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PARAMETRICES AND EXACT PARALINEARISATION OF SEMI-LINEAR BOUNDARY PROBLEMS

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ABSTRACT. The subject is to establish solution formulae for elliptic (and parabolic) semi-linear boundary problems. The results should be new in at least two respects: the desired formulae result from a parametrix construction for semi-linear problems, using only parametrices from the linear theory and the mild assumption that the non-linearity may be decomposed into a suitable solution-dependent linear operator acting on the solution itself. Secondly non-linearities of so-called product type are shown to admit such decompositions via exact paralinearisation. The parametrices give regularity properties under rather weak conditions, with examples of properties that are unobtainable by boot-strap methods. Regularity improvements in submanifolds are deduced from the auxiliary result that operators of type $1,1$ are pseudo-local on large parts of their domains. The framework is flexible, encompassing a broad class of boundary problems and Hölder and Sobolev spaces, or the more general Besov and Triebel–Lizorkin spaces. The examples include the von Karman equation.

1. INTRODUCTION

This article presents a parametrix construction for semi-linear boundary problems along with the resulting regularity properties in L_p-Sobolev spaces. The construction uses systematic investigations of pseudo-differential boundary operators, paramultiplication and function spaces of J.-M. Bony, G. Grubb, V. Rychkov and the author [Bon81, Gru95b, Ryc99b, Joh95a, Joh96], but is also inspired by a joint work with T. Runst [JR97] on solvability of semi-linear problems.

The motivation was first of all to avoid some rather annoying technicalities met earlier in boot-strap arguments, when these were applied under weak assumptions to semi-linear boundary problems; cf [Joh95b]. Secondly it was hoped to find purely analytical proofs, without reiteration, of the regularity properties.

These goals are achieved by means of the parametrix formula presented here. The formula also gives structural information about the solution, and along with stronger a priori regularity of the solution it allows increasingly weaker assumptions on the data. (Boot-strap methods can do neither.)

As a further feature, the parametrix formulae give regularity properties *beyond* those obtainable by boot-strap methods. Indeed, as a gratis consequence of the method, solutions may (depending on the problem and its data) be proved to lie in spaces, on which the non-linear terms lose more derivatives than the linear terms.

This possibility should also be a novelty. It is exemplified in Theorem 7.1 below on quadratic perturbations of polyharmonic operators:

$$(-\Delta)^m u(x) + u(x)^2 = c(x_1^2 + x_2^2)^{-3/4} \quad \text{in} \quad \Omega \subset \mathbb{R}^n,\ n \geq 2. \tag{1.1}$$

Key words and phrases. Exact paralinearisation, moderate linearisation, parameter domain, inverse regularity properties, parametrix, pseudo-differential operators, type $1,1$.

As shown below, there are for sufficiently small $c > 0$ solutions $u \in H_0^m(\overline{\Omega})$, and since $H^m \subset W_1^m$ for Ω bounded, $D^\alpha u$ is a priori known to be an integrable function for $|\alpha| \le m$. But the parametrices yield infinitely many (n,m) for which this holds for $|\alpha| \le 2m$, ie $u \in W_1^{2m}$ then. One example is $n = 5$ and $m = 1$, so already the classical Dirichlét realisation of the Laplacian enters examples of such 'extra' regularity. In comparison boot-strap methods like the extended ones in [Joh95b] do not suffice for this W_1^{2m}-property, since (n,m) are chosen such that $u \mapsto u^2$ is undefined on W_1^{2m} in the distribution sense. Cf Theorem 7.1 ff.

Compared to the paradifferential theory of J.–M. Bony [Bon81], it is on one hand true that the set-up is restricted here to non-linearities of product type, as defined below, but on the other hand it is a main point of the present work that the regularity of non-zero boundary data is taken fully into account (while this was undiscussed in [Bon81]). Special regularity properties in subregions are also carried over to the solutions.

A brief overview of the parametrix results has been given in [Joh03]. The present paper gives the theory in full generality together with the underlying details.

The introduction proceeds to present the results and techniques; notation is settled towards the end. Then the main result follows in a general framework in Section 2. Preliminaries on paramultiplication are given in Section 3. In Section 4 it is verified that non-linearities of product type have the necessary properties. Section 5 presents the consequences for the stationary von Karman problem, and the weak solutions are carried over to general L_p Sobolev spaces; direct proofs are not given in Section 5 because the results follow from those of Section 6. The subject of Section 6 is the parametrix and regularity results obtained for systems of semilinear elliptic boundary problems in vector bundles; this somewhat heavy set-up should be well motivated by the von Karman problem treated in Section 5. The analysis of (1.1) follows in Section 7.

1.1. **The model problem.** The subject is exemplified in the following by means of the below Dirichlét problem on an open set $\Omega \subset \mathbb{R}^n$, which is *bounded* (an essential assumption, made throughout) with C^∞-boundary $\Gamma := \partial\Omega$,

$$\begin{aligned} -\Delta u + u \cdot \partial_{x_1} u &= f \quad \text{in} \quad \Omega, \\ \gamma_0 u &= \varphi \quad \text{on} \quad \Gamma. \end{aligned} \tag{1.2}$$

($\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ is the Laplacian; $\gamma_0 u = u|_\Gamma$ is the trace.) This model problem has been chosen instead of the stationary Navier–Stokes equation, with which it has much in common except that it is not a system, hence is simpler to present.

It is a main point to establish parametrices $P_u^{(N)}$, which for $N \in \mathbb{N}$ are certain linear operators yielding the following new formula for u:

$$u = P_u^{(N)}(R_D f + K_D \varphi) + (R_D L_u)^N u. \tag{1.3}$$

Here $\begin{pmatrix} R_D & K_D \end{pmatrix}$ is the inverse of $\begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix}$ (the subscript D refers to the Dirichlét problem for $-\Delta$), and L_u is an exact paralinearisation of $u \mapsto u\partial_1 u$. In (1.3), $P_u^{(N)}$ can roughly be seen as a modifier of data's contribution to u, while $(R_D L_u)^N$ is an 'error term' analogous to the negligible errors in pseudo-differential calculi. As explained later, u's regularity can be read off directly on the right hand side.

The formula (1.3) should be new even when data and solutions are described in the usual H^s-Sobolev spaces. But the usefulness of the parametrices gets an extra dimension when the L_p-theory is discussed, so it will be natural to consider the Sobolev spaces $H^s_p(\overline{\Omega})$ and the Hölder–Zygmund classes $C^s_*(\overline{\Omega})$.

However, these are contained in the scales of Besov spaces $B^s_{p,q}(\overline{\Omega})$ and Triebel–Lizorkin spaces $F^s_{p,q}(\overline{\Omega})$, since

$$H^s_p = F^s_{p,2} \quad \text{for } 1 < p < \infty \text{ and } s \in \mathbb{R}, \tag{1.4}$$

$$C^s_* = B^s_{\infty,\infty} \quad \text{for } s \in \mathbb{R}. \tag{1.5}$$

(For the well-known W^s_p spaces, $W^s_p = B^s_{p,p}$ for non-integer $s > 0$ and $W^m_p = F^m_{p,2}$ for $m \in \mathbb{N}$). To avoid formulations with many scales, the exposition will be based on the $B^s_{p,q}$ and $F^s_{p,q}$ spaces, and $E^s_{p,q}$ will denote a space which can be either $B^s_{p,q}$ or $F^s_{p,q}$ (in every occurrence within, say the same formula or theorem).

Moreover, $B^s_{p,q}(\overline{\Omega})$ and $F^s_{p,q}(\overline{\Omega})$ are defined for $p,\ q \in]0,\infty]$ ($p < \infty$ for $F^s_{p,q}$) and $s \in \mathbb{R}$, where the incorporation of $p,\ q < 1$ in general is convenient for non-linear problems (the H^s- and H^s_p-scales would be too tight frameworks). The price one pays for this roughly equals the burdening of the exposition that would result from a limitation to $p,\ q \geq 1$.

Furthermore it is noted that $F^m_{p,1}$, $1 < p < \infty$ recently [Joh04b, Joh04a] was shown to play a fundamental role for pseudo-differential operators of type $S^m_{1,1}$, that show up in the linearisations. Cf Section 4.5 below.

If desired, the reader can of course specialise to, say H^s_p, as described above. The main part of the paper deals with the parametrix construction and its consequences, and it does not rely on a specific choice of L_p Sobolev spaces.

For simplicity, (1.2) will in the introduction be discussed in the Besov scale $B^s_{p,q}$. As a basic requirement the spaces should fulfil the two inequalities (where $t_+ := \max(0,t)$ is the positive part of t)

$$s > \tfrac{1}{p} + (n-1)(\tfrac{1}{p} - 1)_+ \tag{1.6a}$$

$$s > \tfrac{1}{2} + n(\tfrac{1}{p} - \tfrac{1}{2})_+. \tag{1.6b}$$

It is known how they allow one to make sense of the trace and the product, respectively. Working under such conditions, a main question for (1.2) is the following *inverse*[1] *regularity* problem:

$$\begin{aligned}&\text{given a solution } u \text{ in one Besov space } B^s_{p,q}(\overline{\Omega})\\ &\text{for data } f \text{ in } B^{t-2}_{r,o}(\overline{\Omega}) \text{ and } \varphi \text{ in } B^{t-\frac{1}{r}}_{r,o}(\Gamma),\\ &\text{will } u \text{ be in } B^t_{r,o}(\overline{\Omega}) \text{ too?}\end{aligned} \tag{IR}$$

Consider eg a solution u in $H^1(\overline{\Omega})$ for data f in $C^\alpha(\overline{\Omega})$ and φ in $C^{2+\alpha}(\Gamma)$ with $\alpha \in]0,1[$. (For $\varphi = 0$ and 'small' $f \in H^{-1}$ solutions exist in H^1_0 for $n = 3$ by the below Proposition 2.4.) The question is then whether u also belongs to $C^{2+\alpha}(\overline{\Omega})$. Here the latter space equals $B^{2+\alpha}_{\infty,\infty}(\overline{\Omega})$ while $H^1 = B^1_{2,2}$, so the above problem (IR) clearly contains a classical issue; actually (IR) is somewhat sharper because of the third parameter.

It is hardly surprising that the answer to (IR) will be affirmative under suitably strong conditions on the parameters (s,p,q) and (t,r,o). But the purpose is to go much further by testing how weak conditions one can impose along with (1.6).

[1] In comparison *direct* regularity properties are used for the collection of mapping properties, for example of $u \mapsto u\partial_1 u$ or of $-\Delta u + u\partial_1 u$.

More importantly, it is described how the *parametrix* formula in (1.3) (cf also (1.18) and Theorems 2.2 and 6.7 below) yields the expected conclusion for (IR) — as well as for a general class of semi-linear problems. The result is a flexible framework implying that $u \in B^t_{r,o}$, also in certain cases when $u \mapsto u\partial_1 u$ has higher order than $-\Delta$ on $B^t_{r,o}$, or when $u\partial_1 u$ is undefined on $B^t_{r,o}$.

Moreover, if in a subregion $\Xi \subset \overline{\Omega}$ the data has additional properties such as $f \in B^{t_1-2}_{r_1,o_1}(\Xi,\mathrm{loc})$ and $\varphi \in B^{t_1-1/r_1}_{r_1,o_1}(\partial\Xi,\mathrm{loc})$, then u is locally in $B^{t_1}_{r_1,o_1}(\Xi)$ too.

Briefly stated, the above programme uses paramultiplication on $\mathbb{R}^n$ in a linearisation of $u\partial_1 u$ together with the parametrix $\begin{pmatrix} R_D & K_D \end{pmatrix}$ of $\begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix}$, belonging to the Boutet de Monvel calculus of pseudo-differential boundary operators — when combined with a Neumann series these ingredients yield $P^{(N)}_u$, and thus the approximative inversion in formula (1.3). This resembles the usual elliptic theory at the place where non-principal terms are included, but one difference is that here a finite series suffices, as in [Bon81], since the error $(R_D L_u)^N u$ in (1.3) only needs to belong to $B^t_{r,o}$. The local improvements in $\Xi \subset \overline{\Omega}$ are deduced from a quite general result about pseudo-locality of type $1,1$ pseudo-differential operators, proved below in Section 4.5 for this purpose.

1.2. **About the linearisations.** It is necessary for the present techniques that the non-linear term allows a *moderate* linearisation, in the sense of Definition 4.6 below (two of the most natural linearisations are not moderate, cf Remark 1.1). Here moderate linerisations are obtained by an exact version of Bony's paradifferential linearisation without regularising remainder terms. Roughly described, a given solution $u \in H^s_p$ leads to operators in the 'exotic' class $\mathrm{OP}(S^{1+(\frac{n}{p}-s)_+ +\varepsilon}_{1,1})$, where the order besides the number 1, coming from ∂_{x_1}, as a novelty contains $(\frac{n}{p}-s)_+ + \varepsilon$ because $u(x)$ may be unbounded on Ω.

More specifically, the linearisation has the following form, for $\Omega = \mathbb{R}^n$ and with $\pi_1(u,\cdot)$ denoting paramultiplication by u (cf (3.13) below)

$$-L_u g = \pi_1(u,\partial_1 g) + \pi_2(u,\partial_1 g) + \pi_3(g,\partial_1 u). \tag{1.7}$$

In the usual paralinearisation, the π_2-term is omitted since it is of higher regularity (leading to the famous formula $F(u(x)) = \pi_1(F'(u(x)),u(x)) +$ smoother terms).

But $\pi_2(u,\partial_1\cdot)$ is first of all *not* regularising in the present context, where u may be given in $B^s_{p,q}$ or $F^s_{p,q}$ also for $s < \frac{n}{p}$ (this is possible by (1.6b)), thus allowing u to be unbounded. Secondly, the only 'non-linear' limitation within the theory (such as (1.6b)) arises because $\pi_2(u,\partial_1 g)$ may or may not be defined; by incorporation of this term into L_u as in (1.7), the resulting limitation is whether or not L_u itself is defined on g. Motivated by this discussion, it might be appropriate to characterise formula (1.7) as the *exact* paralinearisation of $u\partial_1 u$ ('full' or 'complete' could also be used); this terminology is adopted throughout.

As described in connection with the von Karman equation for buckling plates in Section 5 below, it could be important that the theory is established under minimal assumptions, for the well-known weak solutions for this problem are only *barely* covered by a direct application of the present set-up.

1.3. **On the parametrices.** It is perhaps instructive first to review the corresponding linear problem, with u, f and φ as in (IR):

$$\begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix} u = \begin{pmatrix} f \\ \varphi \end{pmatrix}. \tag{1.8}$$

For boundary problems like (1.8), there is a straightforward proof method introduced by G. Grubb in [Gru90, Th 5.4] in a context of H^s_p and classical Besov spaces with $1 < p < \infty$. The advantage is that it altogether avoids the cumbersome boot-strap arguments used earlier on (as in eg V. A. Solonnikov's paper [Sol66]) when neither $B^s_{p,q} \hookrightarrow B^t_{r,o}$ nor $B^t_{r,o} \hookrightarrow B^s_{p,q}$ holds. This problem of non-existing embeddings is not even felt in the passage from (1.8) to (1.10) below.

