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ON THE EQUIVALENCE OF BRUSHLET AND WAVELET BASES

LASSE BORUP AND MORTEN NIELSEN†

Abstract. We prove that the Meyer wavelet basis and a class of brushlet systems associated with exponential type partitions of the frequency axis form a family of equivalent (unconditional) bases for the Besov and Triebel-Lizorkin function spaces. This equivalence is then used to obtain new results on nonlinear approximation with brushlets in Triebel-Lizorkin spaces.

1. Introduction

Wavelet bases for $L_2 := L_2(\mathbb{R})$ provide stable bases for many of the classical function spaces such as Besov and Triebel-Lizorkin spaces, and such systems correspond to dyadic Littlewood-Paley type partitions of the frequency axis. Brushlet bases were introduced by Coifmann and Meyer [13] as a tool for image compression, and brushlets also provide orthonormal bases for L_2, but with a more flexible decomposition of the time-frequency axis. In fact, one can adapt a brushlet basis to any reasonable partition of the frequency axis.

For partitions of exponential type, e.g, $\{\pm[r^j, r^{j+1})\}_{j\in\mathbb{Z}}$ for a fixed $r > 1$, brushlet bases share many properties with wavelet bases. The authors proved in [4] that such brushlet bases form unconditional bases for Triebel-Lizorkin spaces (in particular for L_p, $1 < p < \infty$) and Besov spaces. So a brushlet system of exponential type seems very similar to a wavelet basis, but are the systems actually equivalent bases in other spaces than L_2? We recall that two bases $\{f_n\}_n$ and $\{g_n\}_n$ for a Banach space X are called equivalent if there exists an isomorphism $T\colon X \to X$ satisfying $Tf_n = g_n$.

It is well known that almost any pair of wavelet bases are equivalent bases in L_p, $1 < p < \infty$. In fact, Wojtaszczyk proved in [16] that whenever a wavelet system satisfy a minimal decay condition it is equivalent to the Haar wavelet system. A pair of sufficiently smooth wavelet systems are also equivalent in the Besov and Triebel-Lizorkin spaces, see e.g. [14]. The equivalence of pairs of nice wavelet bases can also be deduced using the powerful φ-transform machinery of Frazier-Jawerth [9]. In [9] the function systems considered are formed by dyadic dilation of a single atom, and the function system is therefore naturally associated with a dyadic decomposition of the frequency axis just as for wavelet bases. So it seems only natural that brushlet bases associated with a partition such as $\{\pm[2^j, 2^{j+1})\}_{j\in\mathbb{Z}}$ should be equivalent to a wavelet system, and this can in fact be deduced from the results in [9]. But it is less obvious what happens for more general

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brushlet systems associated with partitions such as e.g. $\{\pm[2^{j/2}, 2^{(j+1)/2})\}_{j\in\mathbb{Z}}$. Are such systems also equivalent with wavelet bases associated with the standard dyadic dilation?

The purpose of the present paper is to prove that wavelet and a family of brushlet bases of exponential type are in fact equivalent (unconditional) bases for the Besov and Triebel-Lizorkin spaces. The proof will be achieved by constructing the isomorphism T explicitly for the Meyer wavelet/brushlet system. Care has to be taken when defining T - even though the wavelet and brushlets are (unconditional) bases in the respective function spaces, they are far from being symmetric bases so the ordering of the bases is crucial.

Some of the most useful Triebel-Lizorkin and Besov spaces are not Banach spaces, but only *quasi* Banach spaces. For example, L_p and the Hardy spaces H_p are quasi-Banach spaces for $0 < p < 1$, but they still play an important role in, e.g., nonlinear approximation theory with wavelets [11, 5]. For such spaces, we no longer have the notion of an unconditional basis, but we can still study the operator T that takes the wavelet system to the brushlet system, and we will show that T is an isomorphism on the Besov/Triebel-Lizorkin scales. This will still lead to a characterization of the quasi-norm for a given Besov/Triebel-Lizorkin space X in terms of brushlet coefficients in the sense that there exists a corresponding sequence space X_d, a bounded brushlet analysis operator $A\colon X \to X_d$, and a brushlet synthesis operator $S\colon X_d \to X$ for which $S \circ A = \mathrm{Id}_X$ (with Id_X the identity operator on X). As an application of the wavelet/brushlet system equivalence, we deduce new results on nonlinear approximation with brushlet systems in Triebel-Lizorkin spaces, generalizing the results in [4].

The structure of the paper is as follows. In Section 2 we define the brushlet system, and the restricted class of exponential partitions that will be considered. In Section 3 we study the equivalence of brushlets and the Meyer wavelet system. First we prove that the systems are equivalent in the Besov spaces $\dot{B}^s_{p,q}$, $1 < p, q < \infty$, even for brushlets associated with very general exponential partitions. Then we prove the equivalence in L_p, $1 < p < \infty$, for a more restricted class of brushlet systems. The idea is to associate an integral kernel to the map T and then decompose the kernel into a finite sum of (modified) Calderón-Zygmund kernels. We conclude Section 3 by considering the brushlet/wavelet equivalence on the full scale of Triebel-Lizorkin spaces. For this, we use the Frazier-Jawerth theory of almost diagonal matrices [9] to study the properties of the mapping T. Some of the technical lemmas needed for the proofs of the results in Section 3 can be found in Appendix A. In the final section, Section 4, we consider an application of the results to nonlinear approximation with brushlet systems, where we measure the approximation error using the Triebel-Lizorkin (semi)quasi-norms. The idea is to use the results on nonlinear approximation with nice wavelet systems and then "translate" the results to the brushlet system using the mapping T.

2. Brushlet systems

A brushlet basis is associated with a partition of the frequency axis. The partition can be chosen with almost no restrictions, but in order to have good properties of the associated basis we need to impose some growth conditions on the partition. We introduce the following definition of exponential coverings of the frequency axis, where we leave out the origin since we want to study homogeneous function spaces later.

Definition 2.1. A family $\mathcal{I}$ of intervals is called a **disjoint covering** of $\mathbb{R}\setminus\{0\}$ if it consists of a countable set of pairwise disjoint half-open intervals $I=[\alpha_I,\alpha_I')$, $\alpha_I<\alpha_I'$, such that $\cup_{I\in\mathcal{I}} I=\mathbb{R}\setminus\{0\}$. If, furthermore, each interval in $\mathcal{I}$ has a unique adjacent interval in $\mathcal{I}$ to the left and to the right, and there exist two constants $1<\lambda\le\Lambda<\infty$ such that

$$\lambda\le\frac{|I|}{|I'|}\le\Lambda\qquad\text{for all adjacent } I,I'\in\mathcal{I},\tag{2.1}$$

with $|\alpha_I'|<|\alpha_I|$, we call $\mathcal{I}$ an **exponential covering** of $\mathbb{R}$.

Below we will often use the following (non-unique) enumeration of such a disjoint covering $\mathcal{I}$. We number the intervals in $\mathcal{I}$ such that $\mathcal{I}=\mathcal{I}^0\cup\mathcal{I}^1=\{I_m^0\}_{m\in\mathbb{Z}}\cup\{I_m^1\}_{m\in\mathbb{Z}}$, with $I_m^0\subset(-\infty,0)$ and $I_m^1\subset(0,\infty)$ for $m\in\mathbb{Z}$. Furthermore, we require that I_{m+1}^1 is the neighbor to the right of I_m^1 for $m\in\mathbb{Z}$, and I_m^0 is the neighbor to the right of I_{m+1}^0 for $m\in\mathbb{Z}$.

Given a exponential disjoint covering $\mathcal{I}$ of $\mathbb{R}$, assign to each interval $I=[\alpha_I,\alpha_I')\in\mathcal{I}$ a left and right cutoff radius $\varepsilon_I,\varepsilon_I'>0$, satisfying

$$\begin{cases}\text{(i)} & \varepsilon_I'=\varepsilon_{I'}\text{ whenever }\alpha_I'=\alpha_{I'}\\ \text{(ii)} & \varepsilon_I+\varepsilon_I'\le|I|\\ \text{(iii)} & \varepsilon_I\ge c|I|,\end{cases}\tag{2.2}$$

with $c>0$ independent of I.

Example 2.2. If we let $\varepsilon_I=\frac{1}{2\Lambda}|I|$ and ε_I' be given by (i) in (2.2) then (ii) and (iii) are clearly satisfied.

