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December 2003 R-2003-22

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URL: www.math.auc.dk/research/reports/reports.htm

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

DIPATHS AND DIHOMOTOPIES IN A CUBICAL COMPLEX.

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Abstract. In the geometric realization X of a cubical complex, dipaths and dihomotopies may not be combinatorial, i.e., not geometric realizations of combinatorial dipaths and equivalences. When we want to use geometric/topological tools to classify dipaths on the one skeleton, combinatorial dipaths, up to dihomotopy, and in particular up to combinatorial dihomotopy, we need that all dipaths are in fact dihomotopic to a combinatorial dipath. And moreover that two combinatorial dipaths which are dihomotopic are then combinatorially dihomotopic. We prove that any dipath from a vertex to a vertex is dihomotopic to a combinatorial dipath, in a non selfintersecting cubical complex. And that two combinatorial dipaths which are dihomotopic through a non combinatorial dihomotopy are in fact combinatorially dihomotopic, in a geometric cubical complex. Moreover, we prove that in a geometric cubical complex, the d-homotopy introduced in [3] coincides with the dihomotopy in [2]

1. Introduction

The relatively new subject of directed topology and geometry, ditopology, has a combinatorial/algebraic as well as a geometric/topological approach. The subject originates in computer science, where V. Pratt in [4] introduces higher dimensional automata, HDA, as a model for concurrency. This is a both an algebraic and a geometric model: The HDA is a cubical complex, which may be geometrically realized [2] as a locally partially ordered space, or be treated algebraically. As in the development of the non-directed geometry, where one had to compare the more combinatorial approaches and the non-combinatorial approaches to manifolds, we now need to compare the various versions of directed topology. In this paper, we compare the combinatorial directed homotopy classes of combinatorial directed paths, the algebraic side, with directed homotopy classes of directed paths, the geometric side. We prove that in a *geometric* cubical complex, the set of dihomotopy classes of dipaths from a vertex p to a vertex q is isomorphic to the set of combinatorial dihomotopy classes of combinatorial dipaths from p to q. In other words, that a dipath between two vertices is dihomotopic to a combinatorial dipath, i.e., a dipath on the 1-skeleton and also that dipaths on the 1-skeleton are dihomotopic if and only if they are combinatorially dihomotopic, i.e., dihomotopic via a dihomotopy on the 2-skeleton.

A geometric cubical complex is a union of directed n-cubes, such that the intersection of two cubes is a face in both cubes or empty. For simplicial sets, this requirement is quite natural - it means that the subdivision into simplices is a triangulation of the geometric object.

The proofs in this paper are concrete constructions: Given a dipath, we construct a cubical dipath and a dihomotopy between them 4.1. The cubical dipath is not unique, but this is not surprising: The diagonal in a square can be represented both by the two edges running above and by the edges running below the square - and there is not a preferred one, unless there is a consistent numbering of coordinates in all cubes in the complex.

In the last section, we prove that if two combinatorial dipaths are dihomotopic, then they are also cubically dihomotopic. The proof only applies to geometric cubical complexes. We expect that the result is true in more general frameworks, and we give some hints to how a proof of that should go. But we do not give such proofs.

The dihomotopies we give are all d-homotopies [3], and hence we may also conclude that the d-homotopy relation is in fact the same as dihomotopy when we are in a geometric cubical complex. This does not hold for all locally partially ordered spaces. A counterexample is given in [5] p.260; the unreduced suspension $\vec{\Sigma} X = \vec{I} \times X / 0 \times x \sim p_1, 1 \times x \sim p_2$ with partial order only along the suspension coordinate, has only one dihomotopy class of dipaths from p_1 to p_2, if X is connected, but it has a d-homotopy class for each point in X.

We also provide a unique representative of dipaths which traverse the same sequence of carriers. The dimension of the carriers traversed is a measure of the amount of concurrency in an HDA, i.e., how many processes run at a given time.

2. Basic definitions

The definitions here are not new, but we repeat them for the convenience of the reader. Most of them are from [2] and [3]. The carrier sequence and starsequence however are new concepts.

Definition 2.1. *A local po-space is a Hausdorff topological space X with a covering $\mathcal{U} = \{(U_i, \leq_i), i \in J\}$ where $U_i \subset X$ is open and $\leq_i$ is a partial order on U s.t,*

- *$\leq_i$ is closed: For any pair $x, y \in U_i$, with $x \not\leq y$, there are V_x and V_y, open neighborhoods of x and y, s.t. $z \in V_x$ and $w \in V_y$ implies $z \not\leq w$*
- *For all $x \in X$ there is a nonempty open po-neighborhood, $(W_x, \leq_W)$ s.t. whenever $x, y, z \in U_i \cap W_x$, then $y \leq_W z \Leftrightarrow y \leq_{U_i} z$.*
- *Two local partial orders $\mathcal{U}$ and $\mathcal{V}$ on X are equivalent if their union is a local partial order.*

Definition 2.2. *A map $f : X \to Y$ where X and Y are local po-spaces, is a dimap, if f is continuous and for all $x \in X$ there are po-neighborhoods W_x of x and $W_{f(x)}$ of $f(x)$ s.t. for $z, w \in f^{-1}(W_{f(x)}) \cap W_x$ $z \leq_{W_x} w \Rightarrow f(z) \leq_{W_{f(x)}} f(y)$*

Definition 2.3. *A dipath in a local po-space X is a dimap from the ordered unit interval $\vec{I}$ to X. A dimap $H : \vec{I} \times I \to X$ is a dihomotopy with fixed endpoints v and w, if $H(0,s) = v$ and $H(1,s) = w$ for all $s \in I$. Given such a dihomotopy, the dipaths $H_0(t) = H(t,0)$ and $H_1(t) = H(t,1)$ are* dihomotopic.