To give the argument, note that the matrix $\left(\begin{smallmatrix} -\Delta \\ \gamma_0 \end{smallmatrix}\right)$ is an elliptic Green operator belonging to the Boutet de Monvel calculus of such systems; hence it has a parametrix $\left(\begin{smallmatrix} R_D & K_D \end{smallmatrix}\right)$ belonging to the calculus . As first shown by G. Grubb, it is possible to take $\left(\begin{smallmatrix} R_D & K_D \end{smallmatrix}\right)$ such that the class[2] of R_D, by [Gru90, Th 5.4], equals

$$\operatorname{class}(\gamma_0) - \operatorname{order}(-\Delta) = 1 - 2 = -1. \tag{1.9}$$

For such a choice of $\left(\begin{smallmatrix} R_D & K_D \end{smallmatrix}\right)$, the continuity from $B^{t-2}_{r,o}(\overline{\Omega}) \oplus B^{t-\frac{1}{r}}_{r,o}(\Gamma)$ to $B^t_{r,o}(\overline{\Omega})$ follows from [Joh96, Th 5.5] under the assumptions in (1.6a); cf also the concise description in the introduction of [Joh96].

Being a parametrix, $\left(\begin{smallmatrix} R_D & K_D \end{smallmatrix}\right)\left(\begin{smallmatrix} -\Delta \\ \gamma_0 \end{smallmatrix}\right) = I - \mathscr{R}$ for some regularising operator $\mathscr{R}$ with range in $C^\infty(\overline{\Omega})$, and class 1 (although $\left(\begin{smallmatrix} -\Delta \\ \gamma_0 \end{smallmatrix}\right)$ is invertible, so that $\mathscr{R} \equiv 0$ would be possible as in (1.3), $\mathscr{R}$ has been retained here for easier comparison with the general case). So, using the just mentioned continuity, it follows by application of $\left(\begin{smallmatrix} R_D & K_D \end{smallmatrix}\right)$ to both sides of (1.8) that

$$u = R_D f + K_D \varphi + \mathscr{R} u \quad \text{belongs to} \quad B^t_{r,o}(\overline{\Omega}). \tag{1.10}$$

This only requires the continuity of $\left(\begin{smallmatrix} -\Delta \\ \gamma_0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} R_D & K_D \end{smallmatrix}\right)$, and it holds whenever (s,p,q) and (t,r,o) both satisfy the condition in (1.6a). The formula (1.10) should be compared to the corresponding non-linear one in (1.3) or (1.18) below.

For *semi-linear* problems boot-strap techniques have seemingly prevailed. It would be pointless to account for the numerous papers adopting such reiteration methods, but a few remarks are made in order to shed light on the present work. For one thing, there is a boot-strap treatment of the stationary Navier–Stokes equation in [Joh93, Th 5.5.3], with a review in [Joh95b]. It is noteworthy that the methods are rather cumbersome in cases with $r \neq p$ (when r, p play roles as in (IR)). The difficulties come from intermediate spaces, with integral-exponents between r and p, that must be carefully chosen to make sense of the boundary condition ([Joh95b] explains the procedure). In 2 and 3 dimensions, R. Temam [Tem84] derived hypoellipticity (ie C^∞ data yield C^∞ solutions) for the semi-homogeneous Navier–Stokes problem with the Dirichlét condition, for which the mentioned difficulties do not show up. These results are all direct corollaries of the present theory.

For non-linear terms of composition type, ie of the form $g(u(x))$, a longer boot-strap argument (almost a formal algorithm) was introduced together with T. Runst [JR97]. Earlier on eg H. Amann, A. Ambrosetti and G. Mancini [AAM78], A. Ambrosetti and G. Mancini [AM78] and H. Brézis and L. Nirenberg [BN78] obtained

[2]The class is the minimal $r \in \mathbb{Z} \cup \{\pm\infty\}$ with continuity $H^r \to \mathscr{D}'$ of the operator.

hypoellipticity (in various frameworks) by reiteration with several p's. For super-linear Nemytskiĭ operators, S. I. Pohožaev has used analogous arguments [Poh93, Th 1]. (This list is not intended to be complete.) J.-Y. Chemin and C.-J. Xu [CX97] used a boot-strap method to give a simplified proof of the smoothness of weak solutions to the Euler–Lagrange equations of harmonic maps; the basic step was to obtain hypoellipticity of a class of semi-linear problems with terms of the form $\sum a_{j,k}(x,u(x))\partial_j u \partial_k u$. Formally this incorporates both composition and product type non-linearities, but since the weak solutions in this case are proved to be bounded, the difficulties met in [JR97] did not show up in [CX97]. However, this well indicates that even larger families of non-linearities will be relevant and require disturbingly many additional efforts, so since the product type operators defined below lead to treatments of several well-known semi-linear boundary problems, this class should suffice for now.

As indicated the above works had a restricted scope (to hypoellipticity, to $s=1$ or bounded functions) and thereby avoided consideration of $u \mapsto g(u)$ on the *full* H^s_p spaces with $1 < s < \frac{n}{p}$ — which require much sharper arguments. A fortiori there was no need to specify the borderline of the so-called *parameter domain*; cf [JR97, Fig 1] and the discussion of this notion in Section 1.4 below.

The present article does not directly deal with composition type problems (although Section 2 applies to bounded solutions of these), but this sphere of problems could also deserve stronger methods, say to get rid of the algorithmic proof in [JR97]. The paper [JR97] has not only been a source of inspiration, but there is also a common theme of determining a useful parameter domain $\mathbb{D}$ for the semi-linear operators. Although only product type problems are analysed in depth in Section 3 onwards, a minimal set of assumptions is made on $\mathbb{D}$ in Section 2, bearing in mind the more general examples of parameter domains established in [JR97].

In the parametrix construction (which is to be introduced in a general way in Section 2 below) the first step is this: given a solution u of (1.2) as in (IR), find a *linear*, but u-dependent operator L_u such that

$$L_u u = -u\partial_1 u. \tag{1.11}$$

At this point it seems decisive to utilise paramultiplication. On $\mathbb{R}^n$ this yields a decomposition of the usual 'pointwise' product

$$v \cdot w = \pi_1(v,w) + \pi_2(v,w) + \pi_3(v,w), \tag{1.12}$$

where the $\pi_j(\cdot,\cdot)$ are the paraproducts (cf the formulae in (3.13) below).

The difficulty of working on an open set $\Omega \subset \mathbb{R}^n$ is handled here via the operators r_Ω and ℓ_Ω, where r_Ω denotes restriction from $\mathbb{R}^n$ to Ω whilst ℓ_Ω is a universal extension operator for Ω (cf (1.35) below). Using this, the operator L_u is for (1.2) taken as the exact paralinearisation,

$$L_u g = -r_\Omega \pi_1(\ell_\Omega u, \partial_1 \ell_\Omega g) - r_\Omega \pi_2(\ell_\Omega u, \partial_1 \ell_\Omega g) - r_\Omega \pi_3(\ell_\Omega g, \partial_1 \ell_\Omega u). \tag{1.13}$$

It is noteworthy that the last term, in its action on g, is comprised of the operator $r_\Omega \pi_3(\ell_\Omega \cdot, \partial_1 \ell_\Omega u)$ — that *formally* differs from the rest of the right hand side.

However with this definition, L_u has certain mapping properties that are decisive for the argument below (precise assumptions are suppressed here for simplicity's sake). In fact L_u has an order $\omega(s,p,q)$ on all admissible spaces $B^s_{p,q}$, and under mild conditions on u it fulfils $\omega_{\max}(L_u) < 2$ if

$$\omega_{\max}(L_u) := \sup_{s,p,q} \omega(s,p,q). \tag{1.14}$$

This means that L_u is a *moderate* linearisation of $-u\partial_1 u$, in the terminology of Definition 4.6 in below; in fact, even a Δ-moderate decomposition because $\omega_{\max}(L_u)$ is less than the order of $-\Delta$. Cf Remark 1.1 below.

By means of L_u, equation (1.2) implies

$$u - R_D L_u u = R_D f + K_D \varphi + \mathscr{R} u; \tag{1.15}$$

this follows simply by application of $\binom{R_D}{K_D}$ to both sides of (1.2) and insertion of (1.11). The idea is now to apply the finite Neumann series (which will equal the desired parametrix)

$$P_u^{(N)} := I + R_D L_u + \dots + (R_D L_u)^{N-1}. \tag{1.16}$$

Because $(R_D L_u)^j$ is *linear* it follows that

$$P_u^{(N)}(I - R_D L_u) = I - (R_D L_u)^N, \tag{1.17}$$

hence the resulting formula is

$$u = P_u^{(N)}(R_D f + K_D \varphi + \mathscr{R} u) + (R_D L_u)^N(u). \tag{1.18}$$

Note that in comparison with (1.10), there are two extra ingredients here, namely $P_u^{(N)}$ and $(R_D L_u)^N u$ which are manageable in the following way:

A crucial, but not difficult, analysis given in Section 2 below shows two fundamental results, namely that

$$\exists N\colon B^s_{p,q}(\overline{\Omega}) \xrightarrow{(R_D L_u)^N} B^t_{r,o}(\overline{\Omega}) \tag{1.19}$$

$$\forall N\colon B^t_{r,o}(\overline{\Omega}) \xrightarrow{P_u^{(N)}} B^t_{r,o}(\overline{\Omega}). \tag{1.20}$$

Using this, all terms on the right hand side of (1.18) are seen to belong to $B^t_{r,o}$, as desired, provided N is chosen as in (1.19).

Clearly the map $P_u^{(N)}$ is non-local because R_D is so, and u-dependent as one could expect. Moreover, from the family of parametrices $P_u^{(N)}$ one has the freedom to pick a sufficiently regularising one (this situation resembles the Hadamard parametrix construction somewhat [Hör85, 17.4]).

It is not intended to present a symbolic calculus containing the parametrices $P_u^{(N)}$; the difficulties in doing so are elucidated in Section 4.5 below. It is rather a point of the paper that the parametrices (and resulting inverse regularity properties) may be established by simpler means.

Remark 1.1. With the third term of L_u equal to $-r_\Omega \pi_3(\ell_\Omega \cdot, \partial_1 \ell_\Omega u)$, the regularity of $L_u g$ mainly depends on g. More precisely, if $u \in B^{s_0}_{p_0,q_0}(\overline{\Omega})$, then $L_u g$ has in general only $(\frac{n}{p_0} - s_0)_+ + 1$ derivatives less than g, and eg this would simply be 1 in the infinite region where $B^{s_0}_{p_0,q_0} \hookrightarrow L_\infty$, hence $\omega = 1$ there.

The choice $r_\Omega \pi_3(\ell_\Omega u, \partial_1 \ell_\Omega \cdot)$ would have rendered $-L_u g$ equal to $g \mapsto u\partial_1 g$, that might look like a natural linearisation. But since $u\partial_1 g \in B^s_{p,q}$ can be shown to hold if $s \le s_0$, the order of $g \mapsto u\partial_1 g$ fulfils $\omega(t,r,o) \ge t - s_0$ on $B^t_{r,o}$. Clearly this order is larger than that of $-\Delta$ when $t > s_0 + 2$; regardless of whether $B^{s_0}_{p_0,q_0} \hookrightarrow L_\infty$ or not, $t > s_0 + 2$ holds for 'most' spaces (whereas with the above definition of L_u, the action of $-\Delta$ is the dominant one, at least for $B^{s_0}_{p_0,q_0} \hookrightarrow L_\infty$ according to the first part of this remark), and so $g \mapsto u\partial_1 g$ is not even a moderate linearisation because $\omega_{\max} \geq \sup_t t - s_0 = \infty$. These properties of $g \mapsto u\partial_1 g$ do not suffice for the proof of this article's main theorem. For the same reasons it would be equally unfruitful to define $-L_u$ from the differential of $u \mapsto u\partial_1 u$ at u, for this entails taking $g \mapsto u\partial_1 g + g\partial_1 u$. The present definition of $L_u g$ is a suitable choice inbetween the other two just mentioned; and while this choice has been known, its consequences for semi-linear boundary problems have seemingly been unexplored hitherto.

1.4. **Remarks on parameter domains.** When justifying the rather formal steps in (1.15)–(1.18), it is convenient to depart from the *quadratic* standard domain $\mathbb{D}(Q)$. This notion is introduced in Section 4.2 below, and for the quadratic operator $Q(u) := u\partial_1 u$ of the model problem,

$$\mathbb{D}(Q) = \{\, (s,p,q) \mid s > \tfrac{1}{2} + (\tfrac{n}{p} - \tfrac{n}{2})_+ \,\}; \tag{1.21}$$

it is chosen so that Q is well defined on all $B^s_{p,q}$ and $F^s_{p,q}$ in this domain; cf (1.6b).

However, it is equally important that the Dirichlét condition makes sense on all the considered spaces. For this one can (cf (1.36) below, and also (1.6a)) let

$$\mathbb{D}_1 = \{\, (s,p,q) \mid s > \tfrac{1}{p} + (n-1)(\tfrac{1}{p} - 1)_+ \,\}, \tag{1.22}$$

and use this as the parameter domain of the Dirichlét realisation Δ_{γ_0}, ie $\mathbb{D}(\Delta_{\gamma_0}) = \mathbb{D}_1$. (Reference to Δ_{γ_0} is convenient, although (1.2) has an inhomogeneous boundary condition.) It is convenient to introduce $\sigma(s,p,q)$ such that Q is bounded

$$Q\colon B^s_{p,q} \to B^{s-\sigma(s,p,q)}_{p,q}, \tag{1.23}$$

so the crucial question whether Q has order stricly less than that of $-\Delta$ amounts to $\sigma(s,p,q) < 2$. Combining the requirements one is lead to the domain

$$\mathbb{D}(\Delta_{\gamma_0}, Q) = \mathbb{D}(\Delta_{\gamma_0}) \cap \mathbb{D}(Q) \cap \big\{\, (s,p,q) \;\big|\; \sigma(s,p,q) < 2 \,\big\}. \tag{1.24}$$

If for simplicity $n \geq 3$ is assumed, one finds from the general rules in the below Corollary 4.8 that

$$\mathbb{D}(\Delta_{\gamma_0}, Q) = \big\{\, (s,p,q) \;\big|\; s > \tfrac{1}{2} + (\tfrac{n}{p} - \tfrac{3}{2})_+ \,\big\}. \qquad (n \geq 3) \tag{1.25}$$

In the terminology of Section 4.2, Q is Δ_{γ_0}-moderate on every space (with its parameter) in this domain, and accordingly $\mathbb{D}(\Delta_{\gamma_0}, Q)$ is also said to be a domain of Δ_{γ_0}-moderacy for the non-linear operator Q. It will be convenient throughout to say that "u is in $\mathbb{D}(Q)$" when u belongs to $E^s_{p,q}$ for some parameter $(s,p,q) \in \mathbb{D}(Q)$.

The formula $Q(u) = -L_u(u)$ is valid on $\mathbb{D}(Q)$, cf Lemma 4.3 below, but it turns out that $g \mapsto L_u(g)$ for a fixed $u \in B^{s_0}_{p_0,q_0}$ is defined on every space in

$$\mathbb{D}(L_u) = \{\, (s,p,q) \mid s > 1 - s_0 + (\tfrac{n}{p} + \tfrac{n}{p_0} - n)_+ \,\}. \tag{1.26}$$

Here $\mathbb{D}(L_u) \supset \mathbb{D}(Q)$, in general with a large gap, and it is clear that $\mathbb{D}(L_u)$ increases with improving a priori regularity of u (ie with increasing s_0 or p_0). This is exploited in the analysis of (1.1) in Theorem 7.1 below.