We are now ready to define the brushlet system. For each $I\in\mathcal{I}$, we will construct a smooth bell function localized in a neighborhood of this interval. Take a non-negative ramp function $\rho\in C^r(\mathbb{R})$ for some $r\ge1$, satisfying

$$\rho(\xi)=\begin{cases}0 & \text{for }\xi\le-1,\\ 1 & \text{for }\xi\ge1,\end{cases}\tag{2.3}$$

with the property that

$$\rho(\xi)^2+\rho(-\xi)^2=1\qquad\text{for all }\xi\in\mathbb{R}.\tag{2.4}$$

Define for each $I=[\alpha_I,\alpha_I')\in\mathcal{I}$ the **bell function**

$$b_I(\xi):=\rho\Big(\frac{\xi-\alpha_I}{\varepsilon_I}\Big)\rho\Big(\frac{\alpha_I'-\xi}{\varepsilon_I'}\Big).\tag{2.5}$$

Notice that $\operatorname{supp}(b_I)\subset[\alpha_I-\varepsilon_I,\alpha_I'+\varepsilon_I']$ and $b_I(\xi)=1$ for $\xi\in[\alpha_I+\varepsilon_I,\alpha_I'-\varepsilon_I']$. Now the set of local cosine functions

$$\hat{w}_{I,n}(\xi)=\sqrt{\frac{2}{|I|}}b_I(\xi)\cos\Big(\frac{\pi(n+\frac12)}{|I|}(\xi-\alpha_I)\Big),\quad n\in\mathbb{N}_0,\quad I\in\mathcal{I},\tag{2.6}$$

constitute an orthonormal basis for L_2, see e.g. [1]. We call the collection $\{w_{I,n}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ a **brushlet system**. The brushlets also have an explicit representation in the time domain. Define the set of **central bell functions** $\{g_I\}_{I\in\mathcal{I}}$ by

$$\hat{g}_I(\xi) := \rho\bigg(\frac{|I|}{\varepsilon_I}\xi\bigg)\rho\bigg(\frac{|I|}{\varepsilon_I'}(1-\xi)\bigg), \tag{2.7}$$

such that $b_I(\xi) = \hat{g}_I\big(|I|^{-1}(\xi-\alpha_I)\big)$, and let for notational convenience

$$e_{I,n} := \frac{\pi\big(n+\frac{1}{2}\big)}{|I|}, \qquad I\in\mathcal{I},\ n\in\mathbb{N}_0.$$

Then, $w_{I,n}(x) = w_{I,n}^{+}(x) + w_{I,n}^{-}(x)$, with

$$w_{I,n}^{\pm}(x) = \sqrt{\frac{|I|}{2}}e^{i\alpha_I x}g_I\big(|I|(x\pm e_{I,n})\big). \tag{2.8}$$

Thus a brushlet $w_{I,n}$ essentially consists of two "humps" at $\pm e_{I,n}$.

We should remark that our definition of brushlets is slightly different from the definition given by Coifman and Meyer in [13] and more similar to the system considered by Laeng [12].

By a straight forward calculation it can be verified (see [4]) that there exists a constant $C<\infty$ independent of $I\in\mathcal{I}$, such that

$$|g_I(x)| \le C(1+|x|)^{-r}, \tag{2.9}$$

with $r\ge 1$ given by the smoothness of the ramp function. We say that the brushlet basis is r-**localized** if (2.9) is satisfied.

We want to study brushlet systems as bases for homogeneous Triebel-Lizorkin and Besov spaces, so let us first briefly recall the definition of these spaces (see also, *e.g.* [15]). Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be a collection of functions in $\mathcal{S}(\mathbb{R})$ with $\operatorname{supp}(\phi_j)\subset\{x\,|\,2^j\le|x|\le 2^{j+1}\}$, and let $\mathcal{P}$ be the family of polynomials on $\mathbb{R}$. Then

- For $0<p\le\infty$, $s\in\mathbb{R}$, and $0<q\le\infty$, we define the Besov semi-norm for $f\in\mathcal{S}'(\mathbb{R})$,
$$\|f\|_{\dot{B}^s_{p,q}} := \bigg(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\mathcal{F}^{-1}\phi_j\mathcal{F}f\|_{L_p}^q\bigg)^{1/q},$$
with the appropriate modification when $q=\infty$. The homogeneous Besov space is defined as
$$\dot{B}^s_{p,q} := \big\{f : f\in\mathcal{S}'(\mathbb{R})/\mathcal{P}, \|f\|_{\dot{B}^s_{p,q}}<\infty\big\}.$$
- For $0<p<\infty$, $s\in\mathbb{R}$, and $0<q\le\infty$, we define the Triebel-Lizorkin semi-norm for $f\in\mathcal{S}'(\mathbb{R})$,
$$\|f\|_{\dot{F}^s_{p,q}} := \bigg\|\bigg(\sum_{j\in\mathbb{Z}} 2^{jsq}|\mathcal{F}^{-1}\phi_j\mathcal{F}f(\cdot)|^q\bigg)^{1/q}\bigg\|_{L_p},$$
with the appropriate modification when $q=\infty$, and the homogeneous Triebel-Lizorkin space is defined as
$$\dot{F}^s_{p,q} := \big\{f\colon f\in\mathcal{S}'(\mathbb{R})/\mathcal{P}, \|f\|_{\dot{F}^s_{p,q}}<\infty\big\}.$$

As usual, we identify distributions that differ by a polynomial to make $\|\cdot\|_{\dot F^s_{p,q}}$ and $\|\cdot\|_{\dot B^s_{p,q}}$ into (quasi-)norms. It is well-known that for $1<p<\infty$, $\dot F^0_{p,2}\approx L_p$, and for $0<p\le 1$, $\dot F^0_{p,2}\approx H_p$.

Let us also recall the definition of the sequence spaces $\dot f^s_{p,q}$ and $\dot b^s_{p,q}$. A complex-valued sequence $\mathbf{d}=\{d_{j,k}\}$ is said to belong to $\dot f^s_{p,q}$ for $s\in\mathbb{R}$, $0<p<\infty$, and $0<q\le\infty$, if

$$\|d\|_{\dot f^s_{p,q}} := \bigg\|\sum_{j\in\mathbb{Z}}\bigg(\sum_{k\in\mathbb{Z}}\big(2^{j(s+1/2)}|d_{j,k}|\chi_{[2^j,2^{j+1}]}\big)^q\bigg)^{1/q}\bigg\|_{L_p}<\infty,$$

with the appropriate modification when $q=\infty$. Similarly, $\mathbf{d}\in\dot b^s_{p,q}$ for $s\in\mathbb{R}$, and $0<p,q\le\infty$, if

$$\|d\|_{\dot f^s_{p,q}} := \bigg(\sum_{j\in\mathbb{Z}}2^{jq(s+1/2-1/p)}\Big(\sum_{k\in\mathbb{Z}}|d_{j,k}|^p\Big)^{q/p}\bigg)^{1/q}<\infty,$$

with the appropriate modification when $p=\infty$ or $q=\infty$.

Using the ϕ-transform it was proved in [9] and [8] that $\dot F^s_{p,q}$ is a retract of $\dot f^s_{p,q}$ for $s\in\mathbb{R}$, $0<p<\infty$, and $0<q\le\infty$, i.e., there exists an analysis operator $A:\dot F^s_{p,q}\to\dot f^s_{p,q}$ and a synthesis operator $S:\dot f^s_{p,q}\to\dot F^s_{p,q}$ such that $\mathrm{Id}_{\dot F^s_{p,q}}=S\circ A$. Similarly $\dot B^s_{p,q}$ is a retract of $\dot b^s_{p,q}$ for $s\in\mathbb{R}$, and $0<p,q\le\infty$.

A special example of the function used in the theory of ϕ-transform is the Meyer wavelet. Let ψ denote the Meyer wavelet and let $\psi_{j,k}(x):=2^{j/2}\psi(2^jx-k)$. Define the analysis operator $A_\psi\colon L_2\to\ell_2$ by $A_\psi f=\{\langle f,\psi_{j,k}\rangle\}$, and the synthesis operator $S_\psi\colon\ell_2\to L_2$ by $S_\psi d=\sum_{j,k}d_{j,k}\psi_{j,k}$. Since the Meyer wavelet fits into the theory of the ϕ-transform, we have the following commuting diagram.

$$\begin{array}{ccc}
\dot F^s_{p,q}\ [\dot B^s_{p,q}\,] & \xrightarrow{\ \mathrm{Id}_{\dot F^s_{p,q}}\ [\mathrm{Id}_{\dot B^s_{p,q}}\,]\ } & \dot F^s_{p,q}\ [\dot B^s_{p,q}\,] \\
 & {\scriptstyle A_\psi}\searrow \qquad \nearrow{\scriptstyle S_\psi} & \\
 & \dot f^s_{p,q}\ [\dot b^s_{p,q}\,] &
\end{array}$$

In particular, we have the wavelet characterization

$$\|f\|_{\dot F^s_{p,q}}\asymp\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}2^{j(s+\frac12)\tau}|\langle f,\psi_{j,k}\rangle|^\tau\chi_{[2^j,2^{j+1}]}\bigg)^{1/\tau}\bigg\|_{L_p}. \tag{2.10}$$

Given a brushlet system $\{w_{I,n}\}$, define the analysis operator $A_b\colon L_2\to\ell_2$ by $A_bf=\{\langle f,w_{I,n}\rangle\}$, and the synthesis operator $S_b\colon\ell_2\to L_2$ by $S_bd=\sum_{I,n}d_{j,k}w_{I,n}$. One of the purposes of this paper is to show that if the brushlet system is sufficiently nice, the operators A_b and S_b defines similar retracts as shown in the above diagram.