There is a different equivalence relation of dipaths introduced in [3]

Definition 2.4. *A dimap $H : \vec{I} \times \vec{I} \to X$ with $H(0,s) = v$ and $H(1,s) = w$ is a* d-homotopy *with fixed end points and $H_0(t) = H(t,0)$ and $H_1(t) = H(t,1)$ are* d-homotopic. *The d-homotopy relation is the symmetric transitive closure of this relation.*

Definition 2.5. *A semi-cubical complex M is a family of sets $\{M_n | n \geq 0\}$ with face maps $\partial_i^k : M_n \to M_{n-1}$ $(1 \leq i \leq n,\ k = 0,1)$ satisfying the semi-cubical relations:*

$$\partial_i^k \partial_j^l \quad = \quad \partial_{j-1}^l \partial_i^k \quad (i < j)$$

A non-selfintersecting *cubical complex is a semi cubical complex M such that for any $K \in M$, if $\partial_{\mathbf{l}_1}^{\mathbf{k}_1} K = \partial_{\mathbf{l}_2}^{\mathbf{k}_2} K$ where $\mathbf{k}_i$ are multiindices and $\mathbf{l}_i$ are increasing multiindices, then $\mathbf{k}_1 = \mathbf{k}_2$ and $\mathbf{l}_1 = \mathbf{l}_2$.*

A geometric *cubical complex is a semi cubical complex M such that for any pair L_n and K_m of elements of M, there is a (perhaps empty) common face F_r such that any other common face X_k is a face of F_r.*

Example 2.6. *A non-selfintersecting complex is not necessarily geometric: Two copies of the directed unit interval $\vec{I}$ glued at the endpoints to make a circle is non selfintersecting but not geometric. A subdivision would make this a geometric complex. But this is not always possible: Let $M_2 = \{A, B\}$, $M_1 = \{a, b, c, d, e, f\}$ and $M_0 = \{p, q, r, s, t\}$ and set $\partial_2^0 A = \partial_2^0 B = a$, $\partial_1^1 A = \partial_1^1 B = b$ and $\partial_1^0 A = c$, $\partial_1^0 B = d$,$\partial_2^1 A = e$, $\partial_1^0 B = f$ and attach the vertices consistently with the semi cubical relations. Then subdivision will not make the complex geometric: A neighborhood of the vertex $\partial_1^0 b = \partial_1^1 a$ will remain "non geometric" - even after iterated subdivisions.*

The geometric realization of a semi cubical complex is defined as usual: Let $\mathbf{R}(M) = \coprod_n M_n \times \square_n$. The sets M_n have the discrete topology and $\square_n = [0,1]^n$ with the standard topology. Then $| M | = \mathbf{R}(M)/\equiv$ with the quotient topology, where $\equiv$ is the equivalence relation induced by the identities:

$$\forall k, i, n, \forall x \in M_{n+1}, \forall t \in \square_n, n \geq 0, (\partial_i^k(x), t) \equiv (x, \delta_i^k(t))$$

Remark 2.7. In the geometric realization of a geometric cubical complex, the intersection of two cubes is a face in both the cubes or empty. The term "geometric" is in analogy with [1] p. 246.

Definition 2.8. *Let X be the geometric realization of a semi cubical complex and let $p \in X$. The* carrier *of p is the cube of largest dimension which has p as an interior point.*

The central point *of a cube is the point* $(1/2, \dots, 1/2)$.

The minimal vertex, $(0, \dots, 0) \in L$ *of a cube L is denoted $v_-(L)$ and the* maximal vertex $(1, \dots, 1) \in L$ *is denoted $v_+(L)$.*

The lower boundary of L is $\partial^-(L) = \{(x_1, \dots, x_n) \in L | \exists i : x_i = 0\}$.

The upper boundary of L is $\partial^+(L) = \{(x_1, \dots, x_n) \in L | \exists i : x_i = 1\}$. A face in L is a subset $\{(x_1, \dots, x_n) \in L | x_{i_1} = x_{i_2} = \cdots x_{i_k} = 0, for\ i \in I$ and $x_{j_1} = x_{j_2} = \cdots x_{j_k} = 1\ for\ j \in J\}$ for some index sets I and J.

A face is a lower face, if $J = \emptyset$ and it is an upper face, if $I = \emptyset$.

Definition 2.9. *A cubical dipath in the geometric realization of a cubical complex is a dipath γ such that $\gamma(0)$ and $\gamma(1)$ are vertices and for all t, the carrier of $\gamma(t)$ has dimension at most 1.*

This will also be called a combinatorial dipath. Here is the combinatorial/cubical definition of dihomotopy:

Definition 2.10. *Let γ_1 and γ_2 be cubical dipaths, i.e., dipaths on the 1-skeleton, in a cubical complex X. Then γ_1 is combinatorially dihomotopic to γ_2, if they are equivalent under the equivalence relation generated by reparametrization and $\partial_1^1(F) \star \partial_2^0(F) \sim \partial_1^0(F) \star \partial_2^1(F)$ for all 2-cubes $F \in X_2$. Here $\partial_l^k F$ is considered a dipath in the obvious way, $\star$ is concatenation of dipaths and if $\gamma_1 \sim \gamma_2$ then $\mu \star \gamma_1 \star \eta \sim \mu \star \gamma_2 \star \eta$ whenever concatenation is defined.*

Example 2.11. *Cubical dipaths in one cube with the same initial and final points are cubically dihomotopic. The difference between the initial and final point v and w is a set of coordinates, $I \subset \{1, \dots, n\}$,s.t. $v_i = 0$ and $w_i = 1$ for $i \in I$. A cubical dipath increases one coordinate at a time, so the two dipaths only differ in the sequence in which they increase these coordinates. Moreover an elementary dihomotopy, a 2-face, with lower vertex $(p_1, \dots, p_{j-1}, 0, p_{j+1}, \dots, p_{k-1}, 0, p_{k+1}, \dots, p_n)$ and upper vertex $(p_1, \dots, p_{j-1}, 1, p_{j+1}, \dots, p_{k-1}, 1, p_{k+1}, \dots, p_n)$ is a dihomotopy between a dipath which increases v_j and then v_k and a dipath increasing v_k and then v_j. Hence the elementary dihomotopies give rise to transpositions in the sequence of vertices being increased and since these generate the symmetric group, the statement follows.*

Remark 2.12. It is not hard to see that two cubically dihomotopic paths are d-homotopic. And that a d-homotopy is a dihomotopy.

Definition 2.13. *The open star of a point $p \in \mid M \mid$ is*

$$St(p, M) = \{q \in \mid M \mid \ | carrier(p) \text{ is a face of } carrier(q)\}$$

Remark 2.14. A point p in $|M|$ is a vertex if $carrier(p) \in M_0$. The sets $\{St(v)|v \in M_0\}$ define an open covering of $|M|$

Definition 2.15. *Let $\gamma : I \to X$ be a dipath in the realization of a geometric cubical complex. Then a sequence of vertices $v_0, v_1, \ldots, v_N$ defines a* starsequence *$St(v_0), St(v_1), \ldots, St(v_N)$ if there is a sequence $0 = t_0 < t_1 < t_2 \cdots < t_N = 1$ such that $\gamma([t_k, t_{k+1}]) \subset St(v_k)$ for all $k = 0, 1, \ldots, N-1$*

Remark 2.16. All dipaths have a starsequence, since the stars of vertices define a covering of X, and I is compact - just take $N = 1/\mu$ and $t_k = k/\mu$, where μ is a Lebesgue number for the covering of I given by $\{\gamma^{-1}(St(v))\}$. A starsequence is traversed by γ in the order given by the indices. A starsequence is not unique. If γ starts and ends in vertices, then $\gamma(0) = v_0$ and $\gamma(1) = v_N$.