Although parameter domains at first glance may seem to be a notion of minor importance, these four domains and their general counterparts are useful for both the ideas and the exposition of this article. Among them $\mathbb{D}(L_u)$ is a novelty in particular, and it clearly gives a concise explanation of how the properties of Q differ from those of its paralinearisation L_u.

However, in this article, the main motivation for a systematic use of parameter domains is that the parametrices, and the resulting inverse regularity properties, are established in the domain

$$\mathbb{D}_u = \mathbb{D}_1 \cap \mathbb{D}(L_u). \tag{1.27}$$

By (1.24), this is larger than $\mathbb{D}(\Delta_{\gamma_0}, Q)$ since $\mathbb{D}(Q) \subset \mathbb{D}(L_u)$. But the situation is delicate, for the introduction of $\mathbb{D}_u$ presupposes that a solution u is fixed in $\mathbb{D}(\Delta_{\gamma_0}, Q)$, and it is only afterwards one can replace this by $\mathbb{D}_u$. These domains do differ, and they are a little tedious to explain, so for a clarification of the situation it seems instrumental to use parameter domains consistently.

Notice that canonical choices of parameter domains do not exist, since eg Q for different purposes may be considered with $\mathbb{D}(Q)$ or $\mathbb{D}(\Delta_{\gamma_0}, Q)$. The quadratic standard domain $\mathbb{D}(Q)$ is always easy to determine, cf the general rule in Proposition 4.4 below. By comparison, domains of moderacy such as $\mathbb{D}(\Delta_{\gamma_0}, Q)$ do not follow a single rule, for these are obtained from $\mathbb{D}(Q)$ by removal of certain subregions, depending both on the class of the boundary condition and on all orders entering the linear and non-linear operators; cf Corollary 4.8 ff below.

However, using the above formulae, it is not difficult to point to a central point of the main theorem's proof. Introducing the deficit $\delta = 2 - \sigma(s,p,q)$, clearly $\delta > 0$ holds in $\mathbb{D}_u$, and $R_D L_u$ has order $-\delta$ on *all* spaces $B^s_{p,q}$ in $\mathbb{D}_u$. So when u is in a fixed space $B^{s_0}_{p_0,q_0}$ in $\mathbb{D}(\Delta_{\gamma_0}, Q)$, it follows that $(R_D L_u)^N$ has order $-N\delta$ on $\mathbb{D}_u$, and hence has the property (1.19). (This argument breaks down for the other linearisations in Remark 1.1, since they are not moderate.)

For other types of problems the relevant parameter domain will in general be a rather more complicated set than the polygon in (1.25). In particular, it may indeed be non-convex and operators corresponding to $(R_D L_u)^N$ can have orders bounded with respect to N (unlike $-N\delta$). This is exemplified by the composition type problems in [JR97]; cf Figure 1 there.

In view of this, it seems practical to assume that the parameter domain is connected (although a 'path to C^∞' would suffice by (2.20) below). Under this hypothesis it is possible to prove the existence of the desired N in (1.19) by continuous induction along a curve from $(\frac{n}{p}, s)$ to $(\frac{n}{r}, t)$, running inside the parameter domain $\mathbb{D}_u$. Actually any such curve will do, in contrast with a boot-strap argument which for $r \neq p$ relies on the choice of a specific, suitable curve. Cf Section 2.

1.5. **Notation and preliminaries.** The space of smooth functions with compact supported is denoted by $C_0^\infty(\Omega)$ or $\mathscr{D}(\Omega)$, when $\Omega \subset \mathbb{R}^n$ is open; $\mathscr{D}'(\Omega)$ is the dual space of distributions on Ω. $\langle u, \varphi \rangle$ denotes the action of $u \in \mathscr{D}'(\Omega)$ on $\varphi \in C_0^\infty(\Omega)$. The restriction $r_\Omega \colon \mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\Omega)$ is the transpose of the extension by 0 outside of Ω, denoted $e_\Omega \colon C_0^\infty(\Omega) \to C_0^\infty(\mathbb{R}^n)$. Using this, $C^\infty(\overline{\Omega}) = r_\Omega C^\infty(\mathbb{R}^n)$ etc.

The Schwartz space of rapidly decreasing C^∞-functions is written $\mathscr{S}(\mathbb{R}^n)$, while $\mathscr{S}'(\mathbb{R}^n)$ stands for the space of tempered distributions. The Fourier transformation of u is $\mathscr{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,dx$, with inverse $\mathscr{F}^{-1}v(x) = \check{v}(x)$.

For simplicity $t_\pm := \max(0, \pm t)$ for $t \in \mathbb{R}$. As in [Knu92], the bracket $[\![\mathsf{A}]\!]$ stands for 1 and 0 when the assertion A is true and false, respectively.

Norms and quasi-norms are written $\|x\,|X\|$ for x in a vector space X; recall that X is quasi-normed if the triangle inequality is replaced by the existence of $c \geq 1$ such that all x and y in X fulfil $\|x+y\,|X\| \leq c(\|x\,|X\| + \|y\,|X\|)$ ("quasi-" will be suppressed when the meaning is settled by the context). Eg $L_p(\mathbb{R}^n)$ and $\ell_p(\mathbb{N})$ for

$p \in]0,\infty]$ are quasi-normed with $c = 2^{(\frac{1}{p}-1)_+}$; by Hölder's inequality this holds because both ℓ_p and L_p for $0 < p \le 1$ satisfy the following, for $\lambda = p$,

$$\| f+g \| \le (\| f \|^\lambda + \| g \|^\lambda)^{1/\lambda}. \tag{1.28}$$

For brevity $\| f \|_p$ is also used instead of $\| f \,|L_p\|$ for $f \in L_p(\Omega)$, with $\Omega \subset \mathbb{R}^n$ an open set. $X_1 \oplus X_2$ denotes the product space topologised by $\| x_1 \,|X_1\| + \| x_2 \,|X_2\|$. For a bilinear operator $B(\cdot,\cdot)\colon X_1 \oplus X_2 \to Y$, continuity is equivalent to existence of a constant c such that $\| B(x_1,x_2) \,|Y\| \le c \| x_1 \,|X_1\| \, \| x_2 \,|X_2\|$ and to boundedness.

The $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ spaces are defined as follows, with conventions as in [Yam86a]. First a Littlewood–Paley decomposition is constructed using a function Ψ in $C^\infty(\mathbb{R})$ for which $\Psi \equiv 0$ and $\Psi \equiv 1$ holds for $t \ge 13/10$ and $t \le 11/10$, respectively. Then $\Psi_j(\xi) := \Psi(2^{-j}|\xi|)$ and

$$\Phi_j(\xi) = \Psi_j(\xi) - \Psi_{j-1}(\xi) \qquad (\Psi_{-1} \equiv 0) \tag{1.29}$$

gives $\Psi_j = \Phi_0 + \dots + \Phi_j$ for every $j \in \mathbb{N}_0$, hence $1 \equiv \sum_{j=0}^\infty \Phi_j$ on $\mathbb{R}^n$. As a shorthand $\varphi(D)$ will denote the pseudo-differential operator with symbol φ, ie $\varphi(D)u = \mathscr{F}^{-1}(\varphi(\xi)\mathscr{F}u(\xi))$, say for $\varphi \in \mathscr{S}(\mathbb{R}^n)$.

Now, for a *smoothness index* $s \in \mathbb{R}$, an *integral-exponent* $p \in]0,\infty]$ and a *sum-exponent* $q \in]0,\infty]$, the *Besov space* $B^s_{p,q}(\mathbb{R}^n)$ and the *Triebel–Lizorkin space* $F^s_{p,q}(\mathbb{R}^n)$ are defined as

$$B^s_{p,q}(\mathbb{R}^n) = \big\{ u \in \mathscr{S}'(\mathbb{R}^n) \bigm| \big\| \{2^{sj} \| \Phi_j(D)u(\cdot) \,|L_p\|\}_{j=0}^\infty \big|\ell_q \big\| < \infty \big\}, \tag{1.30}$$

$$F^s_{p,q}(\mathbb{R}^n) = \big\{ u \in \mathscr{S}'(\mathbb{R}^n) \bigm| \big\| \| \{2^{sj} \Phi_j(D)u\}_{j=0}^\infty \,|\ell_q\|(\cdot) \big|L_p \big\| < \infty \big\}. \tag{1.31}$$

Throughout this paper it will be tacitly understood that $p < \infty$ whenever Triebel–Lizorkin spaces are under consideration.

The spaces are described in eg [RS96, Tri83, Tri92, Yam86a]. They are quasi-Banach spaces with the quasi-norms given by the finite expressions in (1.30) and (1.31). They have the property (1.28) for $\lambda = \min(1,p,q)$.

Among the embedding properties of these spaces one has $B^s_{p,q} \hookrightarrow B^{s-\varepsilon}_{p,q}$ for $\varepsilon > 0$, and if in the second line $\Omega \subset \mathbb{R}^n$ is open and bounded,

$$B^s_{p,q} \hookrightarrow B^t_{r,o} \quad \text{for} \quad s - \frac{n}{p} = t - \frac{n}{r},\ p > r;\ o = q, \tag{1.32}$$

$$B^s_{p,q}(\overline{\Omega}) \hookrightarrow B^s_{r,q}(\overline{\Omega}) \quad \text{for} \quad p \ge r. \tag{1.33}$$

The analogous holds for $F^s_{p,q}$, except that $F^s_{p,q} \hookrightarrow F^t_{r,o}$ if only $s - \frac{n}{p} = t - \frac{n}{r}$, $p > r$.

Example 1.2. For $b > -n$, the locally integrable function $g = |x|^b$ is in $B^{b+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ for $0 < p \le \infty$, since $\Phi_j(D)g(x) = 2^{(1-j)b}\check{\Phi}_1 * g(2^{j-1}x)$ gives that $\|\Phi_j(D)g\|_p$ equals $2^{(1-j)(\frac{n}{p}+b)}\|\check{\Phi}_1 * g\|_p$. The delta measure $\delta_0 \in B^{\frac{n}{p}-n}_{p,\infty}(\mathbb{R}^n)$ for $0 < p \le \infty$.

A (possibly non-linear) operator T is said to have order ω on $E^s_{p,q}$ if T maps this space into $E^{s-\omega}_{p,q}$ and $\| T(f) \,|E^{s-\omega}_{p,q}\| \le c \| f \,|E^s_{p,q}\|$ for some constant c. The order may depend on the specific $E^s_{p,q}$, hence in general be a function $\omega(s,p,q)$. Typically T is given along with a natural range of parameters (s,p,q) for which it makes sense on $E^s_{p,q}$; then the set of such (s,p,q) is denoted by $\mathbb{D}(T)$ and is called the *parameter domain* of T. If T has an order on every $E^s_{p,q}$ with $(s,p,q) \in \mathbb{D}(T)$

the following quantity is well defined,

$$\omega_{\max}(T) = \sup_{(s,p,q)\in\mathbb{D}(T)} \omega(s,p,q). \tag{1.34}$$

(The order is differently defined if $E^s_{p,q}$ and $E^{s-\omega}_{p,q}$ are considered over underlying manifolds of unequal dimensions, but this will be unnecessary here.)

When $\Omega\subset\mathbb{R}^n$ is open, then $E^s_{p,q}(\overline{\Omega}) := r_\Omega(E^s_{p,q}(\mathbb{R}^n))$ endowed with the infimum norm. An extension operator, ℓ_Ω, is a continuous linear map

$$\ell_\Omega\colon E^s_{p,q}(\overline{\Omega}) \to E^s_{p,q}(\mathbb{R}^n), \tag{1.35}$$

such that $r_\Omega\circ\ell_\Omega = I$. When Ω is a bounded domain for which the boundary locally is the graph of a Lipschitz function, a *universal* extension operator ℓ_Ω exists, that is ℓ_Ω can be constructed such that it has the stated properties for all admissible (s,p,q); ie such that $\mathbb{D}(\ell_\Omega) = \mathbb{R}\times\,]0,\infty]^2$. See V. Rychkov's paper [Ryc99b] (together with [Ryc99a]) for this result, which is convenient here.

The k^{th} parameter domain, $\mathbb{D}_k$, is

$$\mathbb{D}_k = \big\{\, (s,p,q) \;\big|\; s > k + \tfrac{1}{p} - 1 + (n-1)(\tfrac{1}{p}-1)_+ \,\big\}, \tag{1.36}$$

which is the usual choice for elliptic problems of class $k\in\mathbb{Z}$ and for the outward, normal derivative of order $k-1$ at Γ, ie for $\gamma_{k-1}f := ((\frac{\partial}{\partial\vec{n}})^{k-1}f)|_\Gamma$.

For the reader's sake a few lemmas are recalled. They are concerned with convergence of a series $\sum_{j=0}^\infty u_j$ fulfilling the dyadic ball *condition*: for some $A>0$

$$\operatorname{supp}\mathscr{F}u_j \subset \{\, \xi\in\mathbb{R}^n \mid |\xi| \le A2^j \,\}, \quad \text{for} \quad j\ge 0. \tag{1.37}$$

Lemma 1.3 (The dyadic ball criterion)**.** *Let $s > \max(0, \frac{n}{p} - n)$ for $0<p<\infty$ and $0<q\le\infty$ and suppose $u_j\in\mathscr{S}'(\mathbb{R}^n)$ fulfil* (1.37) *and*

$$F(q) := \big\|(\sum_{j=0}^\infty 2^{sjq}|u_j(\cdot)|^q)^{\frac{1}{q}}\big\|_p < \infty. \tag{1.38}$$

Then $\sum_{j=0}^\infty u_j$ converges in $\mathscr{S}'(\mathbb{R}^n)$ to some u lying in $F^s_{p,r}(\mathbb{R}^n)$ for

$$r\ge q, \quad r > \tfrac{n}{n+s}, \tag{1.39}$$

and $\|\,u\,|F^s_{p,r}\| \le cF(r)$ for some $c>0$ depending on n, s, p and r.

As remarked in [Joh04a, Lem 6.1], this follows from the usual version in which $s > \max(0, \frac{n}{p}-n, \frac{n}{q}-n)$ is required, for one can just pass to larger values of q if necessary. The above lemma emphasises that the interrelationship between s and q is inconsequential for the mere existence of the sum. In the Besov case one has

Lemma 1.4 (The dyadic ball criterion)**.** *Let $s > \max(0, \frac{n}{p} - n)$ for $p,q\in\,]0,\infty]$ and suppose $u_j\in\mathscr{S}'(\mathbb{R}^n)$ fulfil* (1.37) *and*

$$B := (\sum_{j=0}^\infty 2^{sjq}\|u_j\|_p^q)^{\frac{1}{q}} < \infty. \tag{1.40}$$

Then $\sum_{j=0}^\infty u_j$ converges in $\mathscr{S}'(\mathbb{R}^n)$ to some u lying in $B^s_{p,q}(\mathbb{R}^n)$ and $\|\,u\,|B^s_{p,q}\| \le cB$ for some $c>0$ depending on n, s, p and q.

It is also well known that the restrictions on s can be entirely removed if $\sum u_j$ fulfils the dyadic *corona* condition: for some $A>0$, $\operatorname{supp}\mathscr{F}u_0 \subset \{\,|\xi|\le A\,\}$ and

$$\operatorname{supp}\mathscr{F}u_j \subset \{\, \xi\in\mathbb{R}^n \mid \tfrac{1}{A}2^j \le |\xi| \le A2^j \,\}, \quad \text{for} \quad j>0. \tag{1.41}$$

Lemma 1.5 (The dyadic corona criterion)**.** *Let $u_j \in \mathscr{S}'(\mathbb{R}^n)$ fulfil* (1.41) *and* (1.40)*. Then $\sum_{j=0}^{\infty} u_j$ converges in $\mathscr{S}'(\mathbb{R}^n)$ to some u for which $\| u \,|B^s_{p,q}\| \le cB$ for some $c > 0$ depending on n, s, p and q. And similarly for $F^s_{p,q}(\mathbb{R}^n)$, if $F(q) < \infty$.*

These lemmas are proved in eg [Yam86a].