3. Equivalent brushlet-wavelet systems in the Besov and Triebel-Lizorkin spaces

In this section we will identify a family of brushlet bases that are equivalent to the Meyer wavelet basis in L_p, or more generally in the Triebel-Lizorkin spaces $\dot{F}^{\alpha}_{p,q}$ and in the Besov spaces $\dot{B}^{\alpha}_{p,q}$. We choose to work with the Meyer wavelet basis since it is equivalent in the Triebel-Lizorkin/Besov spaces to any other wavelet basis with sufficient smoothness and decay. First we will give a result in $\dot{B}^{\alpha}_{p,q}$ for a rather large class of brushlet systems, but for the restricted case $1 < p, q < \infty$. Then we consider the equivalence in L_p, $1 < p < \infty$. Finally, we give a more technical result for the general function spaces $\dot{F}^{\alpha}_{p,q}$ and $\dot{B}^{\alpha}_{p,q}$. The proof of the result in $\dot{F}^{\alpha}_{p,q}$ (and $\dot{B}^{\alpha}_{p,q}$) is more elaborate, we analyze the matrix of a certain decomposition of T in the Meyer wavelet basis and use the Frazier-Jawerth theory of almost diagonal matrices to reach the conclusion. The proof in L_p is more straightforward and consists of a careful analysis of the kernel for the isomorphism T that will be defined below.

3.1. **Wavelet-brushlet equivalence on the Besov Scale.** In this section we will identify a family of brushlet bases that are equivalent to the Meyer wavelet bases in the Besov spaces $\dot{B}^{\alpha}_{p,q}$ for $\alpha \in \mathbb{R}$ and $1 < p, q < \infty$. Compared with the techniques used in Section 3.4 below this is a much easier task than showing the equivalence in the Triebel-Lizorkin spaces. Suppose that $\Lambda < 2$. For $j \in \mathbb{Z}$ and $\varepsilon \in \{0, 1\}$ define $s_{j,\varepsilon} \in \mathbb{Z}$ by $2^j \in I^{\varepsilon}_{s_{j,\varepsilon}}$ ($s_{j,\varepsilon}$ can be shown to be unique, since $\Lambda < 2$), $I^{\varepsilon}_{s_{j,\varepsilon}} \in \mathcal{I}^{\varepsilon}$, and let $p_{j,\varepsilon} := s_{j+1,\varepsilon} - s_{j,\varepsilon}$. Notice that $1 \leq p_{j,\varepsilon} \leq 1/\log_2(\lambda)$. To keep notation simple we will introduce the indices $\Gamma \in \mathbb{Z} \times \mathbb{Z}$ and $\Upsilon \in \mathbb{Z} \times \mathbb{N}_0$. Γ will be used as the index for the wavelets and Υ will be the index of the brushlets. For $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$, $\delta, \varepsilon \in \{0, 1\}$, and $0 \leq m < p_{j,\varepsilon}$ we define

$$\Gamma := \Gamma_{j,k,m,\delta,\varepsilon} := (j, (-1)^{\delta}(2p_{j,\varepsilon}k + 2m + \varepsilon) - \delta)$$

and

$$\Upsilon := \Upsilon_{j,k,m,\delta,\varepsilon} := (s_{j,\varepsilon} + m, 2k + \delta).$$

Notice that there exist constants $0 < c \leq C < \infty$ such that $c2^j < |I_{s_{j,\varepsilon}+m}| < C2^j$ for all $j \in \mathbb{Z}$. Define the operator $T \colon L_2 \to L_2$ by

$$T\psi_{\Gamma} = w^{\varepsilon}_{\Upsilon}. \tag{3.1}$$

It is clear that T is an isomorphism on L_2 since Γ and Υ (considered as functions) injectively run through all of $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{N}_0$, respectively.

We can now state the main result regarding the equivalence on the Besov scale $\dot{B}^{\alpha}_{p,q}$ for $1 < p, q < \infty$.

Proposition 3.1. *The map T defined by* (3.1) *extends to an isomorphism on $\dot{B}^{\alpha}_{p,q}$, $\alpha \in \mathbb{R}$, $1 < p, q < \infty$. In particular, $\{w_{n,I}\}_{I \in \mathcal{I}, n \in \mathbf{N}_0}$ and the Meyer wavelet system are equivalent unconditional bases for $\dot{B}^{\alpha}_{p,q}$.*

Proof. Take any $f \in \dot{B}^\alpha_{p,q}$. Then we have

$$\|f\|_{\dot{B}^\alpha_{p,q}} \asymp \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}\big(2^{j(\alpha+1/2-1/p)}|\langle f,\psi_{j,k}\rangle|\big)^p\Big)^{q/p}\Big)^{1/q}$$
$$= \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}\sum_{\varepsilon,\delta\in\{0,1\}}\sum_{m=0}^{p_{j,\varepsilon}}\big(2^{j(\alpha+1/2-1/p)}|\langle f,\psi_\Gamma\rangle|\big)^p\Big)^{q/p}\Big)^{1/q}$$
$$\asymp \Big(\sum_{j\in\mathbb{Z}}\sum_{\varepsilon,\delta\in\{0,1\}}\sum_{m=0}^{p_{j,\varepsilon}}\Big(\sum_{k\in\mathbb{Z}}\big(2^{j(\alpha+1/2-1/p)}|\langle f,\psi_\Gamma\rangle|\big)^p\Big)^{q/p}\Big)^{1/q},$$

Where we have used that the sum over ε, δ, and m is finite. Also, using the Brushlet characterization of the norm on $\dot{B}^\alpha_{p,q}$ (see [4]), we get

$$\|Tf\|_{\dot{B}^\alpha_{p,q}} \asymp \Big(\sum_{j\in\mathbb{Z}}\sum_{\varepsilon\in\{0,1\}}\sum_{m=0}^{p_{j,\varepsilon}}\Big(\sum_{k\in\mathbb{Z}}\big(|I_{s_{j,\varepsilon}+m}|^{(\alpha+1/2-1/p)}|\langle f,\psi_\Gamma\rangle|\big)^p\Big)^{q/p}\Big)^{1/q}$$
$$\asymp \Big(\sum_{j\in\mathbb{Z}}\sum_{\varepsilon\in\{0,1\}}\sum_{m=0}^{p_{j,\varepsilon}}\Big(\sum_{k\in\mathbb{Z}}\big(2^{j(\alpha+1/2-1/p)}|\langle f,\psi_\Gamma\rangle|\big)^p\Big)^{q/p}\Big)^{1/q}$$

Hence, it follows that $\|f\|_{\dot{B}^\alpha_{p,q}} \asymp \|Tf\|_{\dot{B}^\alpha_{p,q}}$. □

3.2. (a/b)**-regular exponential partition.** Let us now specify the family of brushlet bases that will be considered in the following two sections. For technical reasons we cannot handle general exponential partitions. Suppose that

$$\frac{|I^\varepsilon_{m+1}|}{|I^\varepsilon_m|} = 2^{a/b}, \qquad m\in\mathbb{Z},\ \varepsilon\in\{0,1\} \tag{3.2}$$

for some $b, a \in \mathbb{N}$ with $gcd(b,a) = 1$. We say that such a partition is an (a/b)**-regular exponential partition**. For notational convenience, let $w_{I^\varepsilon_{m,n}} = w^\varepsilon_{m,n}$.

We want to make an bijective mapping of the Meyer wavelet system to the brushlet system. In the following we will focus on the situation where $a < b$ but the results can easily be obtained for $a \geq b$ as well, using the same techniques as below (see Section 3.5). Again, to keep notation simple we will introduce the indicies $\Gamma \in \mathbb{Z}\times\mathbb{Z}$ and $\Upsilon \in \mathbb{Z}\times\mathbb{N}_0$ as follows. For $\ell \in \{0,1,\dots,a-1\}$ let $s_\ell := \lfloor \ell\cdot b/a\rfloor$ and $p_\ell := s_{\ell+1} - s_\ell$ (with $s_a := b$). Notice that $1 \leq p_\ell \leq \lceil b/a\rceil$. For $j\in\mathbb{Z}$, $k\in\mathbb{N}_0$, $\delta,\varepsilon\in\{0,1\}$, $0\leq \ell < a$, and $0 \leq m < p_\ell$ we define

$$\Gamma := \Gamma_{j,k,\ell,m,\delta,\varepsilon} := (aj+\ell, (-1)^\delta(2p_\ell k + 2m + \varepsilon) - \delta) \tag{3.3}$$

and

$$\Upsilon := \Upsilon_{j,k,\ell,m,\delta} := (bj + s_\ell + m, 2k+\delta). \tag{3.4}$$

It is not hard to check that Γ and Υ (considered as functions) are onto $\mathbb{Z}\times\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{N}_0$, respectively, and they are injective on their respective domains. The main justification for introducing these seemingly cumbersome indices is that the length of the interval

$I^{\varepsilon}_{bj+s_\ell+m} \in \mathcal{I}^\varepsilon$ is approximately equal to $2^{aj+\ell}$ up to a constant depending only on ℓ, m and ε. More precisely, define for $0 \le \ell < a$, $0 \le m < p_\ell$, and $\varepsilon \in \{0,1\}$ the constant $q_{\ell,m,\varepsilon} := |I^\varepsilon_{s_\ell+m}|2^{-\ell}$, and observe that

$$|I^{\varepsilon}_{bj+s_\ell+m}| = q_{\ell,m,\varepsilon}2^{aj+\ell}, \qquad \text{for all } j \in \mathbb{Z}.$$

We now want to define the operator $T\colon L_2 \to L_2$. The idea behind the definition of T is very simple, we map a given wavelet onto a brushlet with the same "frequency content" taking care that the mapping is injective and onto. We define

$$T\psi_\Gamma = w^\varepsilon_\Upsilon. \tag{3.5}$$

It is clear that T is an isomorphism on L_2 since Γ and Υ injectively run through all of $\mathbb{Z}\times\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{N}_0$, respectively.