Definition 2.17. *Let $\gamma : \vec{\mathbb{R}}_+ \to X$ be a dipath in the realization of a geometric cubical complex with $\gamma(0) = v$, a vertex. Then there is a unique sequence of cubes $L_0, L_1, \ldots, L_k, \ldots$ and a sequence of real numbers $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_k \le \cdots$ such that*

- *$L_i \neq L_{i+1}$*
- *There is a $t \in [t_i, t_{i+1}]$ with carrier$(\gamma(t)) \in L_i$.*
- *$t \in [t_i, t_{i+1}] \Rightarrow \gamma(t) \in L_i$*
- *$t \in]t_i, t_{i+1}[\Rightarrow$ carrier$(\gamma(t)) = L_i$*
- *carrier$(\gamma(t_i)) \in \{L_{i-1}, L_i\}$ and carrier$(\gamma(t_{i+1})) \in \{L_i, L_{i+1}\}$.*

The sequence $L_0, L_1, \ldots$ is called the carriersequence *for γ.*

Remark 2.18. Since each point $\gamma(t)$ has a unique carrier, L, since the geometric realization of L is a closed subset of X and since γ is continuous, there is a unique carrier sequence for a dipath. In fact we do not need the path to be directed for this.

Definition 2.19. *A semi-cubical complex is locally finite, if for all vertices v, only finitely many cubes have v as a vertex.*

Proposition 2.20. *Let $\gamma : \vec{I} \to X$ be a dipath in a locally finite geometric cubical complex X. Then the carrier sequence for γ is finite.*

Proof. Since I is compact, γ has a finite starsequence.

To see that the sequence of carriers is finite, notice that local finiteness ensures that there are only finitely many cubes in each $St(v)$. Hence, since there are no loops in a star, and hence no paths leaving a cube and returning to it inside a star, the dipath can only traverse finitely many cubes in each star. ☐

3. Center point approximation.

The first approximation of a dipath is not cubical. An approximation of a dipath γ is a dipath μ, which is dihomotopic to γ and such that the two dipaths have a common starsequence. To a general dipath initiating and ending in a vertex, we provide a dihomotopic dipath which traverses the center points of the carriers in the original dipath. This dipath is unique whereas the cubical approximations we will see later are not. It is clear that the dipaths have a common starsequence, since they even have the same carrier sequence.

Theorem 3.1. *Let $\gamma : \vec{I} \to X$ be a dipath, $\gamma(0) = v_0$ and $\gamma(1) = v_1$, where v_i is a vertex. Let $L_0, L_1, \ldots, L_N$ be the sequence of carriers of $\gamma(\vec{I})$, and let $0 = t_0, t_1, \ldots, t_N = 1$ be as in Def. 2.17. Then there is a dipath $\mu : \vec{I} \to X$ such that μ is dihomotopic to γ and $\mu(t_i + t_{i+1})/2 = c(L_i)$ intersects the centerpoint of all the carriers of γ in the order traced out by γ: $c(L_0), c(L_1), \ldots, c(L_N)$.*

Proof. We construct μ inductively using Lemma 3.2. Let $\mu(0) = v_0 = L_0$ and let $s_i = \frac{t_i+t_{i+1}}{2}$. Our induction hypothesis is: $\mu(t)$ is constructed for $t \leq s_{k-1}$ with $\mu(s_l) = c(L_l)$ for $l \leq k-1$. The induction start is $k = 1$: $s_0 = \frac{t_0+t_1}{2} = 0$, since $t_0 = t_1 = 0$ and $\mu(0) = v_0 = L_0 = c(L_0)$

Case 1: $\dim L_{k-1} < \dim L_k$. Then $L_{k-1} = \{(x_1, \ldots, x_n) \in L_k | x_i = 0 \text{ for } i \in I_k\}$, where $\emptyset \neq I_k \subset \{1, \ldots, n\}$ and $\mu(s_{k-1}) = (x_1, \ldots, x_n)$ where $x_j = 0$ for $j \in I_k$ and $x_j = 1/2$ for $j \notin I_k$. Now for $t \in [s_{k-1}, s_k]$, since by Cor. 3.3 $s_i \neq s_{i+1}$, we can define the coordinate functions of $\mu(t)$ as follows

$$\mu(t)_j = \begin{cases} 1/2 & \text{for } j \notin I_k \\ 1/2(\frac{t-s_{k-1}}{s_k-s_{k-1}}) & \text{else} \end{cases}$$

Then $\mu(s_k) = (1/2, \ldots, 1/2)$ and μ is a dipath.

Case 2: $\dim L_{k-1} > \dim L_k$. Then $L_k = \{(x_1, \ldots, x_n) \in L_{k-1} | x_i = 1 \text{ for } i \in I_k\}$ and $\mu(s_{i-1}) = (1/2, \ldots, 1/2)$. Define the coordinate functions of $\mu(t)$ for $t \in [s_{k-1}, s_k]$ as follows:

$$\mu(t)_j = \begin{cases} 1/2 & \text{for } j \notin I_k \\ 1/2(\frac{t+s_k-2s_{k-1}}{s_k-s_{k-1}} & \text{else} \end{cases}$$

Then μ is a dipath and $\mu(s_k) = 1$ if $j \in I_k$ and $\mu(s_k) = 1/2$ otherwise. Hence $\mu(s_k)$ is $c(L_k)$.

The dihomotopy between $\gamma(t)$ and $\mu(t)$ is given by $H(t,r) = \mu(t)r + \gamma(t)(1-r)$ for $t \in [s_{i-1}, s_i], r \in [0,1]$, where addition is in $L_{k-1} \cup L_k = L_k$ in case 1 and in $L_{k-1} \cup L_k = L_{k-1}$ in case 2. This is well defined, since $\gamma([t_{k-1}, t_{k+1}]) \subseteq L_{k-1} \cup L_k$ and $\mu([s_{k-1}, s_k]) \subseteq L_{k-1} \cup L_k$. Since a convex combination of dipaths is a dipath, this defines dimaps from $\overrightarrow{[s_{i-1}, s_i]} \times I$ to X. We have to see, that these pieces of a dihomotopy give a well defined continuous map $H : \vec{I} \times I \to X$, i.e., that $H(s_i, r)$ is well defined.

In the cases $L_{i-1} \subset L_i \subset L_{i+1}$, $L_{i-1} \supset L_i \supset L_{i+1}$ and $L_{i-1} \subset L_i \supset L_{i+1}$, $H(r,t)$ is defined linearly by $H(t,r) = \mu(t)r + \gamma(t)(1-r)$ for $t \in [s_{i-1}, s_{i+1}]$ and addition is in L_{i+1}, L_{i-1} respectively L_i in the three cases, and hence it is well defined at (s_i, r).