In Lemma 1.4 the restrictions on s cannot be improved, for as soon as $q > 1$ on the borderline $s = (\frac{n}{p} - n)_+$, then convergence is not implied by (1.40); cf [Joh95a, Ex 2.4]. A substitute is outlined and used for Theorem 4.11 below, were also the next borderline result, taken from [Joh95a, Prop. 2.5(2)], enters.

Lemma 1.6. *Let $0 < q \le 1 \le p < \infty$ and let $\sum_{j=0}^{\infty} u_j$ be such that $F(q) < \infty$. Then $\sum u_j$ converges in L_p to a sum u fulfilling $\| u \,|L_p\| \le F$.*

Proof. With $\sum |u_j(x)|$ as a majorant (since $F(1) \le F(q)$), $\| \sum_{j=k}^{\infty} |u_j| \,|L_p\| \xrightarrow[k\to\infty]{} 0$. Hence $\sum u_j$ is a fundamental series in L_p, and the estimate follows. □

For the estimates of the exact paralinearisation in Section 4.4 and 4.5, the following vector-valued Nikolskiĭ–Plancherel–Polya inequality will be convenient.

Lemma 1.7. *Let $0 < r < p < \infty$, $0 < q \le \infty$ and $A > 0$. There is a constant c such that for $f_k \in L_r(\mathbb{R}^n) \cap \mathscr{S}'(\mathbb{R}^n)$ with $\operatorname{supp} \mathscr{F} f_k \subset B(0, A2^k)$,*

$$\big\| \big(\sum_{k=0}^{\infty} |f_k|^q\big)^{1/q} \big| L_p \big\| \le c \big\| \sup_k 2^{(\frac{n}{r} - \frac{n}{p})k} |f_k| \,\big| L_r \big\| . \tag{1.42}$$

The usual Nikolskiĭ–Plancherel–Polya inequality results from this if only one f_k is non-trivial. (Lemma 1.7 itself can be reduced to this by [BM01, Lem. 4])

2. The general parametrix construction

2.1. An abstract framwork.

For the applications' sake the below Theorem 2.2 is proved under rather minimal assumptions; examples are given later. If desired the reader may think of the below spaces X^s_p as $H^s_p(\overline{\Omega})$ and consider A to be an elliptic operator like $\binom{-\Delta}{\gamma_0}$ etc.

For the full generality it is assumed that for some $n \in \mathbb{N}$ and $d \in \mathbb{R}$ (playing the role of the dimension and the order of the linear operator A, respectively) the following five conditions are fulfilled:

(I) Two scales X^s_p and Y^s_p of vector spaces are defined with (s,p) belonging to a common parameter set $\mathbb{S}$ lying inside $\mathbb{R} \times]0,\infty]$. In the X^s_p scale there are the usual simple, Sobolev and finite measure embeddings; i.e., for (s,p) and (t,r) in $\mathbb{S}$,

$$X^s_p \subset X^{s-\varepsilon}_p \quad \text{when} \quad \varepsilon > 0, \tag{2.1}$$

$$X^s_p \subset X^t_r \quad \text{when} \quad s \ge t \quad \text{and} \quad s - \tfrac{n}{p} = t - \tfrac{n}{r}, \tag{2.2}$$

$$X^s_p \subset X^s_r \quad \text{when} \quad p \ge r. \tag{2.3}$$

(II) There is a linear map $A_{(s,p)}$, written A for short,

$$A\colon X^s_p \to Y^{s-d}_p \tag{2.4}$$

for (s,p) in a set $\mathbb{D}(A) \subset \mathbb{S}$, termed the parameter domain of A.

There is also, for all $(s,p)\in\mathbb{D}(A)$, a linear map $\widetilde{A}\colon Y^{s-d}_p\to X^s_p$ such that

$$\mathscr{R}:=I_{X^s_p}-\widetilde{A}A \quad\text{has range in}\quad \bigcap_{(s,p)\in\mathbb{D}(A)} X^s_p. \tag{2.5}$$

Inclusions $\bigcup_{\mathbb{D}(A)}X^s_p\subset\mathscr{X}$ and $\bigcup_{\mathbb{D}(A)}Y^{s-d}_p\subset\mathscr{Y}$ hold for some vector spaces $\mathscr{X}$, $\mathscr{Y}$; and for (s,p), $(t,r)\in\mathbb{D}(A)$ there is a commutative diagram

$$\begin{array}{ccc} X^s_p\cap X^t_r & \xrightarrow{\;I\;} & X^s_p \\ {\scriptstyle I}\big\downarrow & & \big\downarrow{\scriptstyle A_{(s,p)}}\,. \\ X^t_r & \xrightarrow{A_{(t,r)}} & \mathscr{Y} \end{array} \tag{2.6}$$

Likewise $\widetilde{A}$ should be unambiguously[3] defined on $Y^{s-d}_p\cap Y^{t-d}_r$.

(III) There is a non-linear operator $\mathscr{N}$, with parameter domain $\mathbb{D}(\mathscr{N})$, which for every (s_0,p_0) in $\mathbb{D}(\mathscr{N})$ and for every $u\in X^{s_0}_{p_0}$ decomposes as $\mathscr{N}(u)=-B_u(u)$, where

$$B_u\colon X^s_p\to Y^{s-d+\delta(s,p)}_p \tag{2.7}$$

is a linear map endowed with a parameter domain $\mathbb{D}(B_u)$ that is required to fulfil $\mathbb{D}(\mathscr{N})\subset\mathbb{D}(B_u)$. For (s,p) and (t,r) both in $\mathbb{D}(\mathscr{N})$ or $\mathbb{D}(B_u)$ there should be a commutative diagram analogous to (2.6) for $\mathscr{N}$ and B_u.

(IV) For u as in (III), the parameter domain $\mathbb{D}(A)\cap\mathbb{D}(B_u)$ is curve-connected with respect to the metric $\operatorname{dist}((s,p),(t,r))$ given by $((s-t)^2+(\frac{n}{p}-\frac{n}{t})^2)^{\frac12}$; ie the Euclidean metric after the transformation $(s,p)\mapsto(\frac{n}{p},s)$.

(V) For u as in (III), the function $\delta(s,p)$ satisfies

$$(s+\delta(s,p),p)\in\mathbb{D}(A)\quad\text{for every}\quad(s,p)\in\mathbb{D}(A)\cap\mathbb{D}(B_u), \tag{2.8}$$

$$\inf\big\{\,\delta(s,p)\,\big|\,(s,p)\in K\,\big\}>0\quad\text{for every}\quad K\Subset\mathbb{D}(A)\cap\mathbb{D}(B_u). \tag{2.9}$$

For the proof of Theorem 2.2 below it is unnecessary to assume that the embeddings in (I) should hold for the Y^s_p spaces too (although they often do hold in practice); as it stands (I) is easier to verify in applications to parabolic initial-boundary problems; cf Remark 2.3 below.

For $X^s_p=H^s_p(\overline{\Omega})$ it is natural to let $\mathbb{S}=\mathbb{R}\times\,]1,\infty[$; the L_2-theory comes out for $\mathbb{S}=\mathbb{R}\times\{1\}$. Besov spaces would often require q to be fixed and $\mathbb{S}=\mathbb{R}\times\,]0,\infty]$. Anyhow $\mathscr{X}=\mathscr{D}'(\Omega)$ could be a typical choice. Continuity of A and $\widetilde{A}$ is not required (although both will be bounded in most applications).

By (2.6) ff, $\mathscr{R}$ may be thought of as an operator from $\bigcup_{\mathbb{D}(A)}X^s_p$ to $\bigcap_{\mathbb{D}(A)}X^s_p$.

For simplicity the arguments s_0, p_0 are usually suppressed in the function δ. By (III), the non-linear map $\mathscr{N}$ sends X^s_p into $Y^{s-d+\delta(s,p)}_p$ for each (s,p) in $\mathbb{D}(\mathscr{N})$ (since $\mathbb{D}(\mathscr{N})\subset\mathbb{D}(B_u)$ for every u in X^s_p). This fact is tacitly used in the following.

Note that $\delta(s,p)>0$ by (2.9), so that (III) implies $\mathscr{N}(u)$ has B_u as a moderate linearisation (according to Definition 4.6 below).

---

[3] Suppressing (s,p) in A is harmless in the sense that A by (2.6) is a well-defined map with domain $\bigcup_{\mathbb{D}(A)}X^s_p$ in $\mathscr{X}$; it is linear on each 'fibre' X^s_p. Similarly $\widetilde{A}$ is a map on $\bigcup_{\mathbb{D}(A)}Y^{s-d}_p$. Moreover, A eg extends to a linear map on the algebraic direct sum $\bigoplus X^s_p\subset\mathscr{X}$ if and only if

$$0={\sum_{\mathbb{D}(A)}}' v_{(s,p)}\implies\sum_{\mathbb{D}(A)}A_{(s,p)}(v_{(s,p)})=0.$$

(' indicates finitely many non-trivial vectors.)

Applying the transformation in condition (IV), the reader should constantly think of $\mathbb{D}(A)$, $\mathbb{D}(\mathcal{N})$ and $\mathbb{D}(B_u)$ as subsets of $[0,\infty[\times\mathbb{R}$.

Since the boundary of $\mathbb{D}(\mathcal{N})$ (or a part thereof) often consists of the (s_0,p_0) for which $\delta\equiv 0$, it may seem natural to require $\mathbb{D}(\mathcal{N})$ to be open in $[0,\infty[\times\mathbb{R}$. However, such an assumption is avoided because it is unnecessary and potentially might exclude application to weak solutions of certain problems; cf the below Section 5.

Evidently the strict positivity in (2.9) is implied by the conjunction of pointwise positivity and lower semi-continuity of $\delta(s,p)$ on $\mathbb{D}(A)\cap\mathbb{D}(B_u)$. However, with respect to (s,p) the function δ is in practice often a constant, which depends effectively on (s_0,p_0). When this is the case and furthermore $\mathcal{N}$ has a natural parameter domain $\mathbb{D}(\mathcal{N})$ on which δ can take both positive and negative values, it is natural to introduce

$$\mathbb{D}(\mathcal{N},\delta)=\{\,(s_0,p_0)\in\mathbb{D}(\mathcal{N})\mid\delta>0\,\} \tag{2.10}$$

and, instead of $\mathbb{D}(\mathcal{N})$, use this subset as the parameter domain of $\mathcal{N}$. In a possibly smaller subset, $\mathcal{N}$ will then be 'dominated' by the linear map A, namely

$$\mathbb{D}(A,\mathcal{N})=\mathbb{D}(A)\cap\mathbb{D}(\mathcal{N},\delta). \tag{2.11}$$

By introducing $\sigma(s,p)=d-\delta(s,p)$, it is clear that $\mathbb{D}(A,\mathcal{N})$ is a generalisation of the domain $\mathbb{D}(\Delta_{\gamma_0},Q)$ in (1.24).

Example 2.1 (The model problem)**.** To elucidate conditions (I)–(V) above, one may in (1.2) set $A=\binom{-\Delta}{\gamma_0}$ and $X^s_p=B^s_{p,q}(\overline{\Omega})$, whereby $q\in]0,\infty]$ is kept fixed. For the operator $\widetilde{A}$ there is a parametrix of A belonging to the Boutet de Monvel calculus (cf Section 6.1 below). Using L_u from (1.13), both B_u and Y^s_p are taken in accordance with

$$B_uv=\begin{pmatrix}L_uv\\0\end{pmatrix}\in\begin{matrix}B^{s-2+\delta(s,p)}_{p,q}(\overline{\Omega})\\ \oplus\\ \{0\}\end{matrix}\subset\left.\begin{matrix}B^{s-2}_{p,q}(\overline{\Omega})\\ \oplus\\ B^{s-\frac{1}{p}}_{p,q}(\Gamma)\end{matrix}\right\}=:Y^{s-2}_p. \tag{2.12}$$

For any $\varepsilon\in]0,1[$ it is possible to take $\delta(s,p)$ as the constant function

$$\delta(s,p)=\begin{cases}1 & \text{for } s_0>\frac{n}{p_0},\\ 1-\varepsilon & \text{for } s_0=\frac{n}{p_0},\\ s_0-\frac{n}{p_0}+1 & \text{for } \frac{n}{p_0}>s_0>\frac{n}{p_0}-1.\end{cases} \tag{2.13}$$

See the below Theorem 4.7. As mentioned in Remark 4.10, this theorem and Corollary 4.8 also gives the parameter domains, for any fixed $u\in X^{s_0}_{p_0}$,

$$\mathbb{D}(A)=\mathbb{D}_1=\{\,(s,p)\mid s>\tfrac{1}{p}+(n-1)(\tfrac{1}{p}-1)_+\,\}, \tag{2.14}$$

$$\mathbb{D}(\mathcal{N})=\big\{\,(s,p)\;\big|\;s>\tfrac12+(\tfrac{n}{p}-\tfrac{n}{2})_+\,\big\}, \tag{2.15}$$

$$\begin{aligned}\mathbb{D}(\mathcal{N},\delta)&=\big\{\,(s,p)\;\big|\;s>\tfrac12+(\tfrac{n}{p}-\tfrac32+\tfrac12[\![n=2]\!])_+\,\big\}\\ &=\mathbb{D}(A,\mathcal{N})\end{aligned} \tag{2.16}$$

$$\mathbb{D}(B_u)=\big\{\,(s,p)\;\big|\;s>1-s_0+(\tfrac{n}{p}+\tfrac{n}{p_0}-n)_+\,\big\}. \tag{2.17}$$

Being isometric to a polygon in $[0,\infty[\times\mathbb{R}$, the set $\mathbb{D}(A)\cap\mathbb{D}(B_u)$ clearly satisfies (IV); when $(s_0,p_0)\in\mathbb{D}(A)\cap\mathbb{D}(B_u)$, then condition (V) may be verified directly.

Altogether this shows that the somewhat lengthy conditions (I)–(V) are uncomplicated to verify for the basic problem in (1.2).

2.2. **The Parametrix Theorem.** Using the above abstract framework, it is now possible to formulate and prove the main result of the article in a widely applicable version.

Theorem 2.2. *Let X^s_p, Y^s_p and the mappings A and $\mathscr{N}$ be given such that conditions* (I)–(V) *above are satisfied.*

(1) *For every*

$$u \in X^{s_0}_{p_0} \quad \text{with } (s_0, p_0) \in \mathbb{D}(A) \cap \mathbb{D}(\mathscr{N}) \tag{2.18}$$

the parametrix $P^{(N)} = \sum_{k=0}^{N-1} (\widetilde{A}B_u)^k$ is for every $N \in \mathbb{N}$ a linear operator

$$P^{(N)}_u : X^s_p \to X^s_p \tag{2.19}$$

for every (s,p) in the set $\mathbb{D}_u := \mathbb{D}(A) \cap \mathbb{D}(B_u)$. And for every (s',p'), $(s'',p'') \in \mathbb{D}_u$ there exists $N' \in \mathbb{N}$ such that the "error term" $(\widetilde{A}B_u)^N$ is a linear map

$$(\widetilde{A}B_u)^N : X^{s'}_{p'} \to X^{s''}_{p''} \quad \text{for} \quad N \geq N'. \tag{2.20}$$

(2) *If some u fulfils* (2.18) *and solves the equation*

$$Au + \mathscr{N}(u) = f \tag{2.21}$$

$$\text{with data} \quad f \in Y^{t-d}_r \quad \text{for some } (t,r) \in \mathbb{D}_u, \tag{2.22}$$

one has for every $N \in \mathbb{N}$ the parametrix formula

$$u = P^{(N)}(\widetilde{A}f + \mathscr{R}u) + (\widetilde{A}B_u)^N u. \tag{2.23}$$

And consequently $u \in X^t_r$ too.