3.3. **The equivalence in L_p.**

For a brushlet system associated with an (a/b)-regular exponential partition, the operator kernel associated to T defined by (3.5) is given by

$$K(x,y) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\sum_{\ell=0}^{a-1}\sum_{m=0}^{p_\ell}\sum_{\delta,\varepsilon\in\{0,1\}} \psi_\Gamma(y)w^\varepsilon_\Upsilon(x).$$

In this section we study T in L_p, $1<p<\infty$, by analyzing the associated kernel $K(x,y)$.

Proposition 3.2. *Let $\{w_{n,I}\}_{I\in\mathcal{I},n\in\mathbf{N}_0}$ be a 2-localized brushlet system associated with an (a/b)-regular exponential partitioning $\mathcal{I}$, $a<b$, and define $T\colon L_2\to L_2$ by (3.5). Then T extends to an isomorphism on L_p, $1<p<\infty$. In particular, $\{w_{n,I}\}_{I\in\mathcal{I},n\in\mathbf{N}_0}$ and the Meyer wavelet system are equivalent unconditional bases for L_p, $1<p<\infty$.*

Proof. It is not hard to verify that $K(x,y)$ is *not* a Calderón-Zygmund kernel, but the idea of the proof is to decompose $K(x,y)$ into a finite number of kernels $K^{\varepsilon,\delta}_{\ell,m}(x,y)$ that are slightly modified Calderón-Zygmund kernels. We split $K(x,y)$ as follows.

$$K(x,y) = \sum_{\delta,\varepsilon\in\{0,1\}}\sum_{\ell=0}^{a-1}\sum_{m=0}^{p_\ell} K^{\varepsilon,\delta}_{\ell,m}(x,y),$$

where

$$K^{\varepsilon,\delta}_{\ell,m}(x,y) := \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\psi_\Gamma(y)w^\varepsilon_\Upsilon(x).$$

Since the wavelet and brushlet system are orthonormal bases for L_2, it is easy to verify that the operator $T^{\varepsilon,\delta}_{\ell,m}$ associated with $K^{\varepsilon,\delta}_{\ell,m}(x,y)$ is bounded on L_2. By lemma A.1 in Appendix A, we have that there exist constants $0<C<\infty$ and $\eta>0$ such that

$$|K^{\varepsilon,\delta}_{\ell,m}(x,y)| \le C\left|(-1)^\delta p_\ell^{-1}y - \frac{q_{\ell,m,\varepsilon}}{\pi}x\right|^{-1},$$

$$|K^{\varepsilon,\delta}_{\ell,m}(x,y) - K^{\varepsilon,\delta}_{\ell,m}(x',y)| \le C|x-x'|^\eta\left|(-1)^\delta p_\ell^{-1}y - \frac{q_{\ell,m,\varepsilon}}{\pi}x\right|^{-1-\eta},$$

if $|x-x'| \leq \frac{1}{2}\left|\frac{(-1)^\delta \pi}{q_{\ell,m,\varepsilon} p_\ell} y - x\right|$, and

$$|K_{\ell,m}^{\varepsilon,\delta}(x,y) - K_{\ell,m}^{\varepsilon,\delta}(x,y')| \leq C|y-y'|^\eta \left|(-1)^\delta p_\ell^{-1} y - \frac{q_{\ell,m,\varepsilon}}{\pi} x\right|^{-1-\eta},$$

if $|y-y'| \leq \frac{1}{2}\left|y - \frac{(-1)^\delta q_{\ell,m,\varepsilon} p_\ell}{\pi} x\right|$.

Notice that $K_{\ell,m}^{\varepsilon,\delta}(x,y)$ is not a standard Calderón-Zygmund kernel since it has the singularity on the line $y = \frac{(-1)^\delta p_\ell q_{\ell,m,\varepsilon}}{\pi} x$. However, this is not a serious problem. Let D_u be the dilation operator defined by $D_u f(x) = f(ux)$. Consider the operator $\tilde{T}_{\ell,m}^{\varepsilon,\delta} := D_{\pi/q_{\ell,m,\varepsilon}} T_{\ell,m}^{\varepsilon,\delta} D_{(-1)^\delta p_\ell^{-1}}$. Clearly, $\tilde{T}_{\ell,m}^{\varepsilon,\delta}$ is bounded on L_2 since D_u is bounded on L_2. Moreover, we see that $\tilde{T}_{\ell,m}^{\varepsilon,\delta}$ has kernel

$$\tilde{K}_{\ell,m}^{\varepsilon,\delta}(x,y) = K_{\ell,m}^{\varepsilon,\delta}\left(\frac{\pi}{q_{\ell,m,\varepsilon}} x, (-1)^\delta p_\ell y\right).$$

By the above estimates on the kernel $K_{\ell,m}^{\varepsilon,\delta}(x,y)$, $\tilde{T}_{\ell,m}^{\varepsilon,\delta}$ is a Calderón-Zygmund operator. Therefore $\tilde{T}_{\ell,m}^{\varepsilon,\delta}$ is bounded on L_p, $1<p<\infty$, and it follows that $T_{\ell,m}^{\varepsilon,\delta} = D_{q_{\ell,m,\varepsilon}/\pi} \tilde{T}_{\ell,m}^{\varepsilon,\delta} D_{(-1)^\delta p_\ell}$ is bounded on L_p since D_u is bounded on L_p, $1<p<\infty$. Now, T can be decomposed as a finite sum $T = \sum_{\ell,m,\varepsilon,\delta} T_{\ell,m}^{\varepsilon,\delta}$, so we may conclude that T is bounded on L_p, $1<p<\infty$.

We can estimate $T^{-1} = T^*$, which has kernel $\overline{K(y,x)}$, using an analogue approach to conclude that T^{-1} is bounded on L_p, $1<p<\infty$, and hence T is an isometry on L_p. □

3.4. **The equivalence in $\dot{F}_{p,q}^\alpha$.** In order to extend the isometry to general homogeneous Triebel-Lizorkin spaces, we need the theory of almost diagonal matrices. Let M be the change of basis matrix from the Meyer wavelet basis to the brushlet system, given by

$$M := [\langle T\psi_{j,k}, \psi_{j',k'}\rangle]_{j,j',k,k' \in \mathbb{Z}}, \tag{3.6}$$

with T given by (3.5). Notice that, M is an isometry on ℓ_2 and $M^{-1} = M^*$. Let $A_b\colon L_2 \to \ell_2$ and $S_b\colon \ell_2 \to L_2$ denote respectively the analysis and synthesis operator associated with the brushlet system, i.e. $A_b f = \{\langle f, w_{I,n}\rangle\}$, and $S_b d = \sum_{I,n} d_{I,n} w_{I,n}$. Since both the wavelet and brushlet system are orthonormal bases, we have the relation

$$A_b = M \circ A_\psi, \qquad \text{and} \qquad S_b = S_\psi \circ M^*. \tag{3.7}$$

We would like to extend A_b and S_b to bounded operators on the Triebel-Lizorkin spaces $\dot{F}_{p,q}^s$ and $\dot{f}_{p,q}^s$, respectively. According to the relation in (3.7) this is equivalent to the property that M and M^* are bounded operators on $\dot{f}_{p,q}^s$.

The strategy is to decompose the operator T into a finite sum of the form

$$T = \sum_u D_u (S_\psi \circ M_u \circ A_\psi),$$

where each M_u is a bounded operator on $\dot{f}_{p,q}^s$.

Let us define the subclass of brushlet systems that we will consider below.

Definition 3.3. Given $N \in \mathbb{N}$ and $\gamma > 0$. We say that a brushlet system $\{w_{I,n}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ is of **class (A)** if it is $(N+\gamma+2)$-localized and associated with an (a/b)-regular exponential partitioning $\mathcal{I}$, $a < b$.