If $L_{i-1} \supset L_i \subset L_{i+1}$, we glue a dihomotopy defined linearly in L_{i-1} with one defined linearly in L_{i+1}. In both cases, the formula at the glueing is $H(s_i, r) = \mu(s_i)r + \gamma(s_i)(1-r)$, and since $\mu(s_i) = c(L_i) \in L_i$ and also $\gamma(s_i) = \gamma(\frac{t_i + t_{i+1}}{2}) \in L_i$, both linear combinations are actually the same convex combinations of points in L_i on the common boundary of L_{i-1} and L_{i+1}, hence the dihomotopies agree at (s_i, r).

□

Lemma 3.2. *Let $L_0, \dots, L_N$ be the sequence of carriers traced out by a dipath γ and let $t_0, \dots, t_N$ be as in Def. 2.17. Then*

(1) *For all $i = 0, \dots, N-1$, $\dim(L_i) \neq \dim(L_{i+1})$ and either L_i is a lower face of L_{i+1} or L_{i+1} is an upper face of L_i.*
(2) *γ intersects L_i transversely ($t_i = t_{i+1}$) only if $\dim(L_{i-1}) > \dim(L_i) < \dim(L_{i+1})$*

Proof. For the proof of 1) notice that $\gamma(t_{i+1}) \in L_i \cap L_{i+1}$. Let F be the carrier of $\gamma(t_{i+1})$ then either $F = L_i$ or $F = L_{i+1}$.

If $F = L_i$ then $\gamma(t_{i+1}) \in \partial^-(L_{i+1}) \setminus \partial^+(L_{i+1})$, since $\gamma(t_{i+1}) \in L_{i+1}$, and L_{i+1} is not its carrier. It is on the *lower* boundary, since γ is increasing into the interior: $\gamma([t_{i+1}, t_{i+2}] \cap \overset{\circ}{L}_{i+1} \neq \emptyset$. Moreover, L_i is a lower face of L_{i+1}, since $\gamma(t_{i+1})$ is an interior point of $F = L_i$ (or F is a vertex, in which case it is also true). The symmetric case when $F = L_{i+1}$ is similar.

Now for 2): Suppose $\dim(L_{i-1}) < \dim(L_i)$, then L_{i-1} is a lower face of L_i and $\gamma(t_i) \in L_{i-1}$, so $\gamma(t_i)$ is not in the interior of L_i. But L_i is a carrier of some point in $\gamma([t_i, t_{i+1}])$, and hence $\gamma([t_i, t_{i+1}]) \cap \overset{\circ}{L}_i \neq \emptyset$. Therefore $t_i \neq t_{i+1}$. This proves that $t_i = t_{i+1} \Rightarrow \dim(L_{i-1}) > \dim(L_i)$, since equality is ruled out by 1). The other relation is proven by a similar argument. □

Corollary 3.3. *Let $L_0, \dots, L_N$ be the sequence of carriers traced out by a dipath γ and let $t_0, \dots t_N$ be defined as above . Define $s_i = \frac{t_i + t_{i+1}}{2}$ for $i = 0, \dots, N$. Then for all i: $s_i \neq s_{i+1}$.*

Proof. $s_i = s_{i+1}$ if and only if $t_i = t_{i+2}$, which contradicts Lemma 3.2.2. □

4. Cubical approximation of dipaths.

A dipath initiating and ending in a vertex is dihomotopy equivalent to a cubical dipath. We give two constructions of such a cubical dipath - one, which "follows behind" the original dipath and one which "runs ahead" of it - see fig. 1 and fig. 2. In both cases, the cubical dipath is an approximation of the original dipath in that they have a common starsequence, the stars of vertices traversed by the cubical dipath.

Theorem 4.1. *Let $\gamma : \vec{I} \to X$ be a dipath, $\gamma(0) = v_0$ and $\gamma(1) = v_1$, where v_i is a vertex. Then there is a dipath $\mu : \vec{I} \to X$ such that μ is dihomotopic to γ and μ is a dipath on the one-skeleton of X, i.e., for all $t \in I$ the dimension of the carrier of $\mu(t)$ is at most 1.*

Proof. As above, let $L_0, L_1, \dots, L_N$ be the sequence of carriers traced out by points in $\gamma(\vec{I})$, i.e., there is a sequence of points $t_0, t_1, \dots, t_N$ such that $\gamma([t_i, t_{i+1}]) \subseteq L_i$. Let $s_i = \frac{t_i + t_{i+1}}{2}$ and $\mu(0) = v_0 = L_0$ (remember that $s_0 = 0$). Now define μ inductively. The induction hypothesis is that μ is constructed for $t \le s_i$ such that $\mu(s_l) = v_-(L_l)$ for $l \le i$:

Case 1): If $\dim L_i < \dim L_{i+1}$ then $\mu(t) = \mu(s_i) = v_-(L_i) = v_-(L_{i+1})$ for $t \in [s_i, s_{i+1}]$.

Case 2): If $\dim L_i > \dim L_{i+1}$, then $v_-(L_{i+1}) \in \partial^+(L_i)$, so there is a *non-unique* cubical dipath $\alpha_i : [0,1] \to L_i$ with $\alpha_i(0) = v_-(L_i)$ and $\alpha_i(1) = v_-(L_{i+1})$. Let $\mu(t) = \alpha_i(\frac{t - s_i}{s_{i+1} - s_i})$, which is allowed by Cor. 3.3. Then $\mu(s_{i+1}) = v_-(L_{i+1})$.

The dihomotopy between a reparametrization of γ and of μ is as above: For $t \in [s_i, s_{i+1}], r \in [0,1]$ let $H(t,r) = \mu(t)r + \gamma(t)(1-r)$ where in case 1, addition is in L_{i+1} and $\mu(t) = (0, \dots, 0)$ and in case 2 addition is in L_i, since $\mu(t) \in L_i \supset L_{i+1}$ and $\gamma(t) \in L_i \cup L_{i+1} = L_i$ for $t \in [t_i, t_{i+2}] \supset [s_i, s_{i+1}]$.

To see that this defines a dihomotopy, proceed as in the proof of Thm. 3.1. For the last case: $L_{i-1} \supset L_i \subset L_{i+1}$ notice that $\mu(s_i) = v_-(L_i)$ and $\gamma(s_i) \in L_i$, so again the addition $H(s_i, r) = \mu(s_i)r + \gamma(s_i)(1-r)$ taking place in L_{i-1} or L_{i+1} actually is in L_i.

A d-homotopy as in [3] is defined as follows:

Case 1): As above - $H_i(t,r) = \mu(t)r + \gamma(t)(1-r) = \gamma(t)(1-r)$, since addition is in L_{i+1} where $\mu(t) = (0, \dots, 0)$ for $t \in [s_i, s_{i+1}]$.

Case 2):

$$H_i(t,r) = \begin{cases} \mu(s_i)r + \gamma(2t - s_i)(1-r) & \text{for } t \in [s_i, \frac{s_i + s_{i+1}}{2}] \\ \mu(2t - s_{i+1})r + \gamma(s_{i+1})(1-r) & \text{for } t \in [\frac{s_i + s_{i+1}}{2}, s_{i+1}] \end{cases}$$

The dihomotopy in case 1) is clearly decreasing in r and increasing in t, so it is a d-homotopy. In case 2), for fixed $t_* \in [s_i, \frac{s_i + s_{i+1}}{2}]$, $\gamma(2t - s_i) \ge \mu(s_i)$, so $H(r, t_*)$

is the line from $\gamma(2t - s_i)$ to $\mu(s_i)$, which is decreasing. In case $t \in [\frac{s_i+s_{i+1}}{2}, s_{i+1}]$, observe that $\gamma(s_{i+1}) \in L_{i+1}$ which is an upper face in L_i and μ runs from $v_-(L_i)$ to $v_-(L_{i+1})$ so it is below $\gamma(s_{i+1})$; that is, the dihomotopy is increasing in t and decreasing in r.

To see that the homotopies are well defined in s_i, notice that in case 2, $H_i(s_i, r) = \mu(s_i)r + \gamma(s_i)(1 - r) = H_{i-1}(s_i, r)$ and in case 1, $H_i(s_i, r) = \gamma(s_i)(1 - r) = \mu(s_i)r + \gamma(s_i)(1 - r)$ and $H_{i-1}(s_i) = \mu(s_i)r + \gamma(s_i)(1 - r)$, since $\mu(s_i) = \mu(s_{i-1})$ in this case.

Again, the dihomotopies are after reparametrization, but reparametrizations are d-equivalences, so we are done.

□

Remark 4.2. The cubical dipaths constructed above have a starsequence provided by the stars of their vertices. This is also a star sequence for the original dipath.

We could have chosen another cubical dipath approximation, namely to let $\mu(s_i) = v_+(L_i)$: In case one, let $\mu([s_i, s_{i+1}])$ be an edge dipath in L_{i+1} from $v_+(L_i)$ to $v_+(L_{i+1})$ and in case 2, $\mu([s_i, s_{i+1}]) = v_+(L_i) = v_+(L_{i+1})$ is constant. The proof that μ is dihomotopic to γ goes as in the above proofs.

Example 4.3. *In Fig. 1 and 2 a dipath γ, its unique centerpoint approximation and the different choices of cubical approximation are displayed. The dipath is the solid curve, the centerpoint approximation is dashed and the cubical approximation is dashed and dotted. The dipath has 14 carriers. In Fig. 1, the cubical approximation is through the lower vertices of the carriers whereas in Fig. 2, we go to the upper vertices.*

In Fig. 1, $\mu(s_0) = \mu(s_1)$ and we can choose to let μ go "above" or "below" the carrier L_7, in order to get from $v_-(L_7)$ to $v_-(L_8) = v_+(L_8) = L_8$.

In Fig. 2, we can choose at two places, namely at L_1 and at L_9. Here $\mu(s_0) \neq \mu(s_1) = v_+(L_1)$.

5. Cubical approximation of dihomotopies.

The main result in this section is, that two cubical dipaths which are dihomotopic are cubically dihomotopic. We restrict to geometric cubical sets, but we believe that the result holds in the larger category of non selfintersecting cubical sets. However our methods needs sharpening to work in that case. In particular, Prop.5.2 is not true for the more general case, but it may very well hold, if we add the assumption that there is a dihomotopy in the common star sequence. bbbbbAnd then the theorem would follow.

Theorem 5.1. *Let μ_1 and μ_2 be cubical dipaths in a locally finite geometric cubical complex X, such that $\mu_1(0) = \mu_2(0) = v$ and $\mu_1(1) = \mu_2(1) = w$. Suppose that*

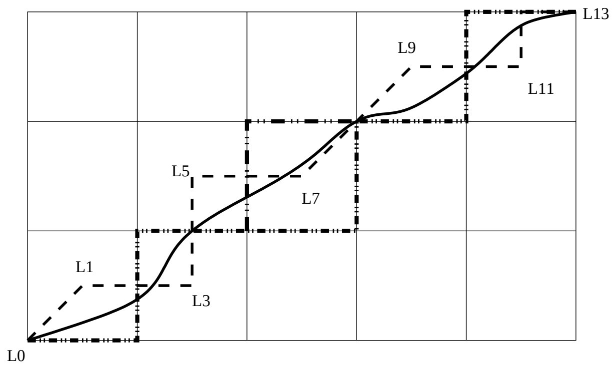

FIGURE 1. A dipath and its approximations

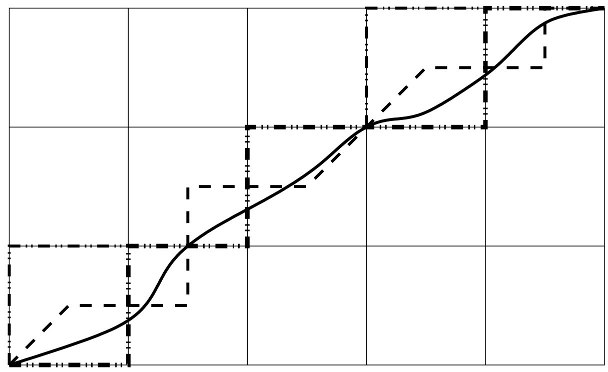

FIGURE 2. The cubical approximation going to upper vertices.

there is a dihomotopy $H : I \times \vec{I} \to X$ *with* $H(s,0) = v$ *and* $H(s,1) = w$ *for all* s *and with* $H(0,t) = \mu_1(t)$ *and* $H(1,t) = \mu_2(t)$*. Then there is a dihomotopy* $\tilde{H} : I \times \vec{I} \to X$ *with the same properties as* H *and moreover,* $\tilde{H}(s,t)$ *is on the 2-skeleton of* X*, i.e., dim(carrier(*$\tilde{H}(s,t)$*))* ≤ 2 *for all* (s,t)*. Furthermore,* $\tilde{H}$ *is a combinatorial dihomotopy.*

Proof. Since $\{H^{-1}(St(v)) | v$ vertex in $X\}$ is an open covering of the compact set $I \times \vec{I}$, let $1/M$ be a Lebesgue number for the covering and deduce that there are sequences $0 = t_0 < t_1 \ldots < t_M = 1$ and $0 = s_0 < s_1 < \cdots < s_M$ such that for all i, j, there is a vertex v_{ij} such that $H([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subset St(v_{ij}))$. Let $\gamma_i(t) = H(s_i, t)$ for $i = 0, \ldots, M$. Then γ_i and γ_{i+1} has a common starsequence, $St(v_{i0}), St(v_{i1}), \ldots, St(v_{iM})$, so choosing cubical approximations from below, $\hat{\gamma_i}$ and $\gamma_{\hat{i+1}}$, these are combinatorially dihomotopic by Lemma 5.3. Hence by induction, $\hat{\gamma_0}$ is combinatorially dihomotopic to $\hat{\gamma}_M$. Since $\gamma_0 = \mu_1$ and $\gamma_M = \mu_2$, which are both already cubical, the theorem follows. □

Proposition 5.2. *Let γ_1 and γ_2 be dipaths in a geometric locally finite cubical complex X such that $\gamma_1(0) = \gamma_2(0) = v$ and $\gamma_1(1) = \gamma_2(1) = w$ are both vertices. Suppose moreover, that γ_1 and γ_2 have a common starsequence. Then γ_1 is dihomotopic to γ_2*

This follows immediately from

Lemma 5.3. *With the assumptions in Prop. 5.2, let $\tilde{\gamma}_i$ be a cubical approximation from below of γ_i for $i \in \{1, 2\}$. Then $\tilde{\gamma}_1$ is cubically dihomotopic to $\tilde{\gamma}_2$.*

Proof. Denote the carriersequence of γ_1 by $L_0, L_1, \ldots, L_m$ and the carriersequence of γ_2 by $K_0, K_1, \ldots, K_n$. By Lemma 5.4, there are subsequences $L_{r_0}, L_{R_0}, L_{r_1} \ldots, L_{R_M}, L_{r_{M+1}}$ and $K_{u_0}, K_{U_0}, \ldots, K_{U_N}, K_{u_{N+1}}$, which are locally minimal/maximal dimensional. Suppose - by weeding out some vertices- that the common star sequence, $v_0, v_1, \ldots, v_r$ is minimal, i.e., that $v_j \neq v_{j+1}$ and for each v_k, either there are i, j such that $v_k \in K_{U_j} \cap K_{U_{j+1}} \cap L_{R_i} = K_{u_{j+1}} \cap L_{R_i}$ or $v_k \in K_{U_j} \cap L_{R_i} \cap L_{R_{i+1}} = K_{U_j} \cap L_{r_{i+1}}$ or v_k is one of the extreme vertices, i.e., $k \in \{0, r\}$. This weeding can be done, since the minimal cubes are in the star of some vertex and all other cubes in the carrier sequence contain a minimal cube.

The cubical approximation from below, $\tilde{\gamma_1}$ is a cubical dipath through the minimal vertices $v_-(L_i) = v_-(L_{R_{j(i)}})$, where $L_{R_{j(i)}}$ is the first maximal cube above L_i. And similarly for $\tilde{\gamma}_2$. We will prove the lemma by iteratively providing cubical homotopies from $\tilde{\gamma}_i$ to a common cubical dipath.

Let $p_0 = v_r$. Since $v_{r-1} \in (K_{U_{N-1}} \cap K_{U_N} \cap L_{R_M}) \cup (K_{U_N} \cap L_{R_{M-1}} \cap L_{R_M})$, we can choose $p_1 \in \{v_-(K_{U_{N-1}} \cap K_{U_N} \cap L_{R_M}), v_-(K_{U_N} \cap L_{R_{M-1}} \cap L_{R_M})\}$. Then

(1) There is a cubical dipath ϕ_1 from p_1 to p_0 in $L_{R_M} \cap K_{U_N}$
(2) There is a cubical dipath from $v_-(K_{U_N})$ to p_1 in K_{U_N}
(3) There is a cubical dipath from $v_-(L_{R_M})$ to p_1 in L_{R_M}

This all follows from the fact that $p_1 \in K_{U_N} \cap L_{R_M}$, and that the complex is geometric, so that the intersection is a face.

Let $\hat{\gamma}_1^1$ be a cubical dipath which follows $\hat{\gamma}_1^0 = \tilde{\gamma}_1$ until it reaches $v_-(L_{R_M})$ then runs to p_1 along edges in L_{R_M} and finally follows ϕ_1 to p_0. Then $\hat{\gamma}_1^1$ is cubically dihomotopic to $\hat{\gamma}_1^0$, since they only differ by the way they get from $v_-(L_{R_M})$ to p_0 in L_{R_M}.

Similarly, $\hat{\gamma}_2^1$ follows $\hat{\gamma}_2^0$ until $v_-(K_{U_N})$ then it runs to p_1 on edges in K_{U_N} and follows ϕ_1 to p_0. This is cubically dihomotopic to $\hat{\gamma}_2^0$, since they only differ by the way they get from $v_-(K_{U_M})$ to p_0 in K_{U_M}.

Now define p_k and $\hat{\gamma}_i^k$ iteratively. First p_k:

If $\mathbf{p_k = v_-(K_{U_i} \cap K_{U_{i+1}} \cap L_{R_j}})$ and $i \neq 0 \neq j$. Then choose $p_{k+1} \in \{v_-(K_{U_{i-1}} \cap K_{U_i} \cap L_{R_j}), v_-(K_{U_i} \cap L_{R_{j-1}} \cap L_{R_j})\}$. This is possible (i.e., the set is nonempty) by the following argument: Let $v_\alpha \in K_{U_i} \cap K_{U_{i+1}} \cap L_{R_j}$ and suppose that $v_{\alpha-1} \notin$

$K_{U_i} \cap K_{U_{i+1}} \cap L_{R_j}$. Then, since $St(v_{\alpha-1}) \cap St(v_\alpha) \neq \emptyset$ and γ_i increases from $St(v_{\alpha-1})$ to $St(v_\alpha)$, we conclude $v_{\alpha-1} \in (K_{U_{i-1}} \cap K_{U_i} \cap L_{R_j}) \cup (K_{U_i} \cap L_{R_{j-1}} \cap L_{R_j})$.

Now if $p_{k+1} = v_-(K_{U_{i-1}} \cap K_{U_i} \cap L_{R_i})$ then

(1) There is a cubical dipath ϕ_{k+1} from p_{k+1} to p_k in $K_{U_i} \cap L_{R_j}$
(2) There is a cubical dipath from $v_-(K_{U_i})$ to p_{k+1} in $K_{U_{i-1}} \cap K_{U_i}$
(3) There is a cubical dipath from $v_-(L_{R_j})$ to p_{k+1} in L_j.

2) and 3) hold, since $p_{k+1} \in K_{U_{i-1}} \cap K_{U_i} \cap L_{R_j}$. To see that 1) holds, we use that the complex is geometric -i.e., that intersections are faces:

In $K_{U_i} = \{(x_1, \dots, x_n) \in [0,1]^n\}$

- $K_{U_i} \cap K_{U_{i+1}} = \{(x_1, \dots, x_n) | x_{\mu_1} = \dots = x_{\mu_l} = 1\}$
- $K_{U_i} \cap K_{U_{i-1}} = \{(x_1, \dots, x_n) | x_{\nu_1} = \dots = x_{\nu_m} = 0\}$
- $K_{U_i} \cap L_{R_j} = \{(x_1, \dots, x_n) | x_{\alpha_1} = \dots = x_{\alpha_r} = 0 \wedge x_{\beta_1} = \dots = x_{\beta_s} = 1\}$
- p_k has $x_{\mu_1} = \dots = x_{\mu_l} = 1 = x_{\beta_1} = \dots = x_{\beta_s}$ and all other coordinates are 0.
- p_{k+1} has $x_{\beta_1} = \dots = x_{\beta_s} = 1$ and all others 0.

There is a cubical dipath ϕ_{k+1} in K_{U_i} from p_{k+1} to p_k raising coordinates x_{μ_i} where $\mu_i \notin \{\beta_1, \dots, \beta_s\}$ from 0 to 1 - one coordinate at a time. We have to see that this dipath is in L_{R_j}. Since $L_{R_j} \cap K_{U_i} \cap K_{U_{i+1}} \neq \emptyset$ we have $\{\mu_1, \dots, \mu_l\} \cap \{\alpha_1, \dots, \alpha_r\} = \emptyset$ and hence ϕ_{k+1} is in L_{R_j}.

If $p_{k+1} = v_-(K_{U_i} \cap L_{R_{j-1}} \cap L_{R_j})$ Then

(1) There is a cubical dipath from p_{k+1} to p_k in $K_{U_i} \cap L_{R_j}$
(2) There is a cubical dipath from $v_-(K_{U_i})$ to p_{k+1} in K_{U_i}.
(3) There is a cubical dipath from $v_-(L_{R_j})$ to p_{k+1} in $L_{R_{j-1}} \cap L_{R_j}$

Again 2) and 3) are trivial. To prove 1), we study the situation in K_{U_i} again with the same notation for p_k, $K_{U_i} \cap L_{R_j}$ and $K_{U_i} \cap K_{U_{i+1}}$ plus the following

- $K_{U_i} \cap L_{R_{j-1}} = \{(x_1, \dots, x_n) | x_{\delta_1} = \dots = x_{\delta_t} = 0 \wedge x_{\varepsilon_1} = \ldots = x_{\varepsilon_s} = 1\}$
- p_{k+1} has $x_{\beta_1} = \dots = x_{\beta_s} = 1$ and all other coordinates are 0. (Actually we also know that $x_{\varepsilon_1} = \ldots = x_{\varepsilon_u} = 1$, but since $L_{R_{j-1}} \cap L_{R_j}$ is a lower face in L_{R_j}, $\{\varepsilon_1, \dots, \varepsilon_u\} \subseteq \{\beta_1, \dots, \beta_s\}$)

A cubical dipath ϕ_{k+1} from p_{k+1} to p_k in K_{U_i} then raises all x_{μ_i} from 0 to 1 unless $\mu_i \in \{\beta_1, \dots, \beta_s\}$ in which case the coordinate is already 1 in p_{k+1}. To see that ϕ_{k+1} is in L_{R_j}, observe that $\{\mu_1, \dots, \mu_l\} \cap \{\alpha_1, \dots, \alpha_r\} = \emptyset$, since $K_{U_{i+1}} \cap L_{R_j} \cap K_{U_i} \neq \emptyset$.

If $\mathbf{p_k = v_-(K_{U_i} \cap L_{R_j} \cap L_{R_{j+1}})}$, then by interchanging K and L in the above argument, we get $p_{k+1} \in \{v_-(K_{U_i} \cap L_{R_{j-1}} \cap L_{R_j}), v_-(K_{U_{i-1}} \cap K_{U_i} \cap L_{R_j})\}$ and there is a cubical dipath ϕ_{k+1} from p_{k+1} to p_k in $K_{U_i} \cap L_{R_j}$.

To define $\hat{\gamma}_i$, suppose that $p_k = v_-(K_{U_i} \cap K_{U_{i+1}} \cap L_{R_j})$ then by the iterative definition above, $p_{k-1} \in \{v_-(K_{U_{i+1}} \cap K_{U_{i+2}} \cap L_{R_j}), v_-(K_{U_{i+1}} \cap L_{R_j} \cap L_{R_{j+1}})\}$. Suppose

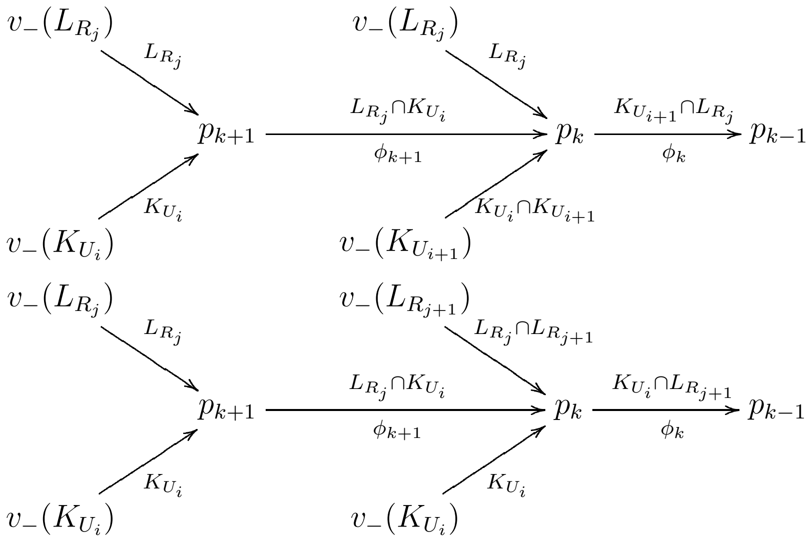

FIGURE 3. The iteration when $p_k = v_-(K_{U_i} \cap K_{U_{i+1}} \cap L_{R_j})$ in the two cases $p_{k-1} = v_-(K_{U_{i+1}} \cap K_{U_{i+2}} \cap L_{R_j})$ and $p_{k-1} = v_-(K_{U_i} \cap L_{R_j} \cap L_{R_{j+1}})$. The arrows are cubical dipaths and the labels on the arrows indicates where the cubical path runs.

that $\hat{\gamma}_1^k$ follows $\tilde{\gamma}_1$ until $v_-(L_{R_j})$, then runs along edges in L_{R_j} to p_k and follows ϕ_k from p_k to p_{k-1} in $K_{U_{i+1}} \cap L_{R_j}$ and via ϕ_j $j = k, \dots, 0$ to p_0.

Then let $\hat{\gamma}_1^{k+1}$ follow $\hat{\gamma}_1^k$ until $v_-(L_{R_j})$ then along edges in L_{R_j} to p_{k+1} and to p_k following ϕ_{k+1}. Finally it follows $\hat{\gamma}_1^k$, i.e., ϕ_j, $j = k, \dots, 0$ to p_0. We notice that $\hat{\gamma}_1^k$ and $\hat{\gamma}_1^{k+1}$ are cubically dihomotopic, since they only differ in the way they get from from $v_-(L_{R_j})$ to p_k in L_{R_j}.

Suppose that $\hat{\gamma}_2^k$ follows $\tilde{\gamma}_2$ until $v_-(K_{U_{i+1}})$. Then it runs in $K_{U_{i+1}}$ to p_k. Since $v_-(K_{U_{i+1}}), p_k \in K_{U_i} \cap K_{U_{i+1}}$, this piece of path is actually in $K_{U_i} \cap K_{U_{i+1}}$. Then it follows the ϕ_j to p_0. Let $\hat{\gamma}_2^{k+1}$ follow $\hat{\gamma}_2^k$ until $v_-(K_{U_i})$. Then to p_{k+1} in K_{U_i} and along ϕ_{k+1} to p_k - again in K_{U_i}. - and follow the ϕ_j to p_0. Then $\hat{\gamma}_2^{k+1}$ and $\hat{\gamma}_2^k$ differ only on the way they get from $v_-(K_{U_i})$ to p_k in K_{U_i}, so they are cubically dihomotopic.

The iteration stops when the starsequence stops, i.e., when $v_k \in K_{U_i} \cap K_{U_{i+1}} \cap L_{R_j}$ for all $k \leq \alpha$ (or symmetrically $v_k \in K_{U_i} \cap L_{R_j} \cap L_{R_{j+1}}$). As the diagram shows, either the iteration step lowers the index of L_{R_j} or of K_{U_i}, so when it stops, both dipaths $\tilde{\gamma}_i$ have been deformed through combinatorial dihomotopy to the dipath composed of the ϕ_j, which proves the theorem.

□

Lemma 5.4. *Let γ be a dipath with carrier sequence $L_0, L_1, \dots, L_{m_{max}}$. Then there are subsequences $L_{r_0}, L_{R_0}, L_{r_1} \dots, L_{R_{max}}, L_{r_{max}}$. which are locally minimal/maximal dimensional in the following sense:*

- $0 = r_0$, $r_{max} = m_{max}$
- $r_j < \mu < R_j \Rightarrow \dim L_{r_j} < \dim L_\mu < \dim L_{R_j}$
- $R_j < \mu < r_{j+1} \Rightarrow \dim L_{R_j} > \dim L_\mu > \dim L_{r_{j+1}}$.

Let $St(v_0), \dots, St(v_{max})$ be a starsequence for γ. Then $L_{R_{max}-1} \cap St(v_{max}) = \emptyset$.

Proof. The existence of subsequences as claimed follow from Lemma 3.2: The dimension of consequtive cubes in the carrier sequence is never the same, hence there are local extrema as wanted. Notice that $v_{max} = L_{m_{max}} = L_{r_{max}}$ and that $v_+(L_{R_{max}}) = v_{max}$, since γ is increasing. Moreover, $L_{R_{max}-1} \subset \partial^-(L_{R_{max}})$, so $v_{max} \notin L_{R_{max}-1}$ and consequently $St(v_{max}) \cap L_{R_{max}-1} = \emptyset$. □

The following lemma is not true in the non-directed case - for a counter example consider the graph of $|x|$, the absolute value, in the plane, where the plane is subdivided in 2-cubes parallel to the axes.

Lemma 5.5. *With notation as in Lem. 5.4 we have $L_{R_j} \cap L_{R_{j+1}} = L_{r_j}$*

Proof. We consider the intersection $A = L_{R_j} \cap L_{R_{j+1}}$ as subset of each cube:

- Since γ runs from $\overset{\circ}{L}_{R_j}$ to A we have $A = \{(x_1, \dots, x_n) \in L_{R_j} | x_{i_1} = \dots = x_{i_k} = 1\}$ for some set of indices $i_1, \dots, i_k$, $k \leq n$.
- Since γ runs from A to $\overset{\circ}{L}_{R_{j+1}}$ we have $A = \{(y_1, \dots, y_m) \in L_{R_{j+1}} | y_{j_1} = \dots = y_{j_l} = 0\}$ for some set of indices $j_1, \dots, j_l$, $l \leq n$.

Clearly $L_{r_j} \subseteq A$, since γ intersects $\overset{\circ}{L}_{r_j}$, so if A is a vertex, we have $L_{r_j} = A$. If $L_{r_j} \neq A$, then, since both A and L_{r_j} are cubes, $L_{r_j} \subset \partial A$. Thus, there is a $\hat{t}$ s.t. $\gamma(\hat{t}) \notin \overset{\circ}{A}$ on the way from $\overset{\circ}{L}_{R_j}$ to $\overset{\circ}{L}_{R_{j+1}}$. i.e.

1. Since γ comes from $\overset{\circ}{L}_{R_j}$, considered in L_{R_j} we have $\gamma(\hat{t}) = (\hat{x}_1, \dots, \hat{x}_n)$, satisfies $\hat{x}_{i_1} = \dots = \hat{x}_{i_k} = 1$ and $\hat{x}_\alpha = 1$ for some $\alpha \notin \{i_1, \dots, i_k\}$
2. Similarly $\gamma(\hat{t}) = (\hat{y}_1, \dots, \hat{y}_n)$ satisfies $\hat{y}_{j_1} = \dots = \hat{y}_{j_l} = 0$ and $\hat{y}_\beta = 0$ for some $\beta \notin \{j_1, \dots, j_l\}$.

Hence, by 1) $\gamma(\hat{t}) \in \partial^+(A)$ (and by 2) $\gamma(\hat{t}) \in \partial^-(A)$) and it follows that there is an $\tilde{\alpha}$ s.t. $\hat{x}_{\tilde{\alpha}} = 0$ which contradicts that γ comes from $\overset{\circ}{L}_{R_j}$. □

The connection between d-homotopy and dihomotopy is

Theorem 5.6. *Let γ_i, $i = 0, 1$ be dipaths in a geometric cubical complex both initiating in a common vertex and ending in a common vertex. Suppose that γ_1 is dihomotopic to γ_2. Then γ_1 is d-homotopic to γ_2.*

Proof. Use Thm.4.1 to give a d-homotopy from γ_i to a cubical approximation of γ_i. Then Thm.5.1 provides a cubical dihomotopy between the cubical approximations, but a cubical dihomotopy is clearly a d-homotopy. □

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