Proof. For arbitrary $(s,p) \in \mathbb{D}_u$, one can use (II) and (2.8) to see that $\widetilde{A}$ is defined on $Y^{s-d+\delta(s,p)}_p$, hence that $\widetilde{A}B_u$ is a well defined composite

$$X^s_p \xrightarrow{B_u} Y^{s-d+\delta(s,p)}_p \xrightarrow{\widetilde{A}} X^{s+\delta(s,p)}_p. \tag{2.24}$$

Since $X^{s+\delta}_p \hookrightarrow X^s_p$ by (I), the operator $\widetilde{A}B_u$ is of order 0 on X^s_p; hence

$$P^{(N)} := \sum_{j=0}^{N-1} (\widetilde{A}B_u)^j \tag{2.25}$$

is a linear map $X^s_p \longrightarrow X^s_p$. This shows the claim on $P^{(N)}$.

Concerning $(\widetilde{A}B_u)^N$, there is, by (IV), a continuous map $k \colon I \to \mathbb{D}_u$, with $I = [a,b]$, such that

$$k(a) = (s',p'), \qquad k(b) = (s'',p''). \tag{2.26}$$

Clearly $\delta_k := \inf\{\, \delta(s,p) \mid (s,p) \in k(I)\,\} > 0$, and for $(s,p) \in k(I)$

$$X^s_p \xrightarrow{\widetilde{A}B_u} X^{s+\delta(s,p)}_p \hookrightarrow X^{s+\delta_k}_p. \tag{2.27}$$

With the convention that $X_{k(\tau)} := X^s_p$ when $k(\tau) = (s,p)$, let

$$T = \{\, \tau \in I \mid \exists N \in \mathbb{N} \colon (\widetilde{A}B_u)^N (X^{s'}_{p'}) \subset X_{k(\tau)}\,\} \tag{2.28}$$

and $M := \sup T$. Then $a \leq M \leq b$ since $\widetilde{A}B_u(X^{s'}_{p'}) \subset X^{s'+\delta}_{p'} \subset X_{k(a)}$. It would now suffice to show that $b \in T$, for then $(\widetilde{A}B_u)^N(X^{s'}_{p'}) \subset X_{k(b)} = X^{s''}_{p''}$ for some $N \in \mathbb{N}$; and since $(\widetilde{A}B_u)^{N'} = (\widetilde{A}B_u)^{N'-N} (\widetilde{A}B_u)^N$ for $N' > N$, the full claim on $\widetilde{A}B_u$ would follow because $(\widetilde{A}B_u)^{N'-N}$ is of order 0 on $X^{s''}_{p''}$ by (2.27).

For one thing $M\in T$: by continuity of k there is a $\tau'<M$ in T such that

$$|k(\tau')-k(M)|<\delta_k/2 \tag{2.29}$$

and $(\widetilde{A}B_u)^{N-1}(X^{s'}_{p'})\subset X_{k(\tau')}$ for some N. But by (2.27) this entails that $(\widetilde{A}B_u)^N(X^{s'}_{p'})$ is a subset of a space with upper index at least δ_k higher than that of $X_{k(\tau')}$, so the embeddings in (I) show that $(\widetilde{A}B_u)^N(X^{s'}_{p'})$ is contained in any space in the intersection of $\mathbb{S}$ and a convex polygon; cf the dashed line in Figure 1 below. It follows that $(\widetilde{A}B_u)^N(X^{s'}_{p'})$ is contained in every X^s_p lying in $\mathbb{S}$ and fulfilling

$$|k(\tau')-(s,p)|<\delta_k/\sqrt{2}, \tag{2.30}$$

so in particular $(\widetilde{A}B_u)^N(X^{s'}_{p'})\subset X_{k(M)}$ is found from (2.29).

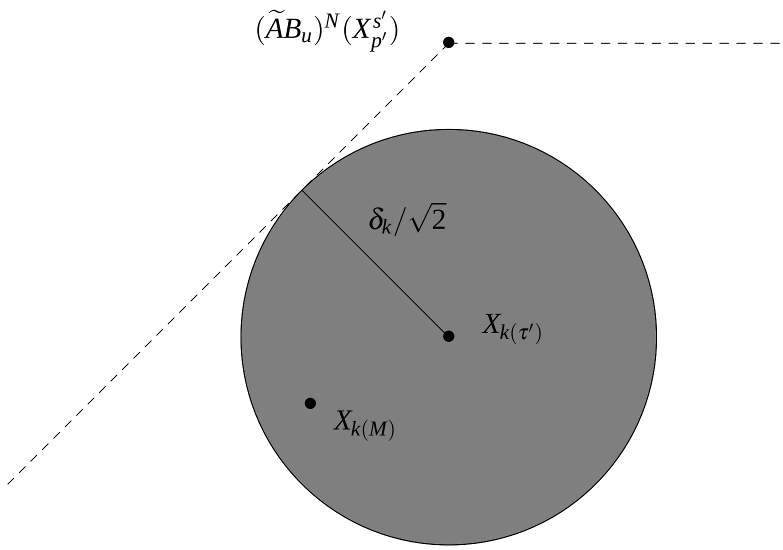

FIGURE 1. The ball specified by (2.30) and a polygon in the $(\frac{n}{p},s)$-plane of spaces containing $(\widetilde{A}B_u)^N(X^{s'}_{p'})$.

Secondly, $M=b$ follows from $k(I)$'s connectedness: every $k(\tau)$ with $M<\tau\le b$ has an open neighbourhood U_τ disjoint from the open $\frac{\delta_k}{2}$-ball around $k(M)$, denoted $B(k(M),\frac{\delta_k}{2})$; for if not, $|k(\tau)-k(M)|\le\frac{\delta_k}{2}<\frac{\delta_k}{\sqrt{2}}$ would hold, so that $M\in T$ would imply (as above) that $(\widetilde{A}B_u)^N(X^{s'}_{p'})\subset X_{k(\tau)}$, contradicting that $\tau\notin T$. Thus $k([M,b])$ is covered by the disjoint open sets $B(k(M),\frac{\delta_k}{2})$ and $\bigcup_{\tau>M}U_\tau$, and since the former is non-empty, no $\tau>M$ exists in $[a,b]$.

According to (II), (III) and the assumptions in the theorem, the mapping $\widetilde{A}$ has the same meaning on both sides of (2.21), regardless of whether one refers to $Y^{s_0-d}_{p_0}$ or to Y^{t-d}_r (on the left and the right hand sides, respectively). Therefore (2.5) and the assumption $(s_0,p_0)\in\mathbb{D}(\mathscr{N})$ entail

$$(I-\mathscr{R})u-\widetilde{A}B_uu=\widetilde{A}f. \tag{2.31}$$

For the given u and f, it follows from (2.25) that

$$P^{(N)}(I-\widetilde{A}B_u)u=(I-(\widetilde{A}B_u)^N)u, \tag{2.32}$$

and moreover that $P^{(N)}$ has the same meaning on both sides of (2.31). Hence (2.31) and (2.32) yield (2.23).

Note that the term $P^{(N)}(\widetilde{A}f+\mathscr{R}u)$ in (2.23) is in X^t_r in view of (2.5) and the proved fact that $P^{(N)}$ has order 0 on X^t_r. By (2.20) also $(\widetilde{A}B_u)^N u$ is in X^t_r, so this holds for u too. □

Applications of Theorem 2.2 to elliptic boundary problems are developed in Section 6 below. The condition (2.8) in (V) may seem strange for an elliptic problem, for with $\mathbb{D}(A)$ equal to one of the standard domains $\mathbb{D}_k$ it is for $\eta>0$ always the case that $(s+\eta,p)$ belongs to $\mathbb{D}(A)$ when (s,p) does so. But first of all there are non-linearities that do not allow arbitrarily high values of s in the parameter domains, eg $|u|^a$ with non-integer $a>0$ for which $\mathbb{D}(B_u)$ cannot contain s much higher than a (depending on the order of the A in play), so a condition like (2.8) will be needed in these cases. Secondly, (2.8) is also relevant for the problems in the next remark.

Remark 2.3. Parabolic initial-boundary problems are also covered by Theorem 2.2, by taking A as the full parabolic system $(\partial_t - a(x,D_x), r_0, T)$ acting in anisotropic spaces (r_0 is restriction to $t=0$, and T a trace operator defining the boundary conditions). Concerning the linear problems, the reader is referred to [Gru95b, Sect. 4] for the L_p-theory (using classical Besov and Bessel potential spaces) with a complete set of compatibility conditions on fully inhomogeneous data. In particular Corollary 4.5 there applies because the underlying manifold $]0,b[\times\Omega$ for $0<b<\infty$ is bounded, so that the solution spaces X^s_p fulfil (I) above. Because of the stronger data norms introduced to control the compatibility of the boundary- and initial-data for exceptional values of s, cf [Gru95b, (4.16)], it is here convenient that the Y^s_p-scale is not required to fulfil (2.1)–(2.3). (Of course the compatibility conditions forces one to work with rather small parameter domains, once the data are given. But even so the present results may well allow considerable improvements of the solution's integrability.) For the non-linear terms, the product type operators of Section 3 below should be straightforward to treat in the corresponding anisotropic spaces, since the necessary paramultiplication estimates have been established in this generality [Yam86a, Joh95a].

The L_p-results for the time-dependent Navier–Stokes equation of G. Grubb [Gru95a] may also be extended by inverse regularity results using the present theory. However, this requires some additional efforts because the underlying linear problem is only degenerately parabolic, but one can overcome this difficulty by adapting the reduction of Grubb and Solonnikov [GS91] to a truely parabolic pseudodifferential problem.

2.3. **A solvability result.** As an addendum to the Parametrix Theorem it is now shown that bilinear perturbations of linear homeomorphisms always give well-posed problems locally, that is for sufficiently small data.

A proof of this may be based on the fixed-point theorem of contractions, that also apply in a quasi-Banach space X for which $\|\cdot\|^\lambda$ is subadditive for some $\lambda\in]0,1]$. (To the experts of topological vector spaces such λ is known to exist in any case, but for Besov and Triebel–Lizorkin spaces it follows from (1.28) ff.)

Proposition 2.4. *Let $A\colon X\to Y$ be a linear homeomorphism between two quasi-Banach spaces and $B\colon X\oplus X\to Y$ be a bilinear bounded map. When $\|\cdot|X\|^\lambda$ is* *subadditive for some* $\lambda \in]0,1]$ *and* $y \in Y$ *fulfills*

$$\|A^{-1}y\,|X\| \le \frac{1}{\|A^{-1}B\|4^{1/\lambda}}, \tag{2.33}$$

then the equation

$$Ax + B(x) = y \tag{2.34}$$

has a unique solution $x \in X$ *in the ball* $\|x\,|X\| \le \frac{1}{\|A^{-1}B\|2^{1/\lambda}}$*, and* x *depends continuously on such* y*.*

Proof. When $R := A^{-1}$, the equation is equivalent to $x = Ry - RB(x)$, where also $RB =: B'$ is bilinear and $\|B'\| \le \|R\|\|B\|$. For $F(x) = Ry - B'(x)$ bilinearity gives

$$\|F(x) - F(z)\|^\lambda \le \|B'\|^\lambda(\|x\|^\lambda + \|z\|^\lambda)\|x - z\|^\lambda. \tag{2.35}$$

Since $d(x,z) = \|x - z\,|X\|^\lambda$ is a complete metric on X, the map F is by (2.35) a contraction on the closed ball $K_a = \{\, x \in X \mid \|x\| \le a \,\}$ if a fulfills

$$2\|B'\|^\lambda a^\lambda < 1. \tag{2.36}$$

In addition F is a map $K_a \to K_a$ for sufficiently large a. In fact, for $x \in K_a$, $\|F(x)\|^\lambda \le \|Ry\|^\lambda + \|B'\|^\lambda a^{2\lambda}$; and $D = 1 - 4\|Ry\|^\lambda\|B'\|^\lambda > 0$ holds by the assumptions, so

$$\|Ry\|^\lambda + \|B'\|^\lambda t^2 < t \iff t \in \Big]\frac{1-\sqrt{D}}{2\|B'\|^\lambda}, \frac{1+\sqrt{D}}{2\|B'\|^\lambda}\Big[, \tag{2.37}$$

where the interval contains a^λ, when this is arbitrarily close to $(2\|B'\|^\lambda)^{-1}$.

Hence F has a unique fixed-point in the closed ball K_a. If also $Ax' + B(x') = y'$ for some $x' \in K_a$,

$$\|x - x'\|^\lambda \le \|R(y - y')\|^\lambda + 2a^\lambda\|B'\|^\lambda\|x - x'\|^\lambda, \tag{2.38}$$

so $d(x,x') \le cd(Ry,Ry')$ for $c = (1 - 2a^\lambda\|B'\|^\lambda)^{-1} < \infty$. This gives the well-posedness in K_a, but with the leeway in the choice of a the proposition follows. □

The proof above is elementary (and could well be folklore), but it is given for the reader's convenience since the result plays a key role for Theorem 7.1 below.

3. Preliminaries on products

For the reader's sake, a brief review of results on pointwise multiplication is given before the non-linear operators are introduced in Section 4 below.

First of all, a non-linear operator of *product type* is roughly a map

$$u \mapsto P_2(D)(P_0(D)u \cdot P_1(D)u), \tag{3.1}$$

where the $P_j(D)$ are partial differential operators, linear with constant coefficients and of orders $d_j \in \mathbb{N}_0$; cf Definition 4.1 below.

For simplicity's sake $P_2 = I$ is often considered, and $P_0(D)u \cdot P_1(D)u$ may then be viewed as a homogeneous second order polynomial $p(z_1, \dots, z_N)$ composed with a jet $J_k u = (D^\alpha u)_{|\alpha| \le k}$, $k = \max(d_0, d_1)$. But in general this jet description is too rigid, for a given operator of product type with $P_2 = I$ may be the restriction of one with $P_2 \ne I$, cf Example 3.1. And conversely $P_2(D)(P_0(D)u \cdot P_1(D)u)$ may be an extension of another one of the type in (3.1).

These differences lie not only in the various expressions such operators can be shown to have, but also in the parameter domains that they *may* be given (in analogy with maximal domains of differential operators in $L_2(\Omega)$). Consider eg

$$u \mapsto u \cdot \partial_1 u, \qquad u \mapsto \frac{1}{2}\partial_1(u^2). \tag{3.2}$$

The latter coincides with the former for $u \in C^\infty$, and in general it does so on the u on which $u\partial_1 u$ is defined. But as it stands, $u\partial_1 u$ does not make sense as a bounded bilinear map $L_4(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$, whereas $\partial_1(u^2)$ clearly does so. Hence $\frac{1}{2}\partial_1(u^2)$ is a non-trivial extension of $u\partial_1 u$, and the natural parameter domains differ for the two expresssions (for the latter it includes eg L_4).