We begin with the observation that the mapping given by $\psi_\Gamma \to w^{\pm}_{\Upsilon}$ has a sparse wavelet representation. A proof of the following lemma is given in Appendix A.

Lemma 3.4. *Suppose the brushlet system $\{w_{I,n}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ is of class (A). Let Γ and Υ be the indicies defined by* (3.3) *and* (3.4). *Then for $j \geq j'$ we have*

$$|\langle D_{\frac{\pi}{p_\ell q_{\ell,m,\varepsilon}}} w^{\pm}_{\Upsilon}, \psi_{\Gamma'}\rangle| \leq \frac{C2^{-(a(j-j')+\ell-\ell')(N+1/2)}}{\big(1+\big|\kappa' \pm 2^{a(j'-j)+\ell'-\ell}|\kappa|\big|\big)^{\gamma}},$$

and for $j < j'$

$$|\langle D_{\frac{\pi}{p_\ell q_{\ell,m,\varepsilon}}} w^{\pm}_{\Upsilon}, \psi_{\Gamma'}\rangle| \leq \frac{C2^{-(a(j'-j)+\ell'-\ell)(N+1/2)}}{\big(1+\big|2^{a(j-j')+\ell-\ell'}\kappa' \pm |\kappa|\big|\big)^{\gamma}},$$

where $\kappa := (-1)^{\delta}(2p_\ell k + 2m + \varepsilon + \delta)$, $\kappa' := (-1)^{\delta'}(2p_{\ell'}k' + 2m' + \varepsilon' + \delta')$, $\Upsilon = (bj + s_\ell + m, 2k + \delta)$ and $\Gamma' = (aj' + \ell', \kappa')$.

The two inequalities in Lemma 3.4 bear some resemblance with the concept of almost diagonal matrices. In order to use the result on the operator T we need to write it as the sum of two operators $T = T^+ + T^-$, using the relation $w_{I,n} = w^+_{I,n} + w^-_{I,n}$.

Define the two operators P_+ and P_- by $P_\pm w_{j,n} = w^{\pm}_{j,n}$. It is easy to see that the L_2-norm of the compound operator $D_{\frac{\pi}{p_\ell q_{\ell,m,\varepsilon}}} P_\pm$ is bounded by the ℓ^2-norm of the sequence $\{\langle D_{\pi/(p_\ell q_{\ell,m,\varepsilon})} w^{\pm}_{\Upsilon}, \psi_{\Gamma'}\rangle\}_{\Upsilon,\Gamma'}$. Since this value is finite according to the estimates in Lemma 3.4, $D_{\frac{\pi}{p_\ell q_{\ell,m,\varepsilon}}} P_\pm$ is bounded on L_2. But this means that $P_\pm$ itself is a bounded operator on L_2 since the dilation operator is bounded on L_2.

As a result of Lemma 3.4, we have the following corollary based on the theory of almost diagonal matrices.

Corollary 3.5. *Suppose the brushlet system $\{w_{I,n}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ is of class (A). Let $S^{\varepsilon,\delta}_{\ell,m,\pm} := D_{\frac{\pi}{p_\ell q_{\ell,m,\varepsilon}}} P_\pm T^{\varepsilon,\delta}_{\ell,m}$. Then for $i, i', \kappa, \kappa' \in \mathbb{Z}$ we have*

$$|\langle S^{\varepsilon,\delta}_{\ell,m,\pm}\psi_{i,\kappa}, \psi_{i',\kappa'}\rangle| \leq \frac{C2^{-|i-i'|(N+1/2)}}{\big(1+\min\{2^i, 2^{i'}\}\big|2^{-i'}\kappa' \pm 2^{-i}(-1)^{\delta}\kappa\big|\big)^{\gamma}}. \tag{3.8}$$

Moreover, $S^{\varepsilon,\delta}_{\ell,m,\pm}$ extends to a bounded operator on the Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$, for parameters $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q < \infty$, satisfying $1/\min(1,p,q) < \gamma$ and $\max\{s, 1/\min(1,p,q) - 1 - s\} < N$.

Proof. The inequality (3.8) is an immediate consequences of Lemma 3.4. In order to conclude the boundedness result we define two operators Q_{-1} and Q_1 by $Q_\zeta\psi_{j,k} := \psi_{j,\zeta k}$. From the wavelet characterization of the Triebel-Lizorkin spaces (see (2.10)) it is easy to see that Q_ζ, is a bounded operator on these spaces. Let $\zeta = \mp(-1)^{\delta}$, and define the matrix $M^{\varepsilon,\delta}_{\ell,m,\pm} := [\langle S^{\varepsilon,\delta}_{\ell,m,\pm} Q_\zeta \psi_{i,\kappa}, \psi_{i',\kappa'}\rangle]_{i,i',\kappa,\kappa'\in\mathbb{Z}}$. This is an almost diagonal matrix for $\dot{f}^s_{p,q}$ as defined by Frazier and Jawerth in [9], and thus bounded on $\dot{f}^s_{p,q}$, provided

$1/\min(1,p,q) < \gamma$ and $\max\{s, 1/\min(1,p,q)-1-s\} < N$, see Theorem 3.3 in [9]. The corollary now follows, since $S^{\varepsilon,\delta}_{\ell,m,\pm} = S_\psi \circ M^{\varepsilon,\delta}_{\ell,m,\pm} \circ A_\psi \circ Q_\zeta$. □

Notice that $T^{\varepsilon,\delta}_{\ell,m} = D_{\frac{p_\ell q_{\ell,m,\varepsilon}}{\pi}}(S^{\varepsilon,\delta}_{\ell,m,+} + S^{\varepsilon,\delta}_{\ell,m,-})$. Thus, we can use the result in Corollary 3.5 to analyze the boundedness of the operator $T^{\varepsilon,\delta}_{\ell,m}$, and since T is given by a finite linear combination of $T^{\varepsilon,\delta}_{\ell,m}$ we can obtain boundedness results for T as well. We have the following proposition.

Proposition 3.6. *Suppose the brushlet system $\{w_{I,n}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ is of class (A). Then the operator T defined by* (3.5) *is an isomorphism on $\dot{F}^s_{p,q}$ for parameters $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ satisfying $1/\min(1,p,q) < \gamma$ and $\max\{s, 1/\min(1,p,q) - s - 1\} < N$. Equivalently, M and M^* defined by* (3.6) *extend to bounded operators on $\dot{f}^s_{p,q}$, for s, p and q in the same range.*

Proof. We recall that T has the finite decomposition

$$T = \sum T^{\varepsilon,\delta}_{\ell,m} = \sum D_{\frac{p_\ell q_{\ell,m,\varepsilon}}{\pi}}(S^{\varepsilon,\delta}_{\ell,m,+} + S^{\varepsilon,\delta}_{\ell,m,-}).$$

Corollary 3.5 shows that $S^{\varepsilon,\delta}_{\ell,m,\pm}$ are bounded on $\dot{F}^s_{p,q}$, and the dilation $D_{\frac{p_\ell q_{\ell,m,\varepsilon}}{\pi}}$ is also bounded on $\dot{F}^s_{p,q}$, so it follows that $T^{\varepsilon,\delta}_{\ell,m}$ is bounded on $\dot{F}^s_{p,q}$. Therefore T is bounded on $\dot{F}^s_{p,q}$, and M is consequently bounded on $\dot{f}^s_{p,q}$.

We now consider $T^* = T^{-1}$ (in the L_2-sense). We have

$$T^* = \sum (T^{\varepsilon,\delta}_{\ell,m})^* = \sum \big((S^{\varepsilon,\delta}_{\ell,m,+})^* + (S^{\varepsilon,\delta}_{\ell,m,-})^*\big) D^*_{\frac{p_\ell q_{\ell,m,\varepsilon}}{\pi}}.$$

Using the same notation as in the proof of Corollary 3.5, we notice that $(S^{\varepsilon,\delta}_{\ell,m,\pm})^* = Q_\zeta \circ S_\psi \circ (M^{\varepsilon,\delta}_{\ell,m,\pm})^* \circ A_\psi$. The estimate of the matrix elements given by Corollary 3.5 is symmetric in (i,κ) and (i',κ') so we deduce that $(M^{\varepsilon,\delta}_{\ell,m,\pm})^* = [\langle \psi_{i,\kappa}, S^{\varepsilon,\delta}_{\ell,m,\pm} Q_\zeta \psi_{i',\kappa'}\rangle]_{i,i',\kappa,\kappa'\in\mathbb{Z}}$ is an almost diagonal matrix for $\dot{f}^s_{p,q}$. Therefore, $(S^{\varepsilon,\delta}_{\ell,m,\pm})^*$ extend to bounded operators on $\dot{F}^s_{p,q}$, and then from the relation $D^*_u = u^{-1}D_{u^{-1}}$, it follows that T^* also extends to a bounded operator on $\dot{F}^s_{p,q}$. However, since $T^* = T^{-1}$ on L_2 and, e.g., $\mathcal{S}(\mathbb{R})$ is dense in both L_2 and $\dot{F}^s_{p,q}$, we conclude that T^{-1} is bounded on $\dot{F}^s_{p,q}$. This implies that the matrix representation M^* of T^* extends to a bounded operator on $\dot{f}^s_{p,q}$. □

