge connected graph of order n at least $\max\left\{\frac{(a+b)^2}{b}, \frac{3(a+b)}{2}\right\}$, and of minimum degree δ at least $\frac{an}{a+b}$. Then G has an even $[a, b]$-factor.*

Example 1. Take even integers a, b such that $a \ge 12$, $b = 2a^2$, let $\delta = \frac{3a}{2} + 4$ and let G be the graph which consists of $2a - 2$ disjoint copies of a complete graph $K_{\delta+1}$, each copy joined by one edge to a common vertex y.

The order of G is $n = (\frac{3a}{2} + 5)(2a - 2) + 1 = 3a^2 + 7a - 9$, and it is easy to see that $n \ge \max\left\{\frac{(a+b)^2}{b}, \frac{3}{2}(a+b)\right\}$. The minimum degree of G is $\delta = \frac{3a}{2} + 4$ and the inequality $\delta \ge \frac{an}{a+b}$ follows from $\frac{an}{a+b} = \frac{a(3a^2+7a-9)}{a+2a^2} = \frac{3a^2+7a-9}{2a+1} \le \frac{3a^2+7a}{2a} \le \frac{3}{2}a + \frac{7}{2}$. So G is not 2-edge connected but satisfies all other conditions of Theorem 3. The graph G has no even $[a, b]$-factor F, because F must contain an edge from y to K, one of the complete graphs $K_{\delta+1}$, and the rectriction of F to K should contain exactly one odd vertex, which is impossible.

Example 2. For a positive integer $k \ge 5$, let $a = 2k + 2$ and $b = ka$. Let $n = k(3k + 2) + 1$. We consider a graph G of order n, composed of k vertex disjoint copies of the complete graph K_{3k+2} , and an external vertex x_0 joined to each copy by 3 edges. This graph is 2-edge connected, its minimum degree is $\delta = 3k \ge \frac{an}{a+b}$, and $n \ge \frac{(a+b)^2}{b}, n \ge \frac{3b}{2}$. In an even $[a, b]$-factor F of G the vertex x_0 must be joined to at least $2k + 2$ other vertices, so in F at least one of the K_{3k+2}'s, say K, is joined to x_0 by exactly 3 edges. Thus the graph K should have a subgraph, namely $K \cap F$, with an odd number of odd vertices. Hence G has no even $[a, b]$-factor.

This example shows that even if G is 3-edge connected the conditions $\delta \ge \frac{an}{a+b}$ and $n \ge max\{\frac{(a+b)^2}{b}, 3b/2\}$ are not sufficient for existence of an even $[a, b]$-factor, even if a is much more smaller than b.

Theorem 4 *Let $a \ge 4$ and $b \ge a$ be two even integers. Let G be a 2-edge connected graph of order $n \ge \frac{(a+b)^2}{b}$ and of minimum degree at least $\frac{an}{a+b} + \frac{a}{2}$. Then G has an even $[a, b]$-factor.*

In the following result, we have a weaker condition on the order, but a stronger one on the edge-connectivity.

Theorem 5 *Let $a \geq 4$ and $b \geq a$ be two even integers, and let $k \geq a + \min\left\{\sqrt{a}, \frac{b}{a}\right\}$. Let G be a k-edge-connected graph of order $n \geq \frac{(a+b)^2}{b}$ and of minimum degree at least $\frac{an}{a+b}$. Then G has an even $[a,b]$-factor.*

Example 3. Let a, b, k be integers such that $b > 3a^2$, and $k \leq a-1$; furthermore a, b are even and k is odd. We define a k-connected graph G as follows.
Let Y be a set k independant vertices, and consider a family of $k+2$ complete graphs H_i for $1 \leq i \leq k+2$ such that $H_i = K_{a+2}$ for $i \leq k+1$, and $H_{k+2} = K_{b+3a-(k+1)(a+3)+1}$. Each $y \in Y$ is joined to exactly $a+1$ vertices, one in H_i for each i, $1 \leq i \leq k+1$, and $a-k$ vertices in H_{k+2} so that no two vertices of Y have a common neighbour. So $d_H(y) = a+1$, for each $y \in Y$. The order n of G is $3a+b$. As $b > 3a^2$, one can verify that $\delta \geq \frac{an}{a+b}$. Thus G satisfies all conditions in Theorem 5, except the one on k. Suppose that G has an even $[a,b]$-factor F. Now, let y be any vertex in Y. As $d_G(y) = a+1$ and $a+1$ is odd, it follows that $d_F(y) = a$. Then necessarily, there exists a copy H_t for some $t \leq k$ such that $e_G(Y, H_t) = e_F(Y, H_t)$. It follows that the restriction of the factor F to H_t has k odd vertices; as k is odd, that is impossible. So, the graph G has no even $[a,b]$-factor.