More general classifications of non-linear operators are available in the literature; the reader may consult the set-up and examples in eg [Bon81, Sect. 5] and [Yam88, § 2]. But as discussed in the introduction, the product type operators defined above are adequate for fixing ideas and for important applications.

Example 3.1. For a useful commutation of differentiations to the left of the pointwise product, consider as in Section 5 below the 'von Karman bracket':

$$[v,w] := D_1^2 v D_2^2 w + D_2^2 v D_1^2 w - 2D_{12}^2 v D_{12}^2 w. \tag{3.3}$$

Introducing the expression

$$B(v,w) = D_{12}^2(D_1 v D_2 w + D_2 v D_1 w) - D_1^2(D_2 v D_2 w) - D_2^2(D_1 v D_1 w), \tag{3.4}$$

then $B(v,w) = [v,w]$ whenever v and w are regular enough to justify application of Leibniz' rule. Clearly $B(\cdot,\cdot)$ is a case with $P_2(D) \neq I$.

Example 3.2. It might be important to allow more general expressions; eg, using the solution operator R_0 for the homogeneous Dirichlét problem for Δ^2, a reduction to one unknown in the von Karman problem, cf J.-L. Lions [Lio69, Ch 1.4], leads to the tri-linear map, with $[\cdot,\cdot]$ as in (3.3),

$$u \mapsto [u, R_0[u,u]]. \tag{3.5}$$

So P_0, P_1 could ideally be allowed to be non-local, like R_0. And $R_0[u, R_0[u,u]]$, used in [Cia97, Th 5.8-1], has $P_2 = R_0$ non-local. Such non-linearities are only mentioned to give a perspective on the introduced product type operators.

3.1. **Generalised multiplication.** The non-linearities in (3.1) often involve multiplication of a non-smooth function and a distribution in $\mathscr{D}' \setminus L_1^{\mathrm{loc}}$; as eg in $u\partial_1 u$ when u belongs to $H^{\frac{1}{2}+\varepsilon}$ for small $\varepsilon > 0$. Although it suffices for a mere construction of weak solutions to consider an (ad hoc) extension of $(u,v) \mapsto u \cdot v$ to a bounded bilinear form defined on, say $H^s \times H^{-s}$ for some $s > 0$, the proof of the regularity properties will in general involve extensions to $F^s_{p,q} \times F^{-s}_{p,q}$ for *several* exponents p and q. This clearly causes a problem of consistency among the various extensions introduced during a single proof, and for $q = \infty$ there is, moreover, no density of smooth functions to play on. In the present context, commutative diagrams like (2.6) would then pose problems for the multiplication, hence condition (III) above would be problematic to verify for the product type operators. Therefore a more unified approach to multiplication is desirable.

Since a paper of L. Schwartz [Sch54] it has been known that products with a few reasonable properties cannot be everywhere defined on $\mathscr{D}' \times \mathscr{D}'$. Consequently many notions of multiplication exist, cf the survey [Obe92], but for the present theory it is important to use a product $\pi(\cdot,\cdot)$ that works well together with paramultiplication on $\mathbb{R}^n$ and also allows a localised version π_Ω to be defined on an open set $\Omega\subset\mathbb{R}^n$. A product π with these properties was analysed in [Joh95a], and for the reader's sake a brief review is given.

The product π is defined on $\mathbb{R}^n$ by simultaneous Fourier regularisation of both factors: when $\psi_k(\xi)=\psi(2^{-k}\xi)$ for $\psi\in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighbourhood of $\xi=0$, then

$$\pi(u,v) := \lim_{k\to\infty}(\psi_k(D)u)\cdot(\psi_k(D)v). \tag{3.6}$$

Here u and $v\in\mathscr{S}'(\mathbb{R}^n)$, and they are required to have the properties that this limit should both exist in $\mathscr{D}'(\mathbb{R}^n)$ for *all* ψ of the specified type and be independent of the choice of ψ. ($\psi_k(D)u := \mathscr{F}^{-1}(\psi\hat{u})$ etc.)

This formal definition is from [Joh95a], but consideration of the limit in (3.6) is folklore. It is a point that $\pi(u,v)$ coincides with the usual pointwise multiplication:

$$L_p^{\rm loc}(\mathbb{R}^n)\times L_q^{\rm loc}(\mathbb{R}^n)\overset{\cdot}{\longrightarrow} L_r^{\rm loc}(\mathbb{R}^n), \tag{3.7}$$

$$\mathscr{O}_M(\mathbb{R}^n)\times\mathscr{S}'(\mathbb{R}^n)\overset{\cdot}{\longrightarrow}\mathscr{S}'(\mathbb{R}^n); \tag{3.8}$$

hereby $0\le\frac{1}{r}=\frac{1}{p}+\frac{1}{q}\le 1$ and $\mathscr{O}_M$ denotes the slowly increasing smooth functions. Cf [Joh95a, Sect. 3.1] for the proofs.

For later reference, the main tool for (3.7) and localisation to open sets Ω is recalled from [Joh95a, Prop. 3.7]: if either u or v vanishes in Ω, ie $r_\Omega u=0$ or $r_\Omega v=0$, then any ψ as in (3.6) gives

$$0=\lim_{k\to\infty} r_\Omega(\psi_k(D)u\cdot\psi_k(D)v)) \quad \text{in } \mathscr{D}'(\Omega). \tag{3.9}$$

When $\pi(u,v)$ is defined, (3.9) implies that $\operatorname{supp}\pi(u,v)\subset\operatorname{supp}u\cap\operatorname{supp}v$ (for (3.7)–(3.8) this is obvious). But the limit in (3.9) exists in any case when one of the factors vanish in Ω.

Using (3.9), $\pi_\Omega(u,v)$ is defined for an arbitrary open set $\Omega\subset\mathbb{R}^n$ on those u, v in $\mathscr{S}'(\overline{\Omega})$ for which U, $V\in\mathscr{S}'(\mathbb{R}^n)$ exist such that $r_\Omega U=u$, $r_\Omega V=v$ and

$$\pi_\Omega(u,v) := \lim_{k\to\infty} r_\Omega(\psi_k(D)U)\cdot(\psi_k(D)V)) \quad \text{exists in } \mathscr{D}'(\Omega) \tag{3.10}$$

independently of $\psi\in C_0^\infty(\mathbb{R}^n)$ with $\psi\equiv 1$ near $\xi=0$. Here (3.9) implies that the limit is independent of the 'extension' (U,V), and that the ψ-independence is so (cf [Joh95a, Def 7.1]). However, because $\pi(U,V)$ need not be defined, it is essential that r_Ω is applied before passing to the limit.

3.2. **Boundedness of generalised multiplication.** Using (3.10), it is clear that π_Ω inherits boundedness from $\pi_{\mathbb{R}^n}$:

Proposition 3.3. *Let each of the spaces E_0, E_1 and E_2 be either a Besov space $B^s_{p,q}(\mathbb{R}^n)$ or a Triebel–Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$, chosen so that $\pi(\cdot,\cdot)$ is a bounded bilinear operator*

$$\pi\colon E_0\oplus E_1\to E_2. \tag{3.11}$$

For the corresponding spaces $E_k(\overline{\Omega}) := r_\Omega E_k$ over an arbitrary open set $\Omega\subset\mathbb{R}^n$, endowed with the infimum norm, π_Ω is bounded

$$\pi_\Omega(\cdot,\cdot)\colon E_0(\overline{\Omega})\oplus E_1(\overline{\Omega})\to E_2(\overline{\Omega}). \tag{3.12}$$

In the result above it is a central question under which conditions (3.11) actually holds. This was almost completely analysed in [Joh95a, Sect. 5] by means of paramultiplication. As a preparation for the definition and analysis (further below) of the exact paralinearisation, this will now be recalled.

First, by using (1.29), and by setting $\Phi_j \equiv 0$ for $j < 0$ etc, the paramultiplication operators $\pi_m(\cdot,\cdot)$ with $m = 1, 2, 3$ (in the sense of M. Yamazaki [Yam86a, Yam86b, Yam88]), are defined for those u and $v \in \mathscr{S}'(\mathbb{R}^n)$ for which the series below converge in $\mathscr{D}'(\mathbb{R}^n)$:

$$\pi_1(f,g) = \sum_{j=0}^{\infty} \Psi_{j-2}(D) f \Phi_j(D) g \tag{3.13a}$$

$$\begin{aligned}\pi_2(f,g) = \sum_{j=0}^{\infty} (\Phi_{j-1}(D) f \Phi_j(D) g + \Phi_j(D) f \Phi_j(D) g \\ + \Phi_j(D) f \Phi_{j-1}(D) g)\end{aligned} \tag{3.13b}$$

$$\pi_3(f,g) = \sum_{j=0}^{\infty} \Phi_j(D) f \Psi_{j-2}(D) g \tag{3.13c}$$

Secondly, this applies to (3.6) by consideration of the case $\psi_k = \Psi_k$, for then the formula $\Psi_k = \Phi_0 + \cdots + \Phi_k$ and bilinearity at once give that the right hand side of (3.6) equals $\sum_{m=1,2,3} \pi_m(u,v)$ provided each $\pi_m(u,v)$ exists — but this existence is easily obtained for each m by standard estimates. (In fact $\pi_1(f,g)$ and $\pi_3(f,g)$ both exist for all f, $g \in \mathscr{S}'(\mathbb{R}^n)$, as observed in [MC97, Ch. 16], so in practice $\pi(u,v)$ is defined if and only if the second series $\pi_2(u,v)$ is so.) Thirdly the independence of ψ is established post festum.

Whilst the boundedness of $\pi(\cdot,\cdot)$ was analysed in depth in [Joh95a], it suffices here to review some central conclusions on 'multiplicability'. For convenience $E^s_{p,q}$ denotes a space which (for every value of (s,p,q)) may be either a Besov or a Triebel–Lizorkin space on $\mathbb{R}^n$.

It was proved in [Joh95a, Th 4.2], albeit with (3.15b) and (3.16b) essentially covered by [Fra86b], that if

$$\| fg \,|E^{s_2}_{p_2,q_2}\| \le c \| f \,|E^{s_0}_{p_0,q_0}\| \, \| g \,|E^{s_1}_{p_1,q_1}\| \tag{3.14}$$

holds for all Schwartz functions f and g, then

$$s_0 + s_1 \ge n(\tfrac{1}{p_0} + \tfrac{1}{p_1} - 1), \tag{3.15a}$$

$$s_0 + s_1 \ge 0. \tag{3.15b}$$

As a supplement to this, the following were also established there:

$$s_0 + s_1 = \tfrac{n}{p_0} + \tfrac{n}{p_1} - n \qquad \text{implies} \begin{cases} \tfrac{1}{q_0} + \tfrac{1}{q_1} \ge 1 \text{ in } BB\bullet\text{-cases,} \\ \tfrac{1}{q_0} + \tfrac{1}{p_1} \ge 1 \text{ in } BF\bullet\text{-cases;} \end{cases} \tag{3.16a}$$

$$s_0 + s_1 = 0 \qquad \text{implies } \tfrac{1}{q_0} + \tfrac{1}{q_1} \ge 1. \tag{3.16b}$$

The main interest lies in the $BB\bullet$- and $FF\bullet$-cases and the case with $\max(s_0,s_1) > 0$ (for $s_0 = s_1 = 0$ Hölder's inequality applies). In this situation the sufficiency of the above conditions was entirely confirmed by means of (3.13), cf the following version of [Joh95a, Cor 6.12] for isotropic spaces:

Theorem 3.4. *When* $\max(s_0,s_1) > 0$, *then it holds in the* $BB\bullet$*- and* $FF\bullet$*-cases that* $E^{s_0}_{p_0,q_0}$ *and* $E^{s_1}_{p_1,q_1}$ *on* $\mathbb{R}^n$ *are 'multiplicable' if and only if both* (3.15a)–(3.15b) *and* (3.16a)–(3.16b) *hold.*

The spaces that receive $\pi(E^{s_0}_{p_0,q_0}, E^{s_1}_{p_1,q_1})$ were almost characterised in [Joh95a], departing from at least 8 other necessary conditions, but the below Theorem 4.7 will imply what is needed in this direction.

Remark 3.5. It is used in Section 7 below that multiplication cannot define a continuous map $W^m_1 \oplus W^m_1 \to \mathscr{D}'$ when $2m < n$. When the range is a Besov space this follows from (3.15a), but for the general statement an explicit proof should be in order. If $\rho \in C^\infty_0$ is real and $\rho_k(x) = \frac{1}{k} 2^{k(n-m)}\rho(2^k x)$, it is easy to see that $\| \rho_k \,|W^m_1\| = \mathscr{O}(\frac{1}{k}) \searrow 0$. But for $\varphi \in C^\infty_0$ non-negative with $\varphi(0) = 1$, $2m < n$ implies

$$\langle \rho_k^2, \varphi \rangle = k^{-2} 2^{k(n-2m)} \int \rho^2(y)\varphi(2^{-k}y)\,dy \nearrow \infty. \tag{3.17}$$

This arguments works for open sets $\Omega \ni 0$ and extends to all $\Omega \subset \mathbb{R}^n$ by translation.

3.3. **Extension by zero.** Having presented the product $\pi(\cdot,\cdot)$ formally, the opportunity is taken to make a digression that will be crucial later.

In Section 5-6 the operators A and $\tilde{A}$ of Section 2 will be realised through the Boutet de Monvel calculus of linear boundary problems, so it will be all-important to have commuting diagrams like (2.6) for this calculus. However, this is a little delicate for the truncated pseudo-differential operators, that locally are of the form $P_+ = r^+Pe^+$. Here the extension by zero outside of $\mathbb{R}^n_+$ can be defined, via the characteristic function χ of $\mathbb{R}^n_+$, by the formula $e^+u = \pi(\chi, v)$ when $r^+v = u$; this way e^+ is defined also on some spaces with $s < 0$.

In relation to commuting diagrams, it is an advantage that $v \mapsto \pi(\chi, v)$, by the definition of π, acts without reference to the spaces v belongs to. For its properties one has

Proposition 3.6. *The characteristic function χ of $\mathbb{R}^n_+$ yields a bounded map*

$$\pi(\chi,\cdot)\colon E^s_{p,q}(\mathbb{R}^n) \to E^s_{p,q}(\mathbb{R}^n), \tag{3.18}$$

for Besov and Triebel–Lizorkin spaces with $\frac{1}{p} - 1 + (n-1)(\frac{1}{p} - 1)_+ < s < \frac{1}{p}$.

This is similar to a result of Franke [Fra86a, Cor. 3.4.6] (that extends to Besov spaces with $p < \infty$ by real interpolation), but Franke departed from a less precise notion of the product by χ: for $\operatorname{supp} \hat{v}$ compact he estimated χv and extended by continuity to all of $F^s_{p,q}$ (for $q = \infty$ based on his well-known Fatou property). Here the formal space dependence is harmless because the approximands only refer to v (a truncated Littlewood–Paley decomposition).