Remark 3.7. Recall that $Id_{\dot{F}^s_{p,q}} = S_\psi \circ A_\psi$. Since M and its inverse are bounded on $\dot{f}^s_{p,q}$ we also have

$$Id_{\dot{F}^s_{p,q}} = S_\psi \circ M^* \circ M \circ A_\psi.$$

But $M \circ A_\psi$ and $S_\psi \circ M^*$ are in fact respectively the analysis and synthesis operator associated with the brushlet system, so the brushlet system defines a retract of $\dot{F}^s_{p,q}$ through the sequence space $\dot{f}^s_{p,q}$. We illustrate this by the following commuting diagram.

$$\begin{array}{ccc} \dot{F}^s_{p,q} & \xrightarrow{\ \mathrm{Id}_{\dot{F}^s_{p,q}}\ } & \dot{F}^s_{p,q} \\ & {\scriptstyle A_b}\searrow \qquad \nearrow{\scriptstyle S_b} & \\ & \dot{f}^s_{p,q} & \end{array}$$

According to Theorem 6.20 in [10] we have a similar result for the homogeneous Besov spaces.

Proposition 3.8. *Suppose the brushlet system $\{w_{I,n}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ is of class (A). Then the operator T defined by* (3.5) *is an isomorphism on $\dot{B}^s_{p,q}$ for parameters $s\in\mathbb{R}$, $0<p,q\le\infty$, satisfying $1/\min(1,p)<\gamma$ and $\max\{s,1/\min(1,p)-s-1\}<N$. Moreover, the brushlet system defines a retract of $\dot{B}^s_{p,q}$ through the sequence space $\dot{b}^s_{p,q}$ for s,p and q in the same range.*

3.5. **Other types of partitions $\mathcal{I}$.** In all the calculations in the previous two sections, we have assumed that the brushlet system is based on an (a/b)-regular exponential partition $\mathcal{I}$ where $b,a\in\mathbb{N}$ with $a<b$ and $gcd(b,a)=1$. In this case we have been able to construct an isomorphism between the brushlet system and the Meyer system in the homogeneous Besov and Triebel-Lizorkin spaces. What happens if we have another type of partition $\mathcal{I}$?

The first situation we consider is when $\mathcal{I}$ is an (a/b)-regular exponential partition with $a>b$. In this case we can construct a wavelet to brushlet mapping very similar to the one considered in the previous section. For $\ell\in\{0,1,\dots,b-1\}$ let $s_\ell:=\lfloor \ell\cdot a/b\rfloor$ and $p_\ell:=s_{\ell+1}-s_\ell$ (with $s_b:=a$). Then, for $j\in\mathbb{Z}$, $k\in\mathbb{N}_0$, $\delta,\varepsilon\in\{0,1\}$, $\ell\in\{0,1,\dots,b-1\}$, and $m\in\{0,1,\dots,p_\ell-1\}$ we define

$$\Gamma:=\Gamma_{j,k,\ell,m,\delta,\varepsilon}:=(aj+s_\ell+m,(-1)^\delta(2k+\varepsilon)-\delta)$$

and

$$\Upsilon:=\Upsilon_{j,k,\ell,m,\delta}:=(bj+\ell,2p_\ell k+2m+\delta).$$

Using the same techniques as in the previous section, it can be verified that the map $T\colon L_2\to L_2$ defined by

$$T\psi_\Gamma=w^\varepsilon_\Upsilon,$$

is an isomorphism on the Besov and Triebel-Lizorkin spaces provided the brushlet is sufficiently regular.

The (a/b)-regular exponential partition condition can be slightly generalized by using more than one rational parameter a/b. For example, suppose the quotient in (3.2) equals a_1/b_1 for even m and a_2/b_2 for odd m. Then it can be proven, along the same line as above, that we can construct a brushlet system based on this partition that is equivalent to the Meyer wavelet. But, the indicies Γ and Υ used to obtain the isomorphism will be even messier looking than in the previous construction. We leave the details for the reader.

Finally, let us mention that it is an open problem whether a brushlet system based on an irrational regular exponential partition (or a more general exponential partition) is equivalent in L_p to a wavelet system. Unfortunately, the technique used in this paper fails in that case. In particular, it is not possible to decompose the operator T into a finite sum of modified Calderón-Zygmund operators.

4. Approximation with brushlet systems

In this final section we use the results of Section 3 to study nonlinear approximation with brushlet systems, where we measure the approximation error in a general Triebel-Lizorkin semi-(quasi-)norm. The Triebel-Lizorkin norm is the "natural" measure of the approximation error when it comes to nonlinear approximation with wavelet type systems, see e.g. [11].

The method used below to obtain results about approximation with brushlets is simple, we use the equivalence of the brushlet system and the Meyer wavelet system to "translate" already known results on approximation with wavelet systems to the brushlet case. Let us first introduce the needed notation.

We consider the following brushlet dictionary

$$\mathcal{D}_b := \{w_{I,n} \,|\, I \in \mathcal{I}, n \in \mathbb{N}_0\},$$

associated with an exponential type partitioning $\mathcal{I}$, and the Meyer wavelet dictionary

$$\mathcal{D}_M := \{\psi(2^j \cdot -k) | j, k \in \mathbb{Z}\},$$

The associated nonlinear manifold of all possible m-term expansions by elements from $\mathcal{D} \in \{\mathcal{D}_b, \mathcal{D}_M\}$ is given by

$$\Sigma_m(\mathcal{D}) = \left\{ S \colon S = \sum_{j=1}^{m} a_j g_j, \quad \text{with } a_j \in \mathbf{C}, g_j \in \mathcal{D} \right\}.$$

The error in $\dot{F}^{\beta}_{p,t}$ of the best m-term approximation from $\Sigma_m(\mathcal{D})$ is given by

$$\sigma_m(f, \mathcal{D})_{\dot{F}^{\beta}_{p,t}} := \inf_{S \in \Sigma_m(\mathcal{D})} |f - S|_{\dot{F}^{\beta}_{p,t}}.$$

We let $\mathcal{A}^{\alpha}_s(\dot{F}^{\beta}_{p,t}, \mathcal{D})$, $\alpha > 0$, $0 < s \leq \infty$, denote the approximation space of all functions f such that

$$|f|_{\mathcal{A}^{\alpha}_s(\dot{F}^{\beta}_{p,t}, \mathcal{D})} := \left(\sum_{m=1}^{\infty} \left(m^{\alpha} \sigma_m(f, \mathcal{D})_{\dot{F}^{\beta}_{p,t}} \right)^s \frac{1}{m} \right)^{1/s} < \infty,$$

with the following standard modification when $s = \infty$:

$$|f|_{\mathcal{A}^{\alpha}_{\infty}(X, \mathcal{D})} := \sup_{m \in \mathbb{N}} m^{\alpha} \sigma_m(f, \mathcal{D})_{\dot{F}^{\beta}_{p,t}} < \infty.$$

Now the fundamental question is whether it is possible to characterize $\mathcal{A}^{\alpha}_s(X, \mathcal{D}_b)$ in terms of well known spaces. In [4], the special case $X = \dot{F}^{0}_{p,2} \approx L_p$ was considered, and it was proven that $\mathcal{A}^{\alpha}_s(L_p, \mathcal{D}_b)$ can be identified by (essentially) a Besov space.

It is well known that the main tool in the characterization of $\mathcal{A}^{\alpha}_s(X, \mathcal{D})$ comes from the link between approximation theory and interpolation theory (see *e.g.* [7, 3]). Let Y be an abelian group with semi-(quasi-)norm $|\cdot|_Y$ continuously embedded in $\dot{F}^{\beta}_{p,t}$. Given $\gamma > 0$, the Jackson inequality

$$\sigma_m(f, \mathcal{D})_{\dot{F}^{\beta}_{p,t}} \leq C m^{-\gamma} |f|_Y, \qquad \forall f \in Y, \, \forall m \in \mathbb{N} \tag{4.1}$$

and the Bernstein inequality

$$|S|_Y \le C' m^{\gamma} |S|_{\dot{F}^{\beta}_{p,t}}, \qquad \forall S \in \Sigma_m(\mathcal{D}) \tag{4.2}$$

(with some constants C and C' independent of f, S and m) imply, respectively, the continuous embedding

$$\left(\dot{F}^{\beta}_{p,t}, Y\right)_{\alpha/\gamma,s} \hookrightarrow \mathcal{A}^{\alpha}_{s}(\dot{F}^{\beta}_{p,t}, \mathcal{D})$$

and the converse embedding

$$\left(\dot{F}^{\beta}_{p,t}, Y\right)_{\alpha/\gamma,s} \hookleftarrow \mathcal{A}^{\alpha}_{s}(\dot{F}^{\beta}_{p,t}, \mathcal{D})$$

for all $0 < \alpha < \gamma$ and $s \in (0, \infty]$. Here $(X, Y)_{\theta,q}$ denotes the interpolation space between X and Y obtained using the real method. We refer the reader to [2] for the definition of the real method of interpolation. The following is known about $\mathcal{D}_M$, see e.g. [11] and [6].

Theorem 4.1. *Let $0 < p < \infty$, $0 < t \le \infty$, $\beta < \gamma$, and τ be defined by $1/\tau := (\gamma - \beta) + 1/p$. The following Jackson inequality holds*

$$\sigma_m(f, \mathcal{D}_M)_{\dot{F}^{\beta}_{p,t}} \le C m^{-(\gamma-\beta)} |f|_{\dot{B}^{\gamma}_{\tau,\tau}}, \qquad \forall f \in \dot{B}^{\gamma}_{\tau,\tau},\ \forall m \in \mathbb{N},$$

and the following Bernstein inequality holds

$$|S|_{\dot{B}^{\gamma}_{\tau,\tau}} \le C m^{(\gamma-\beta)} \|S\|_{\dot{F}^{\beta}_{p,t}}, \qquad \forall S \in \Sigma_m(\mathcal{D}_M), m \in \mathbb{N}.$$

We conclude the paper by Proposition 4.2 below on nonlinear approximation with brushlet systems. The proposition will be deduced from Theorem 4.1. Notice that Proposition 3.6 and Proposition 3.8 provide a class of brushlet systems for which the hypotheses of Proposition 4.2 are satisfied.

Proposition 4.2. *Given $0 < p < \infty$, $0 < t \le \infty$, $\beta < \gamma$, and τ satisfying $1/\tau := (\gamma - \beta) + 1/p$. Suppose $\mathcal{D}_b$ is a brushlet dictionary equivalent to the Meyer wavelet basis in both $\dot{F}^{\beta}_{p,t}$ and $\dot{B}^{\gamma}_{\tau,\tau}$. Then the following Jackson inequality holds*

$$\sigma_m(f, \mathcal{D}_b)_{\dot{F}^{\beta}_{p,t}} \le C m^{-(\gamma-\beta)} |f|_{\dot{B}^{\gamma}_{\tau,\tau}}, \qquad \forall f \in \dot{B}^{\gamma}_{\tau,\tau},\ \forall m \in \mathbb{N},$$

and the following Bernstein inequality holds

$$|S|_{\dot{B}^{\gamma}_{\tau,\tau}} \le C m^{(\gamma-\beta)} \|S\|_{\dot{F}^{\beta}_{p,t}}, \qquad \forall S \in \Sigma_m(\mathcal{D}_b), m \in \mathbb{N}.$$

Moreover,

$$\mathcal{A}^{\alpha}_{s}(\dot{F}^{\beta}_{p,t}, \mathcal{D}_b) = \left(\dot{F}^{\beta}_{p,t}, \dot{B}^{\gamma}_{\tau,\tau}\right)_{\frac{\alpha}{\gamma-\beta},s},$$

for $0 < \alpha < \gamma - \beta$ and $s \in (0, \infty]$.

Proof. Let T be the mapping giving the equivalence of $\mathcal{D}_b$ and the Meyer wavelet basis. Let us prove that there is a Bernstein inequality for the brushlet system. Notice that for

$S \in \Sigma_m(\mathcal{D}_b)$ we have $T^{-1}S \in \Sigma_m(\mathcal{D}_M)$. Hence,

$$\begin{aligned} |S|_{\dot{B}^{\gamma}_{\tau,\tau}} &\le C|T^{-1}S|_{\dot{B}^{\gamma}_{\tau,\tau}} \\ &\le C' m^{(\gamma-\beta)} |T^{-1}S|_{\dot{F}^{\beta}_{p,t}} \\ &\le C'' m^{(\gamma-\beta)} |S|_{\dot{F}^{\beta}_{p,t}}, \end{aligned}$$

with C'' independent of S and m. Next we consider the Jackson estimate. Let $f \in \dot{B}^{\gamma}_{\tau,\tau}$. Then $T^{-1}f \in \dot{B}^{\gamma}_{\tau,\tau}$, and we let $g_m \in \Sigma_m(\mathcal{D}_M)$, $m \ge 1$, be a sequence for which

$$|T^{-1}f - g_m|_{\dot{F}^{\beta}_{p,t}} \le 2\sigma_m(T^{-1}f, \mathcal{D}_M)_{\dot{F}^{\beta}_{p,t}}.$$

Then,

$$\begin{aligned} \sigma_m(f, \mathcal{D}_b)_{\dot{F}^{\beta}_{p,t}} &\le |T(T^{-1}f - g_m)|_{\dot{F}^{\beta}_{p,t}} \\ &\le C|T^{-1}f - g_m|_{\dot{F}^{\beta}_{p,t}} \\ &\le 2C\sigma_m(T^{-1}f, \mathcal{D}_M)_{\dot{F}^{\beta}_{p,t}} \\ &\le C' m^{-(\gamma-\beta)} |T^{-1}f|_{\dot{B}^{\gamma}_{\tau,\tau}} \\ &\le C'' m^{-(\gamma-\beta)} |f|_{\dot{B}^{\gamma}_{\tau,\tau}}, \end{aligned}$$

with C'' independent of m and f. The fact that

$$\mathcal{A}^{\alpha}_s(\dot{F}^{\beta}_{p,t}, \mathcal{D}_b) = \left(\dot{F}^{\beta}_{p,t}, \dot{B}^{\gamma}_{\tau,\tau}\right)_{\frac{\alpha}{\gamma-\beta},s}$$

is a direct consequence of the above Bernstein and Jackson inequalities. □

Appendix A. Proof of some technical lemmas

In this appendix, we prove some technical lemmas used in this paper. First we prove the following result which was used in the proof of Proposition 3.2.

Lemma A.1. *The Kernel given in the proof of Proposition 3.2 satisfies*

$$|K^{\varepsilon,\delta}_{\ell,m}(x,y)| \le C\left|(-1)^{\delta} p_{\ell}^{-1} y - \frac{q_{\ell,m,\varepsilon}}{\pi} x\right|^{-1},$$

$$|K^{\varepsilon,\delta}_{\ell,m}(x,y) - K^{\varepsilon,\delta}_{\ell,m}(x',y)| \le C|x-x'|^{\eta}\left|(-1)^{\delta} p_{\ell}^{-1} y - \frac{q_{\ell,m,\varepsilon}}{\pi} x\right|^{-1-\eta}$$

$$\text{if } |x-x'| \le \frac{1}{2}\left|\frac{(-1)^{\delta}\pi}{q_{\ell,m,\varepsilon} p_{\ell}} y - x\right|,$$

and

$$|K^{\varepsilon,\delta}_{\ell,m}(x,y) - K^{\varepsilon,\delta}_{\ell,m}(x,y')| \le C|y-y'|^{\eta}\left|(-1)^{\delta} p_{\ell}^{-1} y - \frac{q_{\ell,m,\varepsilon}}{\pi} x\right|^{-1-\eta},$$

$$\text{if } |y-y'| \le \frac{1}{2}\left|y - \frac{(-1)^{\delta} q_{\ell,m,\varepsilon} p_{\ell}}{\pi} x\right|,$$

for some $\eta > 0$.

Proof. We use $|I^{\varepsilon}_{bj+s_\ell+m}| = q_{\ell,m,\varepsilon}2^{aj+\ell}$, (2.9), (2.8), and $\psi \in \mathcal{S}(\mathbb{R})$, to obtain

$$\begin{aligned}
|K^{\varepsilon,\delta}_{\ell,m}(x,y)| \le C \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} & 2^{aj+l} q^{1/2}_{\ell,m,\varepsilon}(1+|2^{aj+\ell}y-(-1)^{\delta}(2p_\ell k+2m+\varepsilon)+\delta|)^{-1-\gamma} \\
&\cdot (1+||I^{\varepsilon}_{bj+s_\ell+m}|x-\pi(2k+\delta+1/2)|)^{-1-\gamma} \\
\le C' \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} & 2^{aj+l}(1+|2^{aj+\ell}y-(-1)^{\delta}2p_\ell k|)^{-1-\gamma}(1+||I^{\varepsilon}_{bj+s_\ell+m}|x-2\pi k|)^{-1-\gamma} \\
= C' \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} & 2^{aj+l}(1+|2^{aj+\ell}y-(-1)^{\delta}2p_\ell k|)^{-1-\gamma}(1+|2^{aj+\ell}q_{\ell,m,\varepsilon}x-2\pi k|)^{-1-\gamma} \\
\le C'' \sum_{j\in\mathbb{Z}} & 2^{aj+\ell}(1+|2^{aj+\ell}(-1)^{\delta}p_\ell^{-1}y-2^{aj+\ell}q_{\ell,m,\varepsilon}\pi^{-1}x|)^{-1-\gamma} \\
\le C''' & \left|(-1)^{\delta}p_\ell^{-1}y-\frac{q_{\ell,m,\varepsilon}}{\pi}x\right|^{-1}.
\end{aligned}$$

Next we estimate the smoothness of the kernel $K^{\varepsilon,\delta}_{\ell,m}(x,y)$ in each variable separately. Let us consider the x variable first. Given $\eta \in (0,1)$ we have

$$\begin{aligned}
|K^{\varepsilon,\delta}_{\ell,m}(x,y)-K^{\varepsilon,\delta}_{\ell,m}(x',y)| &\le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |w^{\varepsilon}_{\Upsilon}(x)-w^{\varepsilon}_{\Upsilon}(x')|^{\eta}|w^{\varepsilon}_{\Upsilon}(x)|^{1-\eta}|\psi_\Gamma(y)| \\
&\quad + \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |w^{\varepsilon}_{\Upsilon}(x)-w^{\varepsilon}_{\Upsilon}(x')|^{\eta}|w^{\varepsilon}_{\Upsilon}(x')|^{1-\eta}|\psi_\Gamma(y)| \\
&:= J_1 + J_2.
\end{aligned}$$

The functions g_I are uniformly smooth (analytic, actually) so we have

$$|g_I(x)-g_I(x')| \le C|x-x'|^{\alpha} \qquad \text{for all } I \in \mathcal{I}.$$

Assuming that $\eta \le \frac{\gamma}{2(1+\alpha+\gamma)}$, we obtain

$$\begin{aligned}
J_1 \le C \sum_{j,k} & 2^{(aj+\ell)(1+\alpha\eta)}|x-x'|^{\alpha\eta}(1+|2^{aj+\ell}q_{\ell,m,\varepsilon}x-2\pi k|)^{-(1+\gamma)(1-\eta)} \\
&\cdot (1+|2^{aj+\ell}y-(-1)^{\delta}2p_\ell k|)^{-1-\gamma} \\
\le C|x-x'|^{\alpha\eta}\sum_j & 2^{(aj+\ell)(1+\alpha\eta)}(1+|2^{aj+\ell}(-1)^{\delta}p_\ell^{-1}y-2^{aj+\ell}q_{\ell,m,\varepsilon}\pi^{-1}x|)^{-(1+\gamma)(1-\eta)} \\
\le C|x-x'|^{\alpha\eta} & \left|(-1)^{\delta}p_\ell^{-1}y-\frac{q_{\ell,m,\varepsilon}}{\pi}x\right|^{-1-\alpha\eta}.
\end{aligned}$$

Similar estimates give the bound on J_2

$$J_2 \le C|x-x'|^{\alpha\eta}\left|(-1)^{\delta}p_\ell^{-1}y-\frac{q_{\ell,m,\varepsilon}}{\pi}x'\right|^{-1-\alpha\eta}.$$

Hence, if we assume $|x-x'| \le \frac{1}{2}\left|\frac{(-1)^{\delta}\pi}{q_{\ell,m,\varepsilon}p_\ell^{-1}}y-x\right|$, the triangle inequality gives

$$|K^{\varepsilon,\delta}_{\ell,m}(x,y)-K^{\varepsilon,\delta}_{\ell,m}(x',y)| \le C|x-x'|^{\alpha\eta}\left|(-1)^{\delta}p_\ell^{-1}y-\frac{q_{\ell,m,\varepsilon}}{\pi}x\right|^{-1-\alpha\eta}.$$

Likewise, applying the same technique to the variable y, we get

$$|K^{\varepsilon,\delta}_{\ell,m}(x,y) - K^{\varepsilon,\delta}_{\ell,m}(x,y')| \le C|y-y'|^{\alpha\eta}\Big|(-1)^\delta p_\ell^{-1} y - \frac{q_{\ell,m,\varepsilon}}{\pi} x\Big|^{-1-\alpha\eta}.$$

□

A.1. **Proof of Lemma 3.4.** Here we give a proof of Lemma 3.4. We refer to Section 3 for the statement of the lemma.

Proof of Lemma 3.4. Observe that

$$\begin{aligned}\langle D_{\pi/(p_\ell q_{\ell,m,\varepsilon})} w^{\pm}_{\Upsilon}, \psi_{\Gamma'}\rangle &= q^{1/2}_{\ell,m,\varepsilon} 2^{(a(j+j')+\ell+\ell')/2} \int e^{i\frac{\alpha^\varepsilon_{j,\ell,m}\pi}{p_\ell q_{\ell,m,\varepsilon}} y} g(2^{aj+\ell}\tfrac{\pi}{p_\ell} y \pm \pi(2k+\delta+1/2)) \\ &\qquad\qquad \cdot \psi\big(2^{aj'+\ell'} y - \kappa'\big)\, dy \\ &= q^{1/2}_{\ell,m,\varepsilon} 2^{(a(j-j')+\ell-\ell')/2} \int e^{i\beta(y)} g(2^{a(j-j')+\ell-\ell'}\tfrac{\pi}{p_\ell}(y - x_\pm))\psi(y)\, dy,\end{aligned}$$

where $\alpha^\varepsilon_{j,\ell,m} = \alpha_{I^\varepsilon_{bj+s_\ell+m}}$,

$$x_\pm = -(\kappa' \pm 2^{a(j'-j)+\ell'-\ell} p_\ell(2k+\delta+1/2)),$$

and $\beta(y) := \frac{\alpha^\varepsilon_{j,\ell,m}\pi}{p_\ell q_{\ell,m,\varepsilon}} 2^{-(aj'+\ell')}(y+\kappa')$. Fix a $z \in \mathbb{R}$. Since the brushlets have N vanishing moments for any $N \in \mathbb{N}$ and $\psi \in \mathcal{S}(\mathbb{R})$, a Taylor expansion of ψ around z yields

$$\begin{aligned}&\Big|\int e^{i\beta(y)} g(2^{a(j-j')+\ell-\ell'}\tfrac{\pi}{p_\ell}(y-z))\psi(y)\, dy\Big| \\ &\qquad \le C\Big(\int_{|y-z|\le|z|/2} + \int_{|y-z|>|z|/2}\Big)|g(2^{a(j-j')+\ell-\ell'}\tfrac{\pi}{p_\ell}(y-z))||y-z|^N E(z,y)\, dy \\ &\qquad = I_1 + I_2,\end{aligned}$$

where

$$E(z,y) := \sup_{t\in(0,1)} \big|\tfrac{d^N}{dx^N}\psi(z+t(y-z))\big|/N!.$$

Notice that $E(z,y) \le C(1+|z|)^{-\gamma}$ for $|y-z| \le |z|/2$. Since the brushlet system is γ-localized we obtain

$$\begin{aligned} I_1 &\le C\Big(\frac{\pi}{p_\ell}\Big)^{-N}(1+|z|)^{-\gamma}\int (1+2^{a(j-j')+\ell-\ell'}|y|)^{-N-2}|y|^N\, dy \\ &\le C' 2^{-(a(j-j')+\ell-\ell')(N+1)}(1+|z|)^{-\gamma}.\end{aligned}$$

Moreover, since E is bounded, the γ-localization of the brushlet system gives

$$\begin{aligned} I_2 &\le C\int_{|y-z|>|z|/2} \big(1+2^{a(j-j')+\ell-\ell'}\tfrac{\pi}{p_\ell}|y-z|\big)^{-N-2}|y-z|^N\, dy \\ &\le C' 2^{-(a(j-j')+\ell-\ell')(N+1)}(1+|z|)^{-\gamma}.\end{aligned}$$

The bounds on the two integrals I_1 and I_2 yields the following bound on the inner product,

$$\begin{aligned}&|\langle D_{\pi/(p_\ell q_{\ell,m,\varepsilon})} w_\Upsilon^\pm, \psi_{\Gamma'}\rangle| \\ &\qquad \le C 2^{-(a(j-j')+\ell-\ell')(N+1/2)}\big(1+\big|\kappa' \pm 2^{a(j'-j)+\ell'-\ell} p_\ell(2k+\delta+1/2)\big|\big)^{-\gamma} \\ &\qquad = C 2^{-(a(j-j')+\ell-\ell')(N+1/2)}\big(1+\big|\kappa' \pm 2^{a(j'-j)+\ell'-\ell}(|\kappa|+d)\big|\big)^{-\gamma},\end{aligned}$$

where $d := p_\ell(\delta+1/2)-(-1)^\delta(2m+\varepsilon+\delta)$. Now, using the fact that $(a+|x-d|)^{-1} \le a^{-1}(a+|d|)(a+|x|)^{-1}$ for $a>0$, and $x,d\in\mathbb{R}$, we obtain the first inequality in the lemma. Using similar estimates, we obtain the second inequality. □

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