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by

Horia Cornean and Kim Knudsen

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Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 96 35 80 80 ▪ Telefax: +45 98 15 81 29
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Reconstruction from one boundary measurement of a potential homogeneous of degree zero

Horia Cornean* and Kim Knudsen
Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7G
9220 Aalborg Ø
Denmark
E-mail: cornean@math.auc.dk, kim@math.auc.dk

Abstract

We consider the inverse boundary value problem concerning the determination and reconstruction of an unknown potential in a Schrödinger equation in a bounded domain from measurements on the boundary the domain. For the special case of a small potential homogeneous of degree zero we show that one boundary measurement determines the potential uniquely. Moreover, we give a reconstruction procedure.

1 Introduction

In this paper we consider the question of obtaining information about a potential in a Schrödinger equation in a bounded domain from the knowledge of pairs of corresponding Dirichlet and Neumann data on the boundary of the domain. Let $\Omega \in \mathbb{R}^3$ be a smooth, open and bounded domain and assume $q \in L^2(\Omega)$. Consider the boundary value problem

$$(-\Delta + q)u = 0 \ \text{ in } \Omega, \qquad u = f \ \text{ on } \partial\Omega. \tag{1}$$

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To avoid technical difficulties we assume that zero is not a Dirichlet eigenvalue of $(-\Delta + q)$. Then the problem (1) has a unique weak solution $u \in H^1(\Omega)$ for any $f \in H^{1/2}(\partial\Omega)$ (see section 3 for further details on the existence and regularity of solutions to (1)). Moreover, the unique solution admits a normal derivative at the boundary $g = \partial_\nu u|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ defined coherently by

$$\langle g, \phi \rangle = \int_\Omega (\nabla u \cdot \nabla v + quv) dx$$

for any $\phi \in H^{1/2}(\partial\Omega)$, where $v \in H^1(\Omega)$ has trace ϕ. Here ν is the outward unit normal to $\partial\Omega$ defined in a neighborhood of the boundary. The function g is the natural Neumann data for the equation (1).

In this paper we are interested in the inverse problem related to (1) concerning the determination and reconstruction of the unknown potential q from one pair of corresponding Dirichlet and Neumann data (f, g). This problem arises in a number of applications for instance in electric impedance tomography [1].

In general we cannot expect to recover an arbitrary potential everywhere in the domain (this requires infinitely many pairs of boundary data or the full Dirichlet-to-Neumann map, see [2, 3, 4, 1]), but if we a priori have some kind of information, which can be used to reduce the set of admissible potentials, then one measurement may suffice. This is the situation in the case, where the potential is known to be piecewise constant, say $q(x) = 1 + \chi_D(x)$ (χ_D is the characteristic function on the inclusion $D \subset \overline{D} \subset \Omega$). The problem is then to determine the set D. This problem was solved for convex D (under a few additional hypothesis) in [5] and for balls in [6]. A related inverse scattering problem was considered in [7]. There the potential is assumed to have the form $q(x) = 1 + \sum_{j=1}^N k_j \chi_{D_j}(x)$, where the location of the N disjoint scatterers D_j is known but the constants k_j are unknown. The result is that $\{k_j\}_{j=1}^N$ is determined uniquely by one scattering experiment.

A different point of view is presented by Cannon, Douglas and Jones in their classical paper [8]. They consider the related inverse conductivity problem for a class of conductivities homogeneous in one direction. More precisely let $\Omega = D \times [0, a] \subset \mathbb{R}^n$ for a bounded and smooth $D \subset \mathbb{R}^{n-1}$ and $a > 0$, and assume that the smooth conductivity γ is independent of the cylindrical variable, i.e. $\gamma(x', z) = \gamma(x', 0),\ x' \in D,\ 0 < z \leq a$. Then their result is that γ can be recovered from one particular boundary measurement. Note that in this inverse problem the boundary value of the coefficient determines the coefficient everywhere.