As a more subtle point, the full treatment of P_+ in $B^s_{p,q}$ and $F^s_{p,q}$-spaces is based on the paramultiplicative splitting of π in (3.13), so it is important that Franke's product χv equals $\pi(\chi, v)$. This was exploited, albeit without details, in [Joh96], so because of its role in the commuting diagrams here, it is natural to take the opportunity to return to this point:

Proof. In view of (3.13) it suffices for $B^s_{p,q}$ to show bounds

$$\| \pi_m(\chi,u) \,|B^s_{p,q}\| \le C \| u \,|B^s_{p,q}\| \quad \text{for } m = 1,\ 2,\ 3. \tag{3.19}$$

Using well-known estimates (cf the remarks in the proof of Thm. 4.7 below) this holds for $m = 1$ for every s because $\chi \in L_\infty$. Moreover, for $m = 2$ it holds for $s > (\frac{n}{p} - n)_+$, while for $m = 3$ it does so for $s < 0$. The last two restrictions on s will be relaxed using the anisotropic structure of χ.

For brevity $u_k := \Phi_k(D)u$, $u^k := \Psi_k(D)u$ etc. Now $\pi_3(\chi,u) = \sum_{k\ge 2}\chi_k u^{k-2}$. If H is the Heaviside function, $\chi(x) = 1(x')\otimes H(x_n)$ and

$$\chi_k = c\mathscr{F}^{-1}(\Phi_k\delta_0(\xi')\otimes \hat{H}(\xi_n)) = 1(x')\otimes \mathscr{F}^{-1}_{\xi_n\to x_n}(\Phi_k(0,\xi_n)\hat{H}). \tag{3.20}$$

For the second factor, note that $2^k\hat{H}(2^k\xi_n) = \hat{H}(\xi_n)$ since H is homogeneous of degree zero, so

$$\mathscr{F}^{-1}(\Phi_1(0,2^{-k}\cdot)\hat{H})(x_n) = 2^k\mathscr{F}^{-1}(\Phi_1(0,\cdot)\hat{H}(2^k\cdot))(2^kx_n) = H_1(2^kx_n). \tag{3.21}$$

Here H_k refers to the decomposition $1=\sum\Phi_j(0,\xi_n)$ on $\mathbb{R}^{n-1}$. For $k\ge 1$ this gives

$$\| H_k\,|L_p(\mathbb{R})\| = 2^{-(k+1)/p}\,\| H_1\,|L_p(\mathbb{R})\| < \infty. \tag{3.22}$$

Indeed, $\Phi_1(0,\cdot)\hat{H}\in\mathscr{S}(\mathbb{R})$ because $\mathscr{F}H = \frac{-\mathrm{i}}{\tau}\mathscr{F}(\partial_t H(t)) = \frac{1}{\mathrm{i}\,\tau}$ for $\tau\ne 0$; hence $H_1\in L_p$. Note that $\tilde H := H - H_0$ by (3.22) is in $B^{1/p}_{p,\infty}(\mathbb{R})$ for $0<p\le\infty$.

To handle the factor $1(x')$ in (3.20), there is a mixed-norm estimate

$$\|\chi_k u^{k-2}\,|L_p\|^p \le \int (\sup_{t\in\mathbb{R}} |u^{k-2}(x',t)|)^p\,dx'\,\|H_k\,|L_p(\mathbb{R})\|^p \tag{3.23}$$

so that $s-\frac{1}{p}<0$ in view of a summation lemma (cf [Joh96, Lem. 2.5]) yields

$$\begin{aligned}\sum_{k>1} 2^{skq}\|\chi_k u^{k-2}\|_p^q &\le c\sum_{k>1}2^{(s-\frac{1}{p})kq}\Big(\sum_{0\le l\le k}\|u_l\,|L_p(L_\infty)\|^{\min(1,p)}\Big)^{\frac{q}{\min(1,p)}}\|H_1\|_p^q\\ &\le c\|H_1\|_p^q\sum_{k\ge 0}2^{(s-\frac{1}{p})kq}\|u_k\,|L_p(L_\infty)\|^q\\ &\le c\,\|\tilde H\,|B^{\frac{1}{p}}_{p,\infty}\|^q\,\|u\,|B^s_{p,q}\|^q.\end{aligned} \tag{3.24}$$

Indeed, the last line follows from the Nikolskiĭ–Plancherel–Polya inequality, cf Lemma 1.7, when this is used in the x_n-variable (for fixed x' the Paley–Wiener–Schwartz Theorem gives that $u(x',\cdot)$ has its spectrum in the region $|\xi_n|\le 2^{k+1}$). By the dyadic corona criterion, cf Lemma 1.5, this proves $\pi_3(\chi,u)\in B^s_{p,q}$, hence the case $m=3$ for $s<\frac{1}{p}$.

For $m=2$ only $\frac{1}{p}-1<s\le 0$ remains; this implies $1<p\le\infty$. It can be assumed that $u_0=0$, for u may be replaced by $u-u_0-u_1$ because $\chi\in L_\infty$ implies $\pi_2(\chi,u_0+u_1)$ belongs to $\bigcap_{t>0}B^t_{p,q}$. Then $\pi_2(\chi,u)$ is split in three contributions, with details given for $\sum\chi_k u_k$ (terms with $\chi_k u_{k-1}$ and $\chi_{k-1}u_k$ are treated analogously). In the following it is convenient to replace the sequence (u_j) temporarily by $(0,\dots,0,u_N,\dots,u_{N+M},0,\dots)$, in which the entries are also called u_j for simplicity. In this way the below series trivially converge.

Note that the Nikolskiĭ–Plancherel–Polya inequality used in x_n yields

$$\|\Phi_j(D)\sum_{k\ge j-1}\chi_k u_k\|_p \le c\sum_{k\ge j-1}\Big\|\overset{\vee}{\Phi}_j*(\chi_k u_k)\,\Big|L_{p,x'}(L_{1,x_n})\Big\|\,2^{j(1-\frac{1}{p})}. \tag{3.25}$$

In this mixed-norm expression, Fubini's theorem gives for $k\ge 1$

$$\int|\overset{\vee}{\Phi}_j*(\chi_k u_k)(x',x_n)|\,dx_n \le \|H_k\|_1\iint|\overset{\vee}{\Phi}_j(x'-y',x_n)|\,dx_n\sup_{t\in\mathbb{R}}|u_k(y',t)|\,dy'. \tag{3.26}$$

Reading this as a convolution on $\mathbb{R}^{n-1}$, the usual L_p-estimate leads to

$$\| \overset{\vee}{\Phi}_j * (\chi_k u_k) \,|L_p(L_1)\| \leq \|H_k\|_1 \|\overset{\vee}{\Phi}_j\|_1 \| u_k \,|L_p(L_\infty)\|. \tag{3.27}$$

Combined with (3.25) this gives, since $s+1-\frac{1}{p}>0$ and $\operatorname{supp}\mathscr{F}(\chi_k u_k)$ is disjoint from $\operatorname{supp}\Phi_j$ unless $k>j-2$ (and since $u_0=0$),

$$\begin{aligned}\sum_{j\geq 0} 2^{sjq} \|\overset{\vee}{\Phi}_j * \sum_{k\geq 0} \chi_k u_k\|_p^q &\leq c \sum_{j\geq 0} 2^{(s+1-\frac{1}{p})jq} \Big(\sum_{k\geq j-1} \|H_k\|_1 \| u_k \,|L_p(L_\infty)\| \Big)^q \\ &\leq c' \sum_{j\geq 0} 2^{(s+1-\frac{1}{p})jq} \|H_j\|_1^q \| u_j \,|L_p(L_\infty)\|^q \\ &\leq c' \| \tilde{H} \,|B^1_{1,\infty}(\mathbb{R})\|^q \sum_{j\geq 0} 2^{sjq} \|u_j\|_p^q < \infty. \end{aligned} \tag{3.28}$$

For $q<\infty$ the right hand side tends to 0 for $N\to\infty$, so the π_2-series is fundamental in $B^s_{p,q}$. There is also convergence for $q=\infty$, since $u\in B^{s-\varepsilon}_{p,1}$ for all $\varepsilon>0$. The above estimate then also applies to the original (u_j), which yields (3.19) for $m=2$.

To cover the $F^s_{p,q}$-case, note the continuity $B^{s+\varepsilon}_{p,1} \xrightarrow{\pi(\chi,\cdot)} B^{s+\varepsilon}_{p,1} \hookrightarrow F^s_{p,q}$ for $p<\infty$ and sufficiently small $\varepsilon>0$. If Franke's multiplication by χ is denoted M_χ, it follows from his results that $B^{s+\varepsilon}_{p,1} \xrightarrow{M_\chi} F^s_{p,q}$. Since $\mathscr{S}$ is dense in $B^{s+\varepsilon}_{p,1}$ and M_χ extends the pointwise product by χ, it follows that M_χ coincides with $\pi(\chi,\cdot)$ for all Besov spaces with (s,p,q) as in the theorem, if $p<\infty$. But then they coincide on all the $F^s_{p,q}$ spaces, so $\pi(\chi,\cdot)$ is bounded on $F^s_{p,q}$ as claimed. □

Remark 3.7. The above direct treatment of the Besov case should be of interest in its own right, in view of the mixed-norm estimates that allow the unified proof of all cases. (Even for $B^s_{p,q}$, [Tri83, Th 2.8.7(i)] had to go through subdivisions of the parameter region, with duality arguments for $s<0$ due to the lack of a precise definition of χu; avoided by use of $\pi(\cdot,\cdot)$ here.)

4. PRODUCT TYPE OPERATORS

The desired class of non-linear operators and their paralinearisations can now be formally introduced:

Definition 4.1. Operators of *product type* (d_0,d_1,d_2) on an open set $\Omega\subset\mathbb{R}^n$ are (finite sums of) maps of the form

$$(v,w)\mapsto P_2(D)\pi_\Omega(P_0(D)v,P_1(D)w), \tag{4.1}$$

where the $P_j(D)$ are linear differential operators with constant coefficients and of order d_j. The case with $P_2(D)=I$ is throughout indicated by designating the operator as one of type (d_0,d_1). Generally d_0, d_1, d_2 appear in the same order as the $P_j(D)$'s are applied.

It is essential to use π_Ω in this definition, for the involved product cannot in general be reduced to any of the forms in (3.7)–(3.8) when v and w in (4.1) both are in spaces of low regularity.

Definition 4.2. For each choice of a universal extension operator ℓ_Ω, and choice of Ψ_k in (1.29), the *exact paralinearisation* L_u of an operator of product type d_0,d_1 is defined as follows, cf (1.13)

$$\begin{gathered} L_u g = -r_\Omega \pi_1(P_0U, P_1G) - r_\Omega \pi_2(P_0U, P_1G) - r_\Omega \pi_3(P_0G, P_1U) \\ \text{with } U = \ell_\Omega u \text{ and } G = \ell_\Omega g. \end{gathered} \tag{4.2}$$

For $P_2 \neq I$, the composite $P_2(D)L_u$ is the exact paralinearisation.

The rationale is that $L_u g$ has circa the same regularity as g. Indeed, as a well-known fact $\pi_1(f,g)$ has roughly the same regularity as its second argument g, and $\pi_3(f,g) = \pi_1(g,f)$ yields that the π_3-term mainly depends on g; since in general $\pi_2(f,g)$ is of a matching but not lower regularity than the others, altogether $L_u g$ has regularity like g. This inference will be corroborated in Theorem 4.7 below.

Conceptually, Definition 4.2 invokes an interchange of the maps ℓ_Ω and $P_j(D)$, compared to (4.1), where $P_0(D)$ and $P_1(D)$ are applied before the extensions to $\mathbb{R}^n$ in π_Ω; cf (3.10). The reason for this is that $L_u g$ then has the structure of a composite map $r_\Omega \circ P_u \circ \ell_\Omega(g)$ for a certain pseudo-differential operator P_u of type $1,1$; cf Theorem 4.13 below.

Although ℓ_Ω does not commute with the differential operators $P_0(D)$, $P_1(D)$, the convention above is justified by the fact that $P_j \ell_\Omega v = \ell_\Omega P_j v$ in Ω, so that the localisation property in (3.9) implies that $-L_u u$ gives back the original product type operator:

Lemma 4.3. *Let u belong to a Besov or Triebel–Lizorkin space $E^s_{p,q}(\overline{\Omega})$ such that the parameters $(s-d_j,p,q)_{j=0,1}$ fulfil* (3.15)–(3.16) *and $s > \min(d_0,d_1)$. Then*

$$\pi_\Omega(P_0(D)u, P_1(D)u) = -L_u(u). \tag{4.3}$$

This holds for any choice of ℓ_Ω and Ψ_k (or Φ_k) in the definition of L_u.

Proof. According to Theorem 3.4, the parameters $(s-d_j,p,q)_{j=0,1}$ belong to the parameter domain of π on $\mathbb{R}^n$, so it holds for all v, $w \in E^s_{p,q}(\overline{\Omega})$ that

$$\pi_\Omega(P_0(D)v, P_1(D)w) = r_\Omega \lim_{k\to\infty} (\psi_k(D)P_0(D)\ell_\Omega v) \cdot (\psi_k(D)P_1(D)\ell_\Omega w)). \tag{4.4}$$

Indeed, $P_0(D)\ell_\Omega v$ and $P_1(D)\ell_\Omega w$ are not only well-defined extensions, which may be used as U and V in (3.10), but $\pi(P_0(D)\ell_\Omega v, P_1(D)\ell_\Omega w)$ is defined, so the $\mathscr{D}'$-continuity of r_Ω gives that the limit in (4.4) can be taken before restriction to Ω.

By (3.9) and bilinearity, the choice of ℓ_Ω is inconsequential for $\pi_\Omega(P_0 v, P_1 w)$. There is also freedom to choose $\psi_k = \Psi_k$ since the left hand side of (4.4) does not depend on this; cf (3.10). Now (4.3) follows upon insertion of $v = w = u$, for by (3.13) ff and the formula $\Psi_k = \Phi_0 + \dots + \Phi_k$ the right hand side of (4.4) then equals the formula for $-L_u(u)$ in (4.2). □

The above introduction of paralinearisation is not the only possible, but the intention here is to make the relation to the 'pointwise' product on Ω clear.

4.1. **Estimates of product type operators.** Considering a product type operator

$$B(\cdot,\cdot) := \pi(P_0(D)\cdot, P_1(D)\cdot), \tag{4.5}$$

a large collection of boundedness properties now follows from the theory reviewed in Section 3.1–3.2. Indeed, using Theorem 3.4 it is clear that $\pi(P_0(D)\cdot, P_1(D)\cdot)$ is bounded from $E^{s_0}_{p_0,q_0} \oplus E^{s_1}_{p_1,q_1}$ to some Besov or Triebel–Lizorkin space provided

$$s_0 + s_1 > d_0 + d_1 + (\tfrac{n}{p_0} + \tfrac{n}{p_1} - n)_+. \tag{4.6}$$

The *standard* domain $\mathbb{D}(B)$ of the bilinear operator $B(\cdot,\cdot)$ is the set of parameters $(s_j, p_j, q_j)_{j=0,1}$ satisfying this inequality; since it works equally well for the *BBB*- and FFF-cases, the notation is the same in the two cases.

When the two arguments of B are identical, the resulting operator is throughout denoted by Q, ie

$$Q(u) := B(u,u). \tag{4.7}$$

The parameter domain $\mathbb{D}(Q)$ derived from (4.6) is termed the *quadratic* standard domain of Q (or of B). For this domain one has the next result on the *direct* regularity properties of product type operators. It is a special case of [Joh93, Prop. 3.6.1], where also anisotropic spaces and other co-domains are treated.