3 Proofs

We shall use Claims 1-4 below for the proof of Theorem 3. First we establish the truth of (*) for a large class of ordered pairs X, Y.
Let $\tau(X,Y) = h(X,Y) - b|X| + a|Y| - \sum_{y \in Y} d_G(y) + e(X,Y)$.

The hypotheses of Theorem 3 imply that $\delta \geq \max\left\{\frac{3a}{2}, a + \frac{a^2}{b}\right\}$.

Claim 1 *Inequality (*) holds if $-b|X| + a|Y| \leq 0$.*

Proof. Recall, that for any odd component C, $b|V(C)| + e(C,Y)$ is odd; as b is even, that implies $e(C,Y) \geq 1$. Hence, between Y and each odd component of $G - (X \cup Y)$ there is at least one edge, therefore $h(X,Y) + e(X,Y) \leq \sum_{y \in Y} d_G(y)$, and (*) follows as $-b|X| + a|Y| \leq 0$. □

Claim 2 *Inequality (*) holds if* $|Y| \geq a+b$.

Proof. Let $-b|X|+a|Y|=p$. By Claim 1, we may assume $p>0$. By definition of $h(X,Y)$, we have $|X|+|Y|+h(X,Y)\leq n$. Then we obtain

$$|X| = \frac{a|Y|-p}{b} \leq \frac{a(n-h(X,Y)-|X|)-p}{b},$$

and thus

$$|X| \leq \frac{a(n-h(X,Y))-p}{a+b}.$$

So

$$e(X,Y) \leq |X||Y| \leq \frac{a(n-h(X,Y))-p}{a+b}|Y|.$$

By hypothesis on δ we have

$$-\sum_{y\in Y} d_G(y) \leq -\delta|Y| \leq -\frac{an}{a+b}|Y|.$$

That yields the inequality

$$\tau(X,Y) \leq h(X,Y)+p-\frac{an}{a+b}|Y|+\frac{a(n-h(X,Y))-p}{a+b}|Y|.$$

So now, since $|Y|\geq a+b$, we get

$$\tau(X,Y) \leq h(X,Y)+p-\frac{a(h(X,Y)+p)}{a+b}|Y| \leq (1-a)(h(X,Y)+p).$$

As $a\geq 4$ and $p>0$, we conclude that $\tau(X,Y)\leq 0$ and (*) is proven. □

By Claims 1 and 2 we may henceforth assume $0 \leq \frac{b}{a}|X| < |Y| \leq a+b-1$.

Proof of Theorem 3. We assume $0 \leq \frac{b}{a}|X| < |Y| \leq a+b-1$ and, following the different values of $|Y|$, we proceed to prove that $\tau(X,Y)\leq 0$.
As $h(X,Y)\leq n-|X|-|Y|$, $\tau(X,Y)$ is bounded as follows:

$$\tau(X,Y) \leq h(X,Y)-b|X|+a|Y|-\delta|Y|+|X||Y| \leq n-(\delta-a+1)|Y|+|X|(|Y|-b-1),$$

and therefore, to prove $\tau(X,Y)\leq 0$ it suffices to prove that

$$(***) \qquad n-(\delta-a+1)|Y|+|X|(|Y|-b-1)\leq 0.$$

<u>Case $|Y|\geq b+1$.</u>
Let us set

$$\phi(|Y|) = n-(\delta-a+1)|Y|+\frac{a}{b}|Y|(|Y|-b-1).$$

As $|X| < \frac{a}{b}|Y|$, we see that (***) will follow if $\phi(|Y|)\leq 0$.

Claim 3 $\phi(|Y|) \le 0$.

Proof. For $|Y|$ varying in the interval of integers, $[b+1, a+b-1]$, the maximum value of the parabola ϕ is attained at an endpoint of the interval. In both ends we shall show that $\phi(|Y|) \le 0$.

$$\phi(b+1) = n - (\delta - a + 1)(b+1);$$

and as $-\delta \le -\dfrac{an}{a+b}$, we get

$$\phi(b+1) \le n\frac{b-ab}{a+b} + ab + a - b - 1.$$

As $-n \le -\dfrac{(a+b)^2}{b}$, we obtain
$\phi(b+1) \le (a+b)(1-a) - (b+1)(1-a) = -(1-a)^2 \le 0$.
At the other endpoint,
$\phi(a+b-1) = n + \left(-(\delta - a + 1) + \dfrac{a}{b}(a-2)\right)(a+b-1)$.
As $\delta \ge \dfrac{an}{a+b}$, we get $\phi(a+b-1) \le n\dfrac{2a+b-a^2-ab}{a+b} + (a+b-1)(a^2-2a+ab-b)\dfrac{1}{b}$.

Now the inequalities $n \ge \dfrac{(a+b)^2}{b}$ and $2a - a^2 + b - ab = -a(a-2) - b(a-1) \le 0$ imply

$$\phi(a+b-1) \le \frac{2a+b-a^2-ab}{b}(a+b-a-b+1)$$
$$\phi(a+b-1) \le \frac{-a(a-2)-b(a-1)}{b} \le 0.$$

This proves Claim 3. □

Henceforth we may assume $|Y| \le b$ and $|X| \le a-1$, as $|X| < \dfrac{a}{b}|Y|$.

Let H be the set of odd components C of $G - (X \cup Y)$. Then, $H = H_1 \cup H_2$ where H_1 is the set of the odd components C having $e(C, Y) = 1$, and H_2 is the set of those for which $e(C, Y) \ge 3$. Let us set $h = h(X, Y) = |H|$ and $h_i = h_i(X, Y) = |H_i|,\ i = 1, 2$. So $h = h_1 + h_2$.

Claim 4 $h_1 \le \dfrac{n - |Y|}{\delta + 1 - |X|}$.

Proof of Claim 4. A component C in H_1 has at least two vertices. Otherwise $C = \{c\}$ and, the degree of the vertex c could be at most $|X| + 1$; and, as $|X| \leq a - 1$, then $d_G(c) \leq a$; that contradicts $d_G(c) \geq \delta \geq \frac{3a}{2}$. So the component C contains a vertex c' not joined to any vertex in Y, and hence having at least $\delta - |X|$ neighbours in C, therefore $|C| \geq \delta - |X| + 1$ and we obtain $h_1 \leq \frac{n - |Y|}{\delta + 1 - |X|}$. □

We continue with the proof of Theorem 3.

Case $|Y| \leq b$ and $|X| = 0$.

To prove that $\tau(X, Y) \leq 0$ we shall show that

$$h(X, Y) + a|Y| - \sum_{y \in Y} d(y) \leq 0.$$

As G has no bridge, and $|X| = 0$ necessarily $h_1 = 0$, $h = h_2$ and $h \leq \frac{1}{3} \sum_{y \in Y} d(y)$.

Then

$$\tau(X, Y) \leq -\frac{2}{3} \sum_{y \in Y} d(y) + a|Y| \leq |Y|(a - 2\frac{\delta}{3}).$$

As $\delta \geq \frac{3a}{2}$, we conclude $\tau(X, Y) \leq 0$.

From now, $|Y| \leq b$ and $|X| \geq 1$.

Case $|Y| \leq b$ and $1 \leq |X| \leq a - 1$.

We note that

$\sum_{y \in Y} d(y) \geq e(Y, H) + e(X, Y)$, and $e(Y, H) \geq h_1 + 3h_2 = 3h - 2h_1$, so

$$3h \leq \sum_{y \in Y} d(y) - e(X, Y) + 2h_1 ;$$

$$h \leq \frac{\sum_{y \in Y} d(y) - e(X, Y) + 2h_1}{3}.$$

By Claim 4, then

$$h \leq \frac{\sum_{y \in Y} d(y) - e(X, Y)}{3} + \frac{2n}{3(\delta + 1 - |X|)}.$$

Recalling $\tau(X, Y) = h - b|X| + a|Y| - \sum_{y \in Y} d(y) + e(X, Y)$, we obtain the following upper bound for $\tau(X, Y)$.

$$\tau(X,Y) \leq -2\frac{\sum_{y\in Y} d(y) - e(X,Y)}{3} + \frac{2n}{3(\delta+1-|X|)} - b|X| + a|Y|.$$

From $e(X,Y) \leq |X||Y|$ and $\sum_{y\in Y} d(y) \geq \delta|Y|$ we obtain

$$\tau(X,Y) \leq -2\frac{|Y|\delta}{3} + |X|(\frac{2|Y|}{3} - b) + a|Y| + \frac{2n}{3(\delta+1-|X|)}.$$

As $\delta \geq \frac{3a}{2}$, this gives

$$\tau(X,Y) \leq |X|(\frac{2|Y|}{3} - b) + \frac{2n}{3(\delta+1-|X|)}.$$

Inserting $|Y| \leq b$ yields

$$\tau(X,Y) \leq -\frac{b|X|}{3} + \frac{2n}{3(\delta+1-|X|)}.$$

Then τ is strictly positive if and only if

$$b|X| < \frac{2n}{\delta+1-|X|};$$

in other words if

$$(****) \qquad |X|(\delta+1-|X|) < \frac{2n}{b}.$$

Let us consider the left side of this inequality as a function $f(|X|)$ of $|X|$. We have assumed $1 \leq |X| \leq a-1 < \delta$.

For $|X|$ varying in the interval $[1,\delta]$ the function f has its minimum for $|X| = 1$ and $|X| = \delta$, namely $f(1) = f(\delta) = \delta$. Hence inequality $(****)$ implies that $\delta < \frac{2n}{b}$. As $\delta \geq \frac{an}{a+b}$, we should have $b(a-2) < 2a$. But this does not hold for $b \geq a \geq 4$. So we conclude that τ is nonpositive, and Theorem 3 is proven. □

Proof of theorem 4. $\delta \geq \frac{a}{b}(a+b)+\frac{a}{2}$ implies $\delta \geq \frac{a^2}{b}+\frac{3a}{2} \geq \max\left\{\frac{3a}{2}, a+\frac{a^2}{b}\right\}$, and all arguments, including the argument for the case $|Y| \leq b$, can be carried through. □

Proof of theorem 5. Claims 1, 2 and 3 still hold with the hypotheses of Theorem 5, so the proof of Theorem 5 begins analogously to that of Theorem 3, and we reach the assumption $0 \leq \frac{b}{a}|X| < |Y| \leq b$. Now, we examine the missing case.

Case $|Y| \leq b$.

We know that $0 \leq |X| \leq a-1$ (as $|X| < \frac{a}{b}|Y|$). Since G has edge-connectivity at least k, each component of $G-(X \cup Y)$ sends at least $k-|X|$ edges to Y, so $h(X,Y) \leq \frac{\sum_{y\in Y} d(y) - e(X,Y)}{k-|X|}$.

It follows that

$$\tau(X,Y) \leq \frac{\sum_{y\in Y} d(y) - e(X,Y)}{k-|X|} - b|X| + a|Y| + e(X,Y) - \sum_{y\in Y} d(y),$$

$$\tau(X,Y) \leq \frac{k-|X|-1}{k-|X|}(e(X,Y) - \sum_{y\in Y} d(y)) - b|X| + a|Y|.$$

As $0 \leq |X| \leq a-1$ and $k > a$ we have $\frac{k-|X|-1}{k-|X|} > 0$ and inserting $e(X,Y) - \sum_{y\in Y} d(y) \leq |X||Y| - \delta|Y|$ we obtain

$$\tau(X,Y) \leq \frac{k-|X|-1}{k-|X|}(|X||Y| - \delta|Y|) - b|X| + a|Y|,$$

$$\tau(X,Y) \leq |Y|(a - \frac{k-|X|-1}{k-|X|}\delta) + (\frac{k-|X|-1}{k-|X|}|Y| - b)|X|.$$

The last term is nonpositive, since $|Y| \leq b$; so to have $\tau(X,Y) \leq 0$ it will suffice that

(i) $$\delta \geq a\frac{k-|X|}{k-|X|-1}.$$

On one hand, as $\delta \geq k$, it is sufficient that $k \geq a\frac{k-|X|}{k-|X|-1}$; as $|X| \leq a-1$, we see that the inequality (i) is satisfied if $k \geq a + \sqrt{a}$, because $a\frac{k-|X|}{k-|X|-1} = a(1 + \frac{1}{k-|X|-1}) \leq a(1+\frac{1}{\sqrt{a}}) = a + \sqrt{a} \leq k$.

On the other hand, we have

$$\delta \geq \frac{an}{a+b} \geq a(1+\frac{a}{b}).$$

If $k \geq a + \frac{b}{a}$, and as $|X| \leq a-1$, it follows that $k-|X|-1 \geq \frac{b}{a}$ and $a\frac{k-|X|}{k-|X|-1} = a(1+\frac{1}{k-|X|-1}) \leq a(1+\frac{a}{b}) \leq \delta$ and hence, also in this case, the inequality (i) is satified; and $\tau(X,Y) \leq 0$.

This proves Theorem 5. □

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