In some sense the result presented here can be seen as a generalization of the result of Cannon, Douglas and Jones. Indeed, we will simplify the problem by considering only the case when $\Omega = B_a = B(0,a) \subset \mathbb{R}^3$, the ball centered at zero with radius $a > 0$, and by restricting the interest to potentials homogeneous of degree zero. There are two main results: first we derive an equation, which explicitly links one particular boundary measurement to the potential; secondly, we show how the trace can be found by solving this equation in case the potential is small. Previous results using the same ideas can be found in [9] and [10].

The outline of the paper is the following: in the next section we state the exact results and describe the main ideas. Then in section 3 we consider the regularity properties of solutions to (1). These results are well-known but included here because of the lack of a proper reference. The equation relating the data and boundary value of the potential is derived in section 4, and finally in section 5 we see how this equation can be uniquely solved in a particular case.

2 Outline of the method and results

Our first result is an equation relating the boundary data and the potential. Let $-\Delta_D$ be the Friedrichs extension of $(-\Delta)|_{C_0^\infty(B_a)}$, which is selfadjoint on the domain $\mathcal{D}(-\Delta_D) = H_0^1(B_a) \cap H^2(B_a)$. For $q \in L^2(B_a)$ we can define $(-\Delta_D + q)$ as a selfadjoint operator on the same domain (see Lemma 3.1). The spectrum of this operator consists of a countable number of real eigenvalues, and if zero is not an eigenvalue, then $R_q = (-\Delta_D + q)^{-1}$ exists and is a smoothing operator of degree two, see section 3 below. Let now for $q \in L^2(B_a)$, M_q be the multiplication operator $M_q\colon \phi \mapsto q\phi$ defined in $L^2(B_a)$ on the domain $\mathcal{D}(M_q) = \{\phi \in L^2(B_a) \mid q\phi \in L^2(B_a)\}$. Define further the first order differential operator

$$A = \frac{x}{a} \cdot \nabla,$$

and note that if ∂_r is the differential operator with respect to the radial variable $r = |x|$, then $aA = r\partial_r$. Finally, let $\rho_0\colon H^s(\Omega) \to H^{s-1/2}(\partial\Omega),\ s > 1/2$ be the usual trace operator and define the boundary fields

$$\begin{aligned} g_f &= \rho_0 \partial_r u \\ h_f &= \rho_0 \partial_r v, \end{aligned}$$

where u solves (1) and v solves

$$-\Delta v = 0 \text{ in } B_a \quad v = g_f \text{ on } \partial B_a.$$

In particular we use the notation g_1, h_1 for the boundary fields arising from the special choice $f = 1$. Note that h_f can be computed from g_f by solving a boundary value problem, which does not depend on q.

We will consider potentials, which are homogeneous of degree zero, i.e. potentials q with the property that $q(t\hat{x}) = q(\hat{x})$ for $\hat{x} = ax/|x| \in \partial B_a$ and $t \in (0, 1)$. Note that any boundary potential $p \in L^2(\partial\Omega)$ has an extension $1 \otimes p \in L^2(B_a)$ defined by

$$1 \otimes p(x) = p(\hat{x}), \qquad x \in B_a,$$

to a potential homogeneous of degree zero. In the sequel we shall not distinguish between a potential homogeneous of degree zero and its boundary value.

Define the non-linear operator

$$F : p \mapsto \rho_0 \partial_r R_0 M_p A R_p(p). \tag{2}$$

Since $\rho_0 \partial_r R_0 \in \mathcal{B}(H^{-1/2}(B_a), L^2(\partial\Omega))$, the operator F is well-defined in $L^2(\partial B_a)$ for any function p with the property that zero is not an eigenvalue of $(-\Delta_D + p)$. The first result relating the boundary data g_1, h_1 to the potential q is then

Proposition 2.1. *Let $q \in L^2(B_a)$ be homogeneous of degree zero and assume that zero is not an eigenvalue of $(-\Delta_D + q)$. Then in $L^2(\partial B_a)$ we have*

$$q = h_1 + \frac{3}{a} g_1 - F(q) \tag{3}$$

The equation (3) links the potential to the special boundary field $q_0 = h_1 + \frac{3}{a} g_1$. This equation is the starting point for the inverse problem.