Proposition 4.4. *Let $B(v,w)$ be an operator of product type (d_0,d_1) with $d_0 \le d_1$. The quadratic standard domain $\mathbb{D}(Q)$ consists of the (s,p,q) fulfilling*

$$s > \tfrac{d_0+d_1}{2} + (\tfrac{n}{p} - \tfrac{n}{2})_+, \tag{4.8}$$

and for each such (s,p,q) the non-linear operator Q is bounded

$$Q\colon B^s_{p,q} \to B^{s-\sigma(s,p,q)}_{p,q} \tag{4.9}$$

when $\sigma(s,p,q)$, for some $\varepsilon > 0$, is taken equal to

$$\sigma(s,p,q) = d_1 + (\tfrac{n}{p} + d_0 - s)_+ + \varepsilon[\![\tfrac{n}{p} + d_0 = s]\!][\![q > 1]\!]. \tag{4.10}$$

Similar results hold for $F^s_{p,q}$ provided $[\![q > 1]\!]$ is replaced by $[\![p > 1]\!]$.

Analogous results for open sets $\Omega \subset \mathbb{R}^n$ can be derived from Proposition 3.3. Details on this are left out for simplicity, and so is the proof, for it follows from the below Theorem 4.7 (by application of L_u to u, cf Lemma 4.3).

Remark 4.5. It is noteworthy that the standard domain $\mathbb{D}(Q)$ only depends on the two orders through their mean $(d_0+d_1)/2$; cf (4.8). The correction $\frac{n}{p} - \frac{n}{2}$ occurring for $p < 2$ is independent of d_0 and d_1; cf Figure 2.

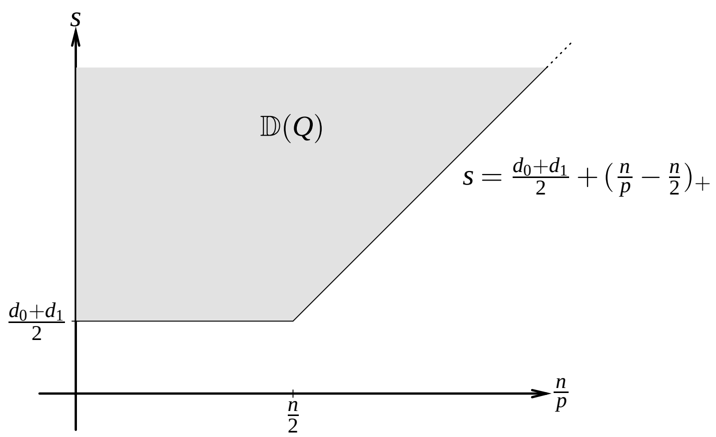

FIGURE 2. The quadratic standard domain $\mathbb{D}(Q)$

4.2. **Moderate linearisations of product type operators.** The next notions are introduced in order to have names for the basic properties of non-linearities, that allow suitable parametrices.

Let $\mathscr{N}$ be a non-linear operator defined on $E^s_{p,q}$ for (s,p,q) running in a parameter domain $\mathbb{D}(\mathscr{N})$, like in (4.8). A linear operator L_u will be called a linearisation of $\mathscr{N}$ if, by convention,

$$\mathscr{N}(u) = -L_u(u) \tag{4.11}$$

for every $u \in E^s_{p,q}$ with (s,p,q) in $\mathbb{D}(\mathscr{N})$. Here L_u should be a meaningful linear operator parametrised by the u in $\mathbb{D}(\mathscr{N})$, and possibly for u in larger domains (this will be the case for the exact paralinearisation in (4.2), as seen below).

It will be required that L_u, for each $u \in E^{s_0}_{p_0,q_0}$, should be of order $\omega(s,p,q)$ on every $E^s_{p,q}$ in $\mathbb{D}(L_u)$, ie be a map $E^s_{p,q} \to E^{s-\omega(s,p,q)}_{p,q}$. Although ω is a function $\omega(s,p,q,s_0,p_0,q_0)$, the arguments s_0,p_0,q_0 are often left out, since u is fixed.

Definition 4.6. For a non-linear operator $\mathscr{N}$, a linearisation L_u with parameter domain $\mathbb{D}(L_u) \supset \mathbb{D}(\mathscr{N})$ is said to be *moderate* if, for every u in an arbitrary space $E^{s_0}_{p_0,q_0}$ in $\mathbb{D}(\mathscr{N})$,

$$\omega_{\max} := \sup_{\mathbb{D}(L_u)\times\mathbb{D}(\mathscr{N})} \omega(s,p,q,s_0,p_0,q_0) < \infty. \tag{4.12}$$

In case there is some (s_0,p_0,q_0) in $\mathbb{D}(\mathscr{N})$ such that $\sup_{(s,p,q)\in\mathbb{D}(L_u)} \omega(s,p,q) < \infty$, then L_u is said to be *moderate on* $E^{s_0}_{p_0,q_0}$.

When A is a linear operator of order $d_A(s,p,q)$ and with parameter domain $\mathbb{D}(A)$, then L_u is said to be A*-moderate* on $E^{s_0}_{p_0,q_0}$ if

$$\omega(s,p,q,s_0,p_0,q_0) < d_A(s,p,q) \tag{4.13}$$

holds for all (s,p,q) in $\mathbb{D}(A)\cap\mathbb{D}(L_{u_0})$.

Similarly $\mathscr{N}$ is called A-moderate on $E^{s_0}_{p_0,q_0}$ in $\mathbb{D}(A)\cap\mathbb{D}(\mathscr{N})$ if (4.13) holds for $(s,p,q) = (s_0,p_0,q_0)$, for since $-L_u u = \mathscr{N}(u)$ holds at (s_0,p_0,q_0) it is trivial that $\mathscr{N}$ is a map $E^{s_0}_{p_0,q_0} \to E^{s_0-\omega(s_0,p_0,q_0)}_{p_0,q_0} \subset E^{s_0-d_A(s_0,p_0,q_0)}_{p_0,q_0}$.

Moderate linearisations could also be described as those that, regardless of the linearisation point u, have bounded orders on their entire parameter domains. A-moderacy of L_u is not an intrinsic property of the non-linearity in the sense that in practice it depends on the linearisation point u; cf the example in (2.15)-(2.16). It is clear that $\mathscr{N}$ is A-moderate on $E^{s_0}_{p_0,q_0}$ if L_u is so. Without linearisations one can define $\mathscr{N}$ to be A-moderate on $E^s_{p,q}$ in $\mathbb{D}(A)\cap\mathbb{D}(\mathscr{N})$ if $\mathscr{N}$ is a map $E^s_{p,q} \to E^{s-\sigma}_{p,q}$ for some $\sigma < d_A$.

This general framework is exemplified by the below theorem and its corollaries. The theorem relies on known estimates of paramultiplication, and it typically leads to operators in $\operatorname{OP}(S^{\omega}_{1,1}(\mathbb{R}^n\times\mathbb{R}^n))$; cf Theorem 4.13 below.

The fact that only product type operators are treated here allows two improvements of the usual linearisation theory in, say [Bon81] and [MC97, Th 16.3]: first of all, the π_2-terms are incorporated into L_u, which is indispensable since, as explained in the introduction, they are not regularising in the present context. Secondly, the mere existence of the operator family L_u is obtained under the very mild assumption $u \in \bigcup B^s_{p,q}$, and it is only for sufficiently regular u, namely in the quadratic standard domain (where $-L_u u = Q(u)$ is a meaningful formula), that L_u serves as a linearisation of Q.

Theorem 4.7 (The Exact Paralinearisation Theorem)**.** *Let $B(v,w)$ be of product type (d_0,d_1,d_2) with $d_0 \le d_1$; and let ℓ_Ω be a universal extension from Ω to $\mathbb{R}^n$.*

Whenever $u \in B^{s_0}_{p_0,q_0}(\overline{\Omega})$ for arbitrary parameters (s_0,p_0,q_0), then the exact paralinearisation in Definition 4.2 yields a linear operator L_u with parameter domain $\mathbb{D}(L_u)$ given by

$$s > d_0 + d_1 - s_0 + (\tfrac{n}{p} + \tfrac{n}{p_0} - n)_+. \tag{4.14}$$

L_u is of constant order ω, ie $L_u\colon B^s_{p,q}(\overline{\Omega}) \to B^{s-\omega}_{p,q}(\overline{\Omega})$, when $(s,p,q) \in \mathbb{D}(L_u)$ and

$$\begin{aligned}\omega = d_2 + d_1 + (\tfrac{n}{p_0} - s_0 + d_0)_+ \\ +\varepsilon[\![\tfrac{n}{p_0} - s_0 + d_0 = 0]\!][\![q_0 > 1]\!], \quad \text{any} \quad \varepsilon > 0.\end{aligned} \tag{4.15}$$

In particular, when $Q(u) := B(u,u)$ and $(s_0,p_0,q_0) \in \mathbb{D}(Q)$, ie fulfils (4.8)*, then L_u is a* moderate linearisation *of Q.*

Corresponding results hold for Triebel–Lizorkin spaces when $u \in F^{s_0}_{p_0,q_0}(\overline{\Omega})$, provided the factor $[\![q_0 > 1]\!]$ in (4.15) *is replaced by $[\![p_0 > 1]\!]$.*

The proof of this result is postponed to Section 4.3 below. It should be noted that the order $\omega(s,p,q,s_0,p_0,q_0)$ not only is independent of (s,p,q), but also gives back the function $\sigma(s,p,q)$ from (4.10) for $(s,p,q) = (s_0,p_0,q_0)$, so σ is the restriction of ω to the diagonal in $\mathbb{D}(B)$.

To shed light on (4.14), one could consider an elliptic problem (A,T), say with A of order $2m$, T of class m and a solution $u \in H^m(\overline{\Omega})$, with $(m,2) \in \mathbb{D}(Q)$, of

$$Au + Q(u) = f \quad \text{in} \quad \Omega \tag{4.16}$$

$$Tu = \varphi \quad \text{on} \quad \Gamma. \tag{4.17}$$

According to (4.14), $\mathbb{D}(L_u)$ then consists of parameters (s,p,q) with

$$s > \frac{d_0+d_1}{2} + (\frac{n}{p} - \frac{n}{2})_+ - (m - \frac{d_0+d_1}{2}), \tag{4.18}$$

so that $\mathbb{D}(L_u)$ is obtained from the quadratic standard domain $\mathbb{D}(Q)$ in (4.8) simply by a downward shift given by the last parenthesis. Therefore $\mathbb{D}(L_u) \supset \mathbb{D}(Q)$, which also holds in general when $(s_0,p_0,q_0) \in \mathbb{D}(Q)$.

The following result is an immediate consequence of (4.15), but it is useful to have easy-to-apply criteria for A-moderacy: since $d_1 \ge d_0$ it is natural first to suppress $P_0(D)$ and ask whether $d_A > d_1 + d_2$, cf (i) below.

Corollary 4.8. *Let $B = P_2(D)\pi_\Omega(P_0(D)\cdot, P_1(D)\cdot)$ be of product type d_0, d_1, d_2 with $d_0 \le d_1$, and let A be linear, of constant order d_A on a parameter domain $\mathbb{D}(A)$. For small $\varepsilon > 0$, the corresponding quadratic operator Q is A-moderate on $E^{s_0}_{p_0,q_0}$ in $\mathbb{D}(A) \cap \mathbb{D}(Q)$ if the conjunction of* (i) *and either* (ii) *or* (iii) *holds:*

(i) $d_A > d_2 + d_1$,
(ii) $d_A > d_2 + d_1 + d_0 + \frac{n}{p_0} - s_0$,
(iii) $d_1 - d_0 \ge n$.

The exact paralinearisation L_u is A-moderate on $E^{s_0}_{p_0,q_0}$ when (i) *and* (ii) *hold.*

Proof. Given (i) and (ii), one has $d_A - d_2 - d_1 > (\frac{n}{p_0} - s_0 + d_0)_+ \ge 0$. So by taking $\varepsilon \in \,]0, d_A - d_2 - d_1[$, clearly $d_A > \omega$ and L_u is moderate on $E^{s_0}_{p_0,q_0}$.

Since $Q(u) = -L_u u$, when u is in any space in $\mathbb{D}(Q) \cap \mathbb{D}(A)$, the above applies also to Q. If (iii) holds, it is clear for both $p_0 < 2$ and $p_0 \ge 2$ that

$$\tfrac{1}{2}(d_0 + d_1) + (\tfrac{n}{p_0} - \tfrac{n}{2})_+ \ge \tfrac{n}{p_0} + d_0. \tag{4.19}$$

The borderline of $\mathbb{D}(Q)$ is given by the left hand side, so $s_0 > \frac{n}{p_0} + d_0$, and $\omega = d_2 + d_1$ because the other terms in (4.15) vanish. Hence Q is A-moderate when (i) holds too. □

One interest of (iii) is that when $d_1 - d_0 \geq n$ and (i) hold, then Q is A-moderate on the entire domain $\mathbb{D}(Q) \cap \mathbb{D}(A)$. In cases with $d_1 - d_0 < n$, there always is a part of the quadratic standard domain $\mathbb{D}(Q)$ where (ii) must be imposed. Indeed, the last two terms in (4.15) contributes to the value of ω in the *slanted slice* of $\mathbb{D}(Q)$ given by

$$\tfrac{1}{2}(d_0 + d_1) + (\tfrac{n}{p} - \tfrac{n}{2})_+ < s \leq \tfrac{n}{p} + d_0, \tag{4.20}$$

and this region is non-empty precisely for $d_1 - d_0 < n$. Hence $\omega > d_2 + d_1$ in this slice.

In other words, $d_A > d_2 + d_1$ is a first criterion for Q to be A-moderate. Then, if $d_1 - d_0 > n$ the domain $\mathbb{D}(A, Q)$ of A-moderacy equals $\mathbb{D}(A) \cap \mathbb{D}(Q)$, and otherwise it is obtained from $\mathbb{D}(A) \cap \mathbb{D}(Q)$ restricting to

$$s > \tfrac{n}{p} - d_A + d_0 + d_1 + d_2. \tag{4.21}$$

This rephrasing of (ii) is of course of great practical importance.

Remark 4.9. One could compare the stationary Navier–Stokes problem (or just (1.2)) with the von Karman problem treated in Section 5 below. They are both fulfil $d_1 - d_0 \leq 1 < n$. In the former problem (ii) is felt, and the quadratic term is only Δ_{γ_0}-moderate on the part of $\mathbb{D}(Q) \cap \mathbb{D}_1$ where additionally $s > \frac{n}{p} - 1$, by (ii) or (4.21). (For the Neumann condition, (ii) gives again $s > \frac{n}{p} - 1$, that now should be imposed on the smaller region $\mathbb{D}(Q) \cap \mathbb{D}_2$ because the boundary condition has class 2.) But in the von Karman problem (ii) is not felt, for it is fulfilled on all of the quadratic standard domain of the form $[\cdot,\cdot]$, and even after this has been extended to the $B(\cdot,\cdot)$ of type $1,1,2$ given in Example 3.1, it *still* holds that $\omega < 4 = d_{\Delta^2}$ on all of $\mathbb{D}(Q)$. But nevertheless a small portion of $\mathbb{D}(Q)$ must be disregarded to have Δ^2-moderacy, simply because the boundary condition in the Dirichlét realisation of Δ^2 is felt; cf Figure 3 below. In view of these observations, Corollary 4.8 is probably the only condition for A-moderacy that is worthwhile working out in general.

Remark 4.10. Concerning the model problem (1.2) and Example 2.1, where $d_0 = 0$, $d_1 = 1$ and $d_A = 2$, the above (4.8) leads to the quadratic standard domains in (1.21) and (2.15). Notice that the more important domains $\mathbb{D}(\Delta_{\gamma_0}, