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Abstract

As a generalization of connected domination in a graph G we consider domination by sets having at most k components. The order $\gamma_c^k(G)$ of such a smallest set we relate to $\gamma_c(G)$, the order of a smallest connected dominating set.
For a tree T we give bounds on $\gamma_c^k(T)$ in terms of minimum valency and diameter. For trees the inequality $\gamma_c^k(T) \leq n - k - 1$ is known to hold, we determine the class of trees, for which equality holds.

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1 Introduction

We consider simple non-oriented graphs. The largest valency in G is denoted by $\Delta(G) = \Delta$, the smallest by $\delta(G) = \delta$. P_n is a path on n vertices and C_n is a circuit on n vertices. In a graph a **leaf** or **pendent vertex** is a vertex of valency one and a **stem** is a vertex adjacent to at least one leaf. In K_2 a vertex is both a leaf and a stem. The set of leaves in T is denoted by $\Omega(T)$. By $K_{1,k}$ we denote a star with one central vertex joined to k other vertices. A **subdivided star** is a star with a subdivision vertex on each edge. A graph G is called a **corona graph** if each vertex of G is a leaf or a stem adjacent to exactly one leaf. For a corona graph we write $G = H \circ K_1$, where H is the subgraph in G spanned by all stems in G. If H is a tree we obtain a **corona tree** $T = H \circ K_1$.

The **eccentricity** $e(x)$ of a vertex x is the distance to a vertex at maximum distance from it, $e(x) = \max\{d(x, y)|y \in V(G)\}$. The **diameter** of G is diam(G)=max$\{e(x)|x \in V(G)\}$. Let $D \subseteq V(G)$, then $N(D)$ is the set of vertices which have a neighbour in D and $N[D]$ is the set of vertices which are in D or have a neighbour in D, $N[D] = D \cup N(D)$. A set $D \subseteq V(G)$ **dominates** G if $V(G) \subseteq N[D]$, i.e. each vertex not in D is adjacent to a vertex in D. The **domination number** $\gamma(G)$ is the cardinality of a smallest dominating set in G.

Ore (1962) proved the inequality below and Payan and Xuong (1982), Fink et al. (1985) determined its extremal graphs.

Theorem (Ore, Payan, Xuong). *Let G be a connected graph with n vertices, $n \geq 2$. Then $\gamma(G) \leq \frac{n}{2}$ and equality holds if and only if G is either a corona graph or a 4-circuit.*

If a tree T has $\gamma(T) = \frac{n}{2}$, then n is even and this Theorem implies that T is a corona tree.

Definition For a positive integer k and a graph G with at most k components we define

$$\gamma_c^k(G) = \min\{|D||D \subseteq V(G), D \text{ has at most } k \text{ components and } D \text{ dominates } G\}.$$

A set D attaining the minimum above is called a γ_c^k-set for G.

Example

$$\gamma_c^k(P_n) = \gamma_c^k(C_n) = \begin{cases} n - 2k & \text{for } n \geq 3k \\ \lceil \frac{n}{3} \rceil & \text{for } 1 \leq n \leq 3k \end{cases}$$

For $k = 1$ we have that γ_c^1 is the usual connected domination number, $\gamma_c^1(G) = \gamma_c(G)$.
For G connected and $k \geq 1$, obviously, $\gamma(G) \leq \gamma_c^k(G) \leq \gamma_c(G)$.

2 General graphs

Let G be a connected graph with n vertices and k a positive integer. Let $\epsilon_F(G)$ be the maximum number of leaves among all spanning forests of G, let $\epsilon_T(G)$ be the maximum number of leaves among all spanning trees of G. Then Niemen (1974) proved statement (i) below about γ and (Hedetniemi and Laskar (1984) generalized it to statement (ii) about γ_c.

(i) $\gamma(G) = n - \epsilon_F(G)$,

(ii) $\gamma_c(G) = n - \epsilon_T(G)$.

We extend these results to γ_c^k.

Theorem 1 *Let G be a connected graph with n vertices and k a positive integer. Let $\epsilon_{F_k}(G)$ be the maximum number of leaves among all spanning forests of G with at most k trees. Then*
$\gamma_c^k(G) = n - \epsilon_{F_k}(G)$.

Proof: In any spanning forest F with at most k trees the leaves will be dominated by their stems, so $\gamma_c^k(G) \leq n - |\Omega(F)|$ and hence $\gamma_c^k(G) \leq n - \epsilon_{F_k}(G)$.
Conversely, let $D = D_1 \cup D_2 \cup \cdots \cup D_t,\ 1 \leq t \leq k$, be a γ_c^k-set for G. Choose for each D_i a spanning tree $T_i, 1 \leq i \leq t$. For each vertex in $V(G) - D$ choose one edge to D. We have constructed a spanning forest F with t components and at least $n - |D| = n - \gamma_c^k(G)$ leaves. Therefore $\epsilon_{F_k}(G) \geq n - \gamma_c^k(G)$ and Theorem 1 is proven. □

Theorem 2 *Let k be a positive integer and G a connected graph. Then*

$$\begin{aligned}\gamma_c^k(G) &= \min\left\{\gamma_c^k(F_k) | F_k \textit{ is a spanning forest of } G \textit{ with at most } k \textit{ trees}\right\}\\ &= \min\left\{\gamma_c^k(T) | T \textit{ is a spanning tree of } G\right\}\end{aligned}$$

Proof: Let F_k be a spanning forest of G with at most k trees. Certainly $\gamma_c^k(G) \le \gamma_c^k(F_k)$ since a set which dominates in F_k also dominates in G. Conversely, we can in G find a spanning forest F_k with at most k components such that $\gamma_c^k(G) = \gamma_c^k(F_k)$: As was also done in the proofs of (i) and (ii) above we construct F_k from a γ_c^k-set $D = D_1 \cup D_2 \cup \cdots \cup D_t,\ 1 \le t \le k$, by choosing a spanning tree T_i in each connected subgraph D_i and joining each vertex in $V(G) - D$ to precisely one vertex in D. Obviously, $\gamma_c^k(F_k) \le |D| = \gamma_c^k(G)$. This proves the first equality. For the second equality we observe that the first minimum is chosen among a larger set, so that $\min \gamma_c^k(F_k) \le \min \gamma_c^k(T)$, and secondly that any F_k by addition of edges renders a tree T with $\gamma_c^k(T) \le \gamma_c^k(F_k)$. □

Hartnell and Vestergaard (2003a) proved the following result.

Theorem (Hartnell, Vestergaard). *For $k \ge 1$ and G connected*

$$\gamma_c(G) - 2(k-1) \le \gamma_c^k(G) \le \gamma_c(G).$$

From this theorem we can easily derive the following classical result proven by Duchet and Meyniel (1982).

Corollary (Duchet, Meyniel) *For any connected graph G, $\gamma_c(G) \le 3\gamma(G) - 2$.*

Proof: Let G be a connected graph with domination number $\gamma(G)$. Choose $k = \gamma(G)$, then $\gamma_c^k(G) = \gamma(G)$. Substituting into Hartnell's and Vestergaard's theorem above we obtain $\gamma_c(G) - 2(k-1) \le \gamma(G)$ and that proves the corollary. □

2.1 Other bounds on γ_c^k

Theorem 3 *For a positive integer k and a connected graph G with maximum valency Δ we have*
(A) $\gamma_c(G) \le n - \Delta$ and for trees T equality holds if and only if T has at most one vertex of valency ≥ 3.
(B) $\gamma_c^k(G) \le n - \frac{(D-1)(\delta-2)}{3} - 2k$ if G has diameter $D \ge 3k-1$ and the minimum valency $\delta = \delta(G)$ is at least 3.
(C) If G is a connected graph with two vertices of valency Δ at distance d apart, $d \ge 3$, then $\gamma_c^k(G) \le n - 2(\Delta - 1) - 2\min\{k-1, \frac{d-2}{3}\}$.

(D) Let $x \in V(G)$ have valency $d(x)$ and eccentricity $e(x)$. Then $\gamma_c^k(G) \leq n - d(x) - 2\min\{k-1, \frac{e(x)-2}{3}\}$.

Proof: (A). Let T be a spanning tree of G with $\Delta(T) = \Delta(G) = \Delta$. T has at least Δ leaves, and hence $\gamma_c(G) \leq \gamma_c(T) \leq n - \Delta$.

If T has two vertices of valency ≥ 3, the number of leaves in T wil be larger than Δ, and we get strict inequality in (A). Clearly, a tree T with exactly one vertex of valency $\Delta \geq 3$ has equality in (A) and for $\Delta = 2$, $\gamma_c(P_n) = n - 2$.

(B). Let $P = v_1v_2v_3\dots v_{3t+u}, \quad k \leq t, 0 \leq u \leq 2$, be a diagonal path in G. P has length $D = 3t + u - 1$. For $i = 1, \dots, t$ let v_{3i-1} have neighbours v_{3i-2}, v_{3i} and $a_{ij}, \quad j = 1, \dots, \quad j \geq \delta - 2 \geq 1$. In $G - \{v_{3i}v_{3i+1} | 1 \leq i \leq k-1\}$ consider the $k-1$ disjoint stars with center v_{3i-1} and neighbours $N(v_{3i-1}), \quad 1 \leq i \leq k-1$, and the tree consisting of the path $v_{3k-2}v_{3k-1}v_{3k}\dots v_{3t+u}$ and leaves $v_{3i-1}a_{3i-1,j}, \quad j = 1, \dots$ from vertices $v_{3i-1}, \quad k \leq i \leq t$.

Extend this forest of k trees to a spanning forest F with k trees in $G - \{v_{3i}v_{3i+1} | 1 \leq i \leq k-1\}$. The number of leaves in F is at least $t(\delta - 2) + 2k$ and hence $\gamma_c^k(G) \leq n - t(\delta-2) - 2k$. From $t = \frac{D+1-u}{3} \geq \frac{D-1}{3}$ we obtain $\gamma_c^k(G) \leq n - \frac{(D-1)(\delta-2)}{3} - 2k$.

(C). Let $d(v_1) = d(v_s) = \Delta$ and let $P = v_1v_2\dots v_s$ be a shortest v_1v_s-path, $s = 3t + 1 + u, t \geq 1, 0 \leq u \leq 2$.

<u>$t \geq k-1$</u>: In $G - \{v_{3i-1}v_{3i} | 1 \leq i \leq k-2\}$ we extend the k trees below to a spanning forest F of G,

1. The star consisting of v_1 joined to all its neighbours,
2. the $k-2$ paths of length two $v_{3i}v_{3i+1}v_{3i+2}, 1 \leq i \leq k-2$,
3. the path $v_{3k-3}v_{3k-2}\dots v_s$ together with all $\Delta - 1$ neighbours of v_s outside of P.

F will have at least $2(\Delta - 1) + 2(k-1)$ leaves.

<u>$t \leq k-2$</u>: $s = 3t + 1 + u, d = d(v_1, v_s) = s - 1 = 3t + u, t - 1 = \frac{d-u}{3} - 1 \geq \frac{d-2}{3} - 1$. As before, we can find a spanning forest F of G whose number of leaves is at least $2\Delta + 2(t-1) \geq 2(\Delta - 1) + 2\frac{d-2}{3}$ and consequently $\gamma_c^k(G) \leq n - 2(\Delta - 1) - 2\frac{d-2}{3}$. The proof of (D) is similar. □

3 Trees

For trees Hartnell and Vestergaard (2003a) found

Theorem (Hartnell, Vestergaard). *Let k be a positive integer and T a tree with $|V(T)| = n, n \geq 2k+1$. Then $\gamma_c^k(T) \leq n-k-1$.*

This inequality is best possible. For $k = 1$ the extremal trees are paths P_n and for $k \geq 2$ extremal trees will be described in the following Theorem 4.

A tree is of type A if T contains a vertex x_0 such that $T - x_0$ is a forest of trees $T_1, T_2, \ldots, T_\alpha, \alpha \geq 1$, such that each tree T_i is a corona tree and x_0 is joined to a stem in each of the trees T_i, $1 \leq i \leq \alpha$. We note that a subdivision of a star is a tree of type A.

A tree is of type B if T contains a path uvw such that $T - \{u, v, w\}$ is a forest of corona trees $T_1, T_2, \ldots, T_s$, $T_{s+1,}, \ldots, T_\alpha$, $\alpha \geq 2, 1 \leq s < \alpha$ and u is joined to a stem in each of the trees $T_1, T_2, \ldots, T_s$, while w is joined to a stem in each of the trees $T_{s+1,}, \ldots, T_\alpha$.

The theorem below was proven by Randerath and Volkmann (1998) and Baogen et al. (2000).

Theorem (Randerath, Volkmann, Baogen, Cockayne, et al.). *If T is a tree with n vertices, n odd, and $\gamma(T) = \lfloor \frac{n}{2} \rfloor$ then T is a tree of type A or B.*

We shall now determine the trees extremal for Hartnell, Vestergaard's Theorem.

Theorem 4 *Let $k \geq 2$ be a positive integer and T a tree with n vertices, $n \geq 2k+1$. Then $\gamma_c^k(T) = n-k-1$ if and only if one of cases (i)-(iii) below occur.*
(i) $k = \frac{n-1}{2}$, $\gamma_c^k(T) = \gamma(T) = \frac{n-1}{2}$ and T is of type A or B.
(ii) $k = \frac{n-2}{2}$, $\gamma_c^k(T) = \gamma(T) = \frac{n}{2}$ and T is a corona tree.
(iii) $k = \frac{n-3}{2}$, $\gamma_c^k(T) = \frac{n+1}{2}$, $\gamma(T) = \frac{n-1}{2}$ and T is a star $K_{1,k+1}$ with a subdvision vertex on each edge.

Proof: Let $k \geq 2$ and a tree T of order n be given such that $n \geq 2k+1$ and $\gamma_c^k(T) = n-k-1$. We shall prove that one of cases (i)-(iii) must occur.

We note that $\gamma(T) \leq k$ as well as $\gamma_c^k(T) \leq k$ implies $\gamma_c^k(T) = \gamma(T)$. We also note that for $k \geq 1$ and a tree T of order $n \geq 2$ we either have $n \geq 2k+1$ and then $\gamma_c^k(T) \leq n-k-1$ by Hartnell, Vestergaard's Theorem or $2 \leq n \leq 2k$ and $\gamma_c^k(T) = \gamma(T) \leq \frac{n}{2}$ by Ore, Payan, Xuong's Theorem.

If $n = 2k+1$ we have $\gamma_c^k(T) = n-k-1 = k$. By the remark above $\gamma(T) = k = \lfloor \frac{n}{2} \rfloor$ and from the Theorem by Randerath et al. we see that T is a tree of type A or B, so (i) occurs. If $n = 2k+2$ we have $\gamma_c^k(T) = n-k-1 = k+1$ and $\gamma(T) = \gamma_c^k(T) = \frac{n}{2}$, so T by Ore, Payan, Xuong's Theorem is a corona tree and (ii) occurs. We may now assume $n \geq 2k+3$.

Let $v_1v_2\ldots v_\alpha$ be a longest path in T. Since $\gamma_c^k(T) = n-k-1 \geq k+2 \geq 4$, T is neither a star nor a bistar, so $\alpha \geq 5$. We have $d_T(v_2) = 2$. Otherwise $d_T(v_2) \geq 3$ and we could from T delete three leaves adjacent to v_2 if $d_T(v_2) \geq 4$ and in case $d_T(v_2) = 3$ we could delete v_2 and two leaves adjacent to it obtaining in both cases a tree T' of order $n-3 \geq 2(k-1)+1$ which by Harntell, Vestergaard's Theorem has $\gamma_c^{k-1}(T') \leq (n-3)-(k-1)-1 \leq n-k-3$. Adding v_2 to a $\gamma_c^{k-1}(T')$-set we would obtain $\gamma_c^k(T) \leq n-k-2$, a contradiction so $d_T(v_2) = 2$. No leaf is adjacent to v_3 because, if c were a leaf adjacent to v_3 let d denote either another leaf adjacent to v_3 or let $d = v_3$ if no other leaf exists. Consider $T' = T - \{v_1, v_2, c, d\}$. T' has order $n-4 \geq 2(k-1)+1$ and by Hartnell, Vestergaard's Theorem $\gamma_c^{k-1}(T') \leq (n-4)-(k-1)-1$ $\leq n-k-4$. Adding v_2, v_3 to a $\gamma_c^{k-1}(T')$-set we obtain $\gamma_c^k(T) \leq n-k-2$, a contradiction, so v_3 is not a stem. On the other hand $d_T(v_3) \geq 3$, for assume $d_T(v_3) = 2$, then $T' = T - \{v_1, v_2, v_3\}$ has $\gamma_c^{k-1}(T') \leq n-k-3$ and addition of v_2 gives $\gamma_c^k(T) \leq n-k-2$, a contradiction. Assume therefore that v_3 besides v_2 and v_4 is adjacent to $a_1, a_2, \ldots, a_t$, $t \geq 1$, where each a_i has valency two and is adjacent to the leaf $b_i, 1 \leq i \leq t$. We have $k-t \geq 1$ because $V(T) - \{v_1, b_1, b_2, \ldots, b_t, v_3\}$ is a connected subgraph with $n-t-2$ vertices which dominate T, so that $n-k-1 = \gamma_c^k(T) \leq n-t-2$ giving $k-t \geq 1$. Consider the tree $T' = T - \{v_1, v_2, a_1, a_2, \ldots, b_1, b_2, \ldots, b_t, v_3\}$ of order $n-2t-3$. If $n-2t-3 \geq 2(k-t)+1$ we obtain by Hartnell, Vestergaard's Theorem that $\gamma_c^{k-t}(T') \leq (n-2tk-3)-(k-t)-1 \leq n-k-t-4$, and adding $t+2$ vertices $\{v_2, v_3, a_1, a_2, \ldots, a_t\}$, forming one component, to a $\gamma_c^{k-t}(T')$-set we obtain $\gamma_c^k(T) \leq n-k-2$, a contradiction. So we have $n-2t-3 \leq 2(k-t)$ and by an earlier remark $\gamma_c^{k-t}(T') \leq \frac{n-2t-3}{2}$. That implies $n-k-1 = \gamma_c^k(T) \leq \frac{n-2t-3}{2}+t+2 = \frac{n+1}{2}$ or $n \leq 2k+3$. Together with the assumption $n \geq 2k+3$ we get $n = 2k+3$. Then $\gamma_c^k(T) = k+2$ and we have $\gamma(T) \leq k+1$ by Ore, Payan, Xuong's Theorem. Thus $\gamma(T) = k+1$ and any $\gamma(T)$-set consists of $k+1$ isolated vertices. As $\gamma(T) = \lfloor \frac{n}{2} \rfloor$ the tree T is of type A or B. But T cannot be of type B, for assume T is of type B. Then T consists of a 3-path , uvw, with each of its ends joined to stems of corona trees, and since we have just seen that $v_3, v_{\alpha-2}$ are neither stems nor leaves, they must play the role of u, w, so $\alpha = 7$ and T consists of two subdivided stars centered at $u = v_3$ and $w = v_5$ and a vertex $v = v_4$ joined to u and w. This graph T has a γ-set with two adjacent vertices v_2 and v_3, a contradiction, so T is of type A . Using, in analogy to v_2, v_3, that $d_T(v_{\alpha-1}) = 2$ and that $v_{\alpha-2}$ is not a stem, we get that $\alpha = 5$ and T is a subdivided star so that (iii) occurs.

Conversely, it is easy to see that if (i), (ii) or (iii) holds then $\gamma_c^k(T) = \gamma(T) = n - k + 1$. This proves Theorem 4.

□

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