The forward problem concerns the computation of q_0 from q. This is done by

$$(I + F)(q) = q_0.$$

The inverse problem then concerns the inversion of $I + F$. We will show that $I + F$ is injective in a neighbourhood of $q = 0$ with inverse given through an iterative scheme.

Introduce the set $W_\lambda \subset L^2(\partial B_a)$ consisting of functions f, which satisfy $\|f\|_{L^2(\partial B_a)} < \lambda$. Assume that q_0 is given. Define the non-linear operator $T \colon W_{\lambda_1} \to L^2(\partial B_a)$ by

$$T(p) = q_0 - F(p). \tag{4}$$

In Lemma 3.4 below we show that when λ_1 is sufficiently small and $p \in W_{\lambda_1}$, then zero is not an eigenvalue of $(-\Delta_D+q)$, and therefore $F(p)$ is well-defined and (3) holds. We then interpret (3) as a fixed point equation for T. Now, to solve the inverse problem we will prove that if $\lambda_0 \leq \lambda_1$ is sufficiently small, then T has a unique fixed point in the set W_{λ_0}, and moreover, that this fixed point can be found by iteration. The following theorem states the result:

Theorem 2.2. *Assume $q(x)$ is a potential homogeneous of degree zero. Then there is a constant λ_0 depending only on the radius a such that if $q \in W_{\lambda_0}$ and $q_0 = h_1 + \frac{3}{a} g_1$, then q is the unique fixed point of T. Moreover, q can be found as the $L^2(\partial B_a)$ limit of the convergent sequence $\{T^n(0)\}_{n\in\mathbb{N}} \subset L^2(\partial B_a)$, i.e.*

$$q = \lim_{n\to\infty} T^n(0).$$

3 Regularity of solutions

In this section we collect some results concerning perturbations of $-\Delta_D$ and inversion of such perturbation operators. Then we use these results to prove regularity properties of solutions to boundary value problems. Note that the results stated here are valid for any smooth, open and bounded domain $\Omega \subset \mathbb{R}^3$.

We start out by defining $-\Delta_D + q$ rigorously as a selfadjoint operator:

Lemma 3.1. *For $q \in L^2(\Omega)$ the operator $(-\Delta_D + q)$ in $L^2(\Omega)$ is selfadjoint on the domain $\mathcal{D}(-\Delta_D + q) = \mathcal{D}(-\Delta_D) = H_0^1(\Omega) \cap H^2(\Omega)$.*

Furthermore, $(-\Delta_D + q)$ has discrete spectrum, and if zero is not an eigenvalue of $(-\Delta_D + q)$, then $R_q = (-\Delta_D + q)^{-1}$ is in $\mathcal{B}(L^2(\Omega), H^2(\Omega))$ and in $\mathcal{B}(H^{-1}(\Omega), H^1(\Omega))$.

Proof. We will prove that the multiplication operator $M_q\colon \phi \mapsto q\phi$ is operator bounded by $(-\Delta_D)$ with bound less than 1, and then apply the Kato-Rellich theorem (see [11, Theorem 4.3]).

Since for $f \in L^\infty(\Omega)$

$$\|qf\|_{L^2(\Omega)} \leq \|q\|_{L^2(\Omega)} \|f\|_{L^\infty(\Omega)}$$

it follows for $\phi \in \mathcal{D}(-\Delta_D)$ and $\mu > 0$ that with $R_\mu = (-\Delta_D + \mu)^{-1}$

$$\begin{aligned} &\|q\phi\|_{L^2(\Omega)} \\ &\quad \leq \|qR_\mu(-\Delta_D + \mu)\phi\|_{L^2(\Omega)} \\ &\quad \leq \|q\|_{L^2(\Omega)} \|R_\mu\|_{\mathcal{B}(L^2(\Omega),L^\infty(\Omega))} \|(-\Delta_D + \mu)\phi\|_{L^2(\Omega)} \\ &\quad \leq \|q\|_{L^2(\Omega)} \|R_\mu\|_{\mathcal{B}(L^2(\Omega),L^\infty(\Omega))} (\| - \Delta_D\phi\|_{L^2(\Omega)} + \mu\|\phi\|_{L^2(\Omega)}). \end{aligned} \tag{5}$$

Let $G_\mu(x,y)$ denote the Dirichlet Green's function for $(-\Delta_D+\mu)$ defined for fixed $x\in\Omega$ as the unique solution to

$$(-\Delta_D+\mu)G_\mu(x,y)=\delta(x-y) \text{ in } \Omega, \qquad G_\mu(x,y)=0 \text{ on } \partial\Omega.$$

Let further $\Phi_\mu(x,y)=e^{-\sqrt{\mu}|x-y|}(4\pi|x-y|)^{-1}$ be the standard fundamental solution. Since $\Phi_\mu(x,y)-G_\mu(x,y)$ is harmonic on Ω and hence bounded, $G_\mu(x,y)$ is positive near the singularity x. By the maximum principle applied to $G_\mu(x,y)$ in $\Omega\setminus B(x,\epsilon),\ x\in\Omega,\epsilon>0$ then shows, that $G_\mu(x,y)$ is non-negative everywhere. Another application of the maximum principle to $\Phi_\mu(x,y)-G_\mu(x,y)$ gives for $x,y\in\Omega,\ x\neq y$ the estimate

$$G_\mu(x,y)<\frac{e^{-\sqrt{\mu}|x-y|}}{4\pi|x-y|}.$$

This shows that

$$\begin{aligned}\|R_\mu\|_{\mathcal{B}(L^2(\Omega),L^\infty(\Omega))} &= \sup_{f\in L^2(\Omega),\ \|f\|_{L^2(\Omega)}=1}\left\|\int_\Omega G_\mu(x,y)f(y)dy\right\|_{L^\infty(\Omega)}\\ &\le \sup_{x\in\Omega}\left(\int_\Omega |G_\mu(x,y)|^2dy\right)^{1/2}\\ &\le \sup_{x\in\Omega}\frac{1}{4\pi}\left(\int_\Omega \frac{e^{-2\sqrt{\mu}|x-y|}}{|x-y|^2}dy\right)^{1/2}\\ &\le \left(\int_0^{2a} e^{-2\sqrt{\mu}r}dr\right)^{1/2}\\ &\le C\mu^{-1/4},\end{aligned}$$

and by (5) we find that $\phi\mapsto q\phi$ is $(-\Delta_D)$-bounded, i.e. for $\phi\in\mathcal{D}(-\Delta_D)$

$$\|q\phi\|_{L^2(\Omega)}\le b\|(-\Delta_D)\phi\|_{L^2(\Omega)}+c\|\phi\|_{L^2(\Omega)}, \tag{6}$$

where $b=C\mu^{-1/4}\|q\|_{L^2(\Omega)}, c=C\mu^{3/4}\|q\|_{L^2(\Omega)}$. The choice $\mu>C\|q\|^4_{L^2(\Omega)}$ implies that $b<1$, and then it follows from the Kato-Rellich theorem that $-\Delta_D+q$ is well-defined and selfadjoint on $\mathcal{D}(-\Delta_D)$.

It is well-known that $(-\Delta_D+q)$ has purely discrete spectrum. When zero is not an eigenvalue, the inverse R_q exists as a bounded operator on $L^2(\Omega)$. The smoothing property of R_q is an easy consequence of the mapping properties of $(-\Delta_D)^{-1}$. □

To solve the problem (1) we apply the standard procedure of transforming the problem into a problem with a source term and zero boundary condition. This reduction relies on well-known properties of solutions to the Laplace equation:

Proposition 3.2. *For any $f \in H^{1/2+s}(\partial\Omega)$, $s \geq 0$, there is a unique solution $v \in H^{1+s}(\Omega)$ to*

$$-\Delta v = 0 \quad \text{in } \Omega, \qquad v = f \quad \text{on } \partial\Omega. \tag{7}$$

Proof. See for instance [12] for a proof. □

We can now prove the result

Proposition 3.3. *Let $q \in L^2(B_a)$ and assume zero is not an eigenvalue of $(-\Delta_D + q)$. Then for $s = 0, 1$ there is for any $f \in H^{1/2+s}(\partial\Omega)$ a unique solution $u \in H^{1+s}(\Omega)$ to (1).*

Proof. Let $v \in H^{1+s}(\Omega)$ solve (7) and introduce $w = u - v$. Then w solves $(-\Delta + q)w = -(qv)$ in Ω and vanishes on $\partial\Omega$, and formally we define $w = u - v = -R_q(qv)$. For $f \in H^{1/2+s}(\partial\Omega)$ Proposition 3.2 gives that $v \in H^{1+s}(\Omega)$. Hence the result follows from the mapping properties of R_q given in Lemma 3.1 provided that $qv \in H^{-1+s}(\Omega)$. For $s = 1$ the Sobolev embedding $H^2(\Omega) \subset L^\infty(\Omega)$ gives that $qv \in L^2(\Omega)$. For $s = 0$ the Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$ and Hölder's inequality implies that $qv \in L^{3/2}(\Omega)$. By the Sobolev embedding $qv \in H^{-1/2}(\Omega) \subset H^{-1}(\Omega)$ we get the result.

Uniqueness follows from the injectivity of R_q. □

It is well-known that the spectrum of $(-\Delta_D)$ is positive. When q is small this property is inherited by $(-\Delta_D + q)$. In particular zero cannot be an eigenvalue.

Lemma 3.4. *Assume $\|q\|_{L^2(\Omega)} \|(-\Delta_D)^{-1}\|_{\mathcal{B}(L^2(\Omega), H^2(\Omega))} < 1$. Then zero is not an eigenvalue of $(-\Delta_D + q)$.*

Proof. To prove the result, we write in $\mathcal{D}(-\Delta_D)$

$$(-\Delta_D + q) = (1 + q(-\Delta_D)^{-1})(-\Delta_D), \tag{8}$$

and hence formally

$$R_q = (-\Delta_D + q)^{-1} = (-\Delta_D)^{-1}(1 + q(-\Delta_D)^{-1})^{-1}.$$

The inverse of $1 + q(-\Delta_D)$ is easily seen to be given by a convergent Neumann series when q is small. □

Note that from the convergent Neumann series we can get the estimate

$$\|R_q\|_{\mathcal{B}(L^2(\Omega), H^2(\Omega))} \leq (1 - \|q\|_{L^2(\Omega)} \|(-\Delta_D)^{-1}\|_{\mathcal{B}(L^2(\Omega), H^2(\Omega))})^{-1}, \tag{9}$$

for $\|q\|_{L^2(\Omega)} \|(-\Delta_D)^{-1}\|_{\mathcal{B}(L^2(\Omega), H^2(\Omega))} < 1$.

4 Derivation of the boundary integral equation

In this section we derive the equation (3). The idea is to establish a relation between h_f and the second order derivative $\rho_0\partial_r^2 u$ of the solution to (1), and then use the partial differential equation for u to express $\rho_0\partial_r^2 u$ as a sum of lower order terms in the special case $f = 1$.

The details will be given in two lemmas:

Lemma 4.1. *Let $q \in L^2(B_a)$ be homogeneous of degree zero and assume that zero is not an eigenvalue of $(-\Delta_D + q)$. Then for any $f \in H^{3/2}(\partial B_a)$, the function $h_f - \rho_0\partial_r^2 u \in L^2(\partial B_a)$, and we have in $L^2(\partial B_a)$ the relation*

$$\rho_0\partial_r^2 u - h_f = \frac{1}{a}g_f - \frac{2}{a}\rho_0\partial_r v + \rho_0\partial_r R_0 M_q A R_q M_q v - \rho_0\partial_r R_0 M_q A v \tag{10}$$

where u, v are solutions to (1) *and* (7) *respectively.*

Proof. Since $\rho_0 A = \rho_0\partial_r$ on ∂B_a we have $\rho_0 A u = g_f \in H^{1/2}(\partial B_a)$, where $u \in H^2(B_a)$ is the solution to (1) according to Proposition 3.3. Let $\tilde{u} \in H^1(B_a)$ be the unique solution to $-\Delta\tilde{u} = 0$ in B_a with $\rho_0\tilde{u} = g_f$. Since an easy calculation shows that in the sense of distributions

$$[-\Delta, A] = -\frac{2}{a}\Delta,$$

the function $Au - \tilde{u} \in H_0^1(B_a)$ solves

$$-\Delta_D(Au - \tilde{u}) = -\Delta(Au - \tilde{u}) = -A\Delta u - \frac{2}{a}\Delta u = -A(qu) - \frac{2}{a}\Delta u. \tag{11}$$

This implies

$$-\Delta_D(Au - \tilde{u} - \frac{2}{a}(u - v)) = -A(qu).$$

Note that since q is homogeneous of degree zero, $A(qu) = M_q(Au) \in H^{-1/2}(B_a)$ in the sense of distributions and therefore

$$Au - \tilde{u} - \frac{2}{a}(u - v) = -R_0 M_q A u \in H^{3/2}(B_a).$$

The operator $\rho_0\partial_r$ takes $R_0 M_q A u$ into $L^2(\partial B_a)$ and by using the fact that

$$\rho_0\partial_r A u = \rho_0\partial_r \frac{r}{a}\partial_r u = \rho_0\partial_r^2 u + \frac{1}{a}g_f,$$

$$\rho_0\partial_r\tilde{u} = h_f,$$

$$\rho_0\partial_r u = g_f.$$

we find that

$$\rho_0\partial_r^2 u - \frac{1}{a}g_f - h_f + \frac{2}{a}\rho_0\partial_r v = -\rho_0\partial_r R_0 M_q A u.$$

From this it follows that

$$\rho_0\partial_r^2 u - h_f = \frac{1}{a}g_f - \frac{2}{a}\rho_0\partial_r v - \rho_0\partial_r R_0 M_q A u \in L^2(B_a).$$

To get (10) we then use the formula

$$u = -R_q(qv) + v.$$

□

For the special choice of boundary field $f = 1$ we can remove the $\rho_0\partial_r^2 u$ term from (10).

Lemma 4.2. *Let $q \in L^2(B_a)$ be homogeneous of degree zero and assume that zero is not an eigenvalue of $(-\Delta_D + q)$. Let $u_1 \in H^2(B_a)$ be the solution to* (1) *with $f = 1$. Then*

$$\rho_0\partial_r^2 u_1 = q - \frac{2}{a}g_1. \tag{12}$$

Note that $\rho_0\partial_r^2 u_1$ a priori is no better than $H^{-1/2}(\partial B_a)$, but the lemma implies that $\rho_0\partial_r^2 u_1 \in L^2(\partial B_a)$.

Proof. Write the Laplace operator in spherical coordinates

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}L_{\theta,\phi}$$

where

$$L_{\theta,\phi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{(\partial\phi)^2}.$$

Hence

$$\begin{aligned}0 &= \rho_0(-\Delta + q)u_1 \\ &= \rho_0(-\partial_r^2 - \frac{2}{r}\partial_r - \frac{1}{r^2}L_{\theta,\phi} + q)u_1 \\ &= -\rho_0\partial_r^2 u_1 - \frac{2}{a}g_1 + q,\end{aligned}$$

since $\rho_0 L_{\theta,\phi}u_1 = L_{\theta,\phi}\rho_0 u_1 = L_{\theta,\phi}1 = 0$. □

Proof of Proposition 2.1. The result follows from (10) and (12) since $v = 1$ is the unique harmonic function with $\rho_0 v = 1$. Therefore $\rho_0\partial_r v = 0$ and $Av = 0$. □

5 Solving the boundary integral equation

In this section the proof of Theorem 2.2 will be given. The first step is the following lemma concerning the contraction properties of the operator F defined in (2):

Lemma 5.1. *There is a* $\lambda_2 > 0$ *depending only on the radius* a *such that if* $p_1, p_2 \in W_{\lambda_2}$, *then*

$$\|F(p_1) - F(p_2)\|_{L^2(\partial B_a)} \leq \frac{1}{2}\|p_1 - p_2\|_{L^2(\partial B_a)}.$$

Proof. From Lemma 3.4 we can find λ_1 such that if $q \in W_{\lambda_1}$, then $R_q = (-\Delta_D + q)^{-1} \in \mathcal{B}(L^2(B_a), H^2(B_a))$. Then for $\lambda \leq \lambda_1$ and $p_1, p_2 \in W_\lambda$ the operator F is well-defined, and we have

$$\begin{aligned} F(p_1) - F(p_2) &= \rho_0 \partial_r R_0 (M_{p_1} A R_{p_1}(p_1) - M_{p_2} A R_{p_2}(p_2)) \\ &= \rho_0 \partial_r R_0 \big(M_{p_1} A R_{p_1}(p_1 - p_2) \\ &\quad + M_{p_1} A (R_{p_1} - R_{p_2})(p_2) && (13) \\ &\quad + (M_{(p_1 - p_2)}) A R_{p_2}(p_2) \big). && (14) \end{aligned}$$

To estimate the norm of $F(p_1) - F(p_2)$ we note that

$$\|\rho_0 \partial_r R_0\|_{H^{-1/2}(B_a), L^2(B_a)} \leq C a^2,$$

for some constant C independent of a. Furthermore, we use the resolvent identity $R_{q_1} - R_{q_2} = -R_{q_1} M_{q_1 - q_2} R_{q_2}$ to estimate

$$\begin{aligned} &\|R_{p_1} - R_{p_2}\|_{\mathcal{B}(L^2(B_a), H^2(B_a))} \\ &\qquad \leq \|R_{p_1}\|_{\mathcal{B}(L^2(B_a), H^2(B_a))} \|R_{p_2}\|_{\mathcal{B}(L^2(B_a), H^2(B_a))} \|p_1 - p_2\|_{L^2(B_a)}. \quad (15) \end{aligned}$$

Finally, we note that for a potential homogeneous of degree zero

$$\|p\|_{L^2(B_a)} = a^{3/2} \|p\|_{L^2(\partial B_a)}. \quad (16)$$

Based on (14), (15) and (16) we can estimate

$$\begin{aligned} &\|F(p_1) - F(p_2)\|_{L^2(\partial B_a)} \leq C a^2 \|p_1 - p_2\|_{L^2(B_a)} \\ &\quad (\|p_1\|_{L^2(B_a)} \|R_{p_1}\|_{\mathcal{B}(L^2(B_a), H^2(B_a))} \\ &\quad + \|p_2\|_{L^2(B_a)} \|R_{p_2}\|_{\mathcal{B}(L^2(B_a), H^2(B_a))} \\ &\quad + \|p_1\|_{L^2(B_a)} \|R_{p_1}\|_{\mathcal{B}(L^2(B_a), H^2(B_a))} \|p_2\|_{L^2(B_a)} \|R_{p_2}\|_{\mathcal{B}(L^2(B_a), H^2(B_a))}) \\ &\quad \leq C a^5 (C(\lambda) + a^{3/2} C(\lambda)^2) \|p_1 - p_2\|_{L^2(\partial B_a)}, \end{aligned}$$

where

$$C(\lambda) = \max_{p \in W_\lambda} \|p\|_{L^2(\partial B_A)} \max_{p \in W_\lambda} \|R_p\|_{\mathcal{B}(L^2(B_a), H^2(B_a))}.$$

The result then follows by using the uniform resolvent estimate (9) and choosing λ_2 such that

$$(C(\lambda) + a^{3/2} C(\lambda)^2) \leq \frac{1}{2Ca^5}.$$

□

Proof of Theorem 2.2. Let λ_2 be given from Lemma 5.1. For the uniqueness, assume $q_1, q_2 \in W_{\lambda_2}$ are fixed points for T. Then from Lemma 5.1 we have

$$\begin{aligned}
\|q_1 - q_2\|_{L^2(\partial B_a)} &= \|T(q_1) - T(q_2)\|_{L^2(\partial B_a)} \\
&= \|F(q_1) - F(q_2)\|_{L^2(\partial B_a)} \\
&\leq \frac{1}{2} \|q_1 - q_2\|_{L^2(\partial B_a)}
\end{aligned}$$

which implies $\|q_1 - q_2\|_{L^2(\partial B_a)} = 0$.

Concerning the convergence of $\{T^n(0)\}_{n \in \mathbb{N}}$ we note that if $q_0 = 0$, then the existence of a fixed point follows by Banach's fixed point theorem [13, section 9.2.1], since F (and hence T) is a contraction in W_{λ_2} by Lemma 5.1. In case $q_0 \neq 0$ we cannot immediately use a general fixed point theorem, so we will have to rely on the smallness assumption. For $q \in W_{\lambda_1}$ it follows from (3) that

$$\|h_1 + \frac{3}{a} g_1\|_{L^2(\partial B_a)} \leq C_a (1 + \|R_q\|_{\mathcal{B}(L^2(B_a), H^2(B_a))} \|q\|_{L^2(\partial B_a)}) \|q\|_{L^2(\partial B_a)},$$

where C_a is a constant depending only on a. Hence there is a constant $\lambda_0 \leq \lambda_2 \leq \lambda_1$ such that if $q \in W_{\lambda_0}$, then $q_0 = T(0) = h_1 + \frac{3}{a} g_1 \in W_{\lambda_2/2}$. By induction one can now show that if

$$\|T^{m-1}(0)\|_{L^2(\partial B_a)} \leq \sum_{k=1}^{m-1} \left(\frac{1}{2}\right)^k \lambda_2,$$

which in particular implies that $T^{m-1}(0) \in W_{\lambda_2}$, then by Lemma 5.1

$$\begin{aligned}
\|T^m(0)\|_{L^2(\partial B_a)} &\leq \|F(T^{m-1}(0)) - F(T^{m-2}(0))\|_{L^2(\partial B_a)} + \|T^{m-1}(0)\|_{L^2(\partial B_a)} \\
&= \sum_{k=1}^{m} \left(\frac{1}{2}\right)^m \lambda_2.
\end{aligned}$$

By the same type of argument we find that

$$\begin{aligned}\|T^{m+1}(0) - T^m(0)\|_{L^2(\partial B_a)} &\le \frac{1}{2}\|T^m(0) - T^{m-1}(0)\|_{L^2(\partial B_a)} \\ &\le \frac{1}{2^{m+1}}\|T(0)\|_{L^2(\partial B_a)}\end{aligned}$$

and hence

$$\begin{aligned}\|T^{m+k}(0) - T^m(0)\|_{L^2(\partial B_a)} &\le \sum_{j=1}^{k}\|T^{m+j}(0) - T^{m+(j-1)}(0)\|_{L^2(\partial B_a)} \\ &\le \sum_{j=1}^{k}\frac{1}{2^{m+j+1}}\|T(0)\|_{L^2(\partial B_a)}.\end{aligned}$$

From this expression it follows that $\{T^n(0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\partial B_a)$, and hence the sequence converges to a $p \in W_{\lambda_2}$. Since

$$\begin{aligned}\|p - T(p)\|_{L^2(\partial B_a)} &\le \lim_{n\to\infty}(\|p - T^n(0)\|_{L^2(\partial B_a)} + \|T^n(0) - T(p)\|_{L^2(\partial B_a)}) \\ &\le \lim_{n\to\infty}(\|p - T^n(0)\|_{L^2(\partial B_a)} + \frac{1}{2}\|T^{n-1}(0) - p\|_{L^2(\partial B_a)}) \\ &= 0,\end{aligned}$$

we see that p is indeed a fixed point for T. □

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