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Addendum to the paper: "A unified approach to resolvent expansions at thresholds"

by

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Addendum to the paper: "A unified approach to resolvent expansions at thresholds"

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This note contains a few remarks concerning the paper in the title [1]. The first remark is that the statement of Corollary 2.2 in [1] is incomplete: For the conclusion to be true one needs that $A_0 S = 0$, which is not the case, when A_0 has a nilpotent part. The correct form is:

Proposition 1. *Let $F \subset \mathbf{C}$ have zero as an accumulation point. Let $A(z)$, $z \in F$, be a family of bounded operators of the form*

$$A(z) = A_0 + zA_1(z), \tag{1}$$

with A_1 uniformly bounded as $z \to 0$. Suppose 0 is an isolated point of the spectrum of A_0, and let S be the corresponding Riesz projection. If

$$A_0 S = 0, \tag{2}$$

then for sufficiently small z the operator $B(z)\colon S\mathcal{H} \to S\mathcal{H}$, defined by

$$B(z) = \frac{1}{z}(S - S(A(z)+S)^{-1}S) = \sum_{j=0}^{\infty} (-z)^j S[A_1(z)(A_0+S)^{-1}]^{j+1} S, \tag{3}$$

is uniformly bounded as $z \to 0$. The operator $A(z)$ has a bounded inverse in $\mathcal{H}$, if and only if $B(z)$ has a bounded inverse inverse in $S\mathcal{H}$, and in this case

$$A(z)^{-1} = (A(z)+S)^{-1} + \frac{1}{z}(A(z)+S)^{-1} S B(z)^{-1} S (A(z)+S)^{-1}. \tag{4}$$

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The rest of the paper is not affected, since everywhere A_0 is self-adjoint, so that (2) holds true.

The second remark is that (2) might be true, even if A_0 is not self-adjoint. As we shall prove below, this is the case, when one considers the asymptotic expansion of the perturbed resolvent around an embedded non-threshold eigenvalue, λ_0, of the unperturbed (self-adjoint) Hamiltonian as in [2].

More precisely (see [2] for details), if the unperturbed resolvent is

$$R_0(z) = \frac{P_0}{\lambda_0 - z} + \tilde{R}_0(z) \tag{5}$$

and $V = |V|^{1/2}U|V|^{1/2}$ is a bounded self-adjoint perturbation (here we choose U to be unitary by defining it to be 1 on $\operatorname{Ker} V$, such that $U^2 = I$), then A_0 turns out to be

$$A_0 = U + \lim_{\eta \searrow 0} |V|^{1/2}\tilde{R}_0(\lambda_0 + i\eta)|V|^{1/2}. \tag{6}$$

It is assumed that the limit exists and is compact, and then either A_0 has a bounded inverse, or 0 is an isolated part of the spectrum of A_0. In the latter case one can define the corresponding Riesz projection, S.

The next Proposition shows that under these conditions (2) holds true.

Proposition 2.

$$SA_0 = SA_0S = 0. \tag{7}$$

Proof. The first equality follows from the fact that S is a Riesz projection. The key observation is that from (6)

$$\operatorname{Im} A_0 \geq 0. \tag{8}$$

Consider now, as an operator in $S\mathcal{H}$,

$$A_1 = SA_0S. \tag{9}$$

On one hand $S\mathcal{H}$ is finite dimensional, and on the other hand for all $\Psi \in S\mathcal{H}$:

$$\operatorname{Im}\langle\Psi, A_1\Psi\rangle = \operatorname{Im}\langle\Psi, SA_0S\Psi\rangle = \operatorname{Im}\langle\Psi, A_0S\Psi\rangle = \operatorname{Im}\langle\Psi, A_0\Psi\rangle \geq 0,$$

i.e.

$$\operatorname{Im} A_1 \geq 0. \tag{10}$$

Since A_1 is nilpotent

$$\operatorname{Tr} A_1 = \operatorname{Tr}\operatorname{Re} A_1 + i\operatorname{Tr}\operatorname{Im} A_1 = 0, \tag{11}$$

which together with (10) implies that

$$\operatorname{Im} A_1 = 0. \tag{12}$$

As a consequence A_1 is self-adjoint, and since $\sigma(A_1) = \{0\}$, it follows that $A_1 = 0$. □

The last remark is that one can generalize Corollary 2.2, and then the whole procedure in [1], to non-self-adjoint A_0, under the additional assumption that A_0 is a Fredholm operator with index zero [3] (i.e. $\dim \operatorname{Ker} A_0 = \dim(\operatorname{Ran} A_0)^{\perp}$). Then if $A_0 = W|A_0|$ is the polar decomposition of A_0, one can extend (in a non-unique way) W to a unitary operator U (just take $\{f_j\}$ and $\{g_j\}$ orthonormal bases in $\operatorname{Ker} A_0$ and $(\operatorname{Ran} A_0)^{\perp}$ respectively, and define $Uf_j = g_j$). Then write

$$A(z) = U(|A_0| + zU^{-1}A_1(z)), \tag{13}$$

and apply Corollary 2.2 to $|A_0| + zU^{-1}A_1(z)$.

There is a price to pay in this case: Due to the non-unicity of U the obtained expansions are not "canonical". Of course, the coefficients of the expansion do not depend on U, but various identities have to be used in order to see that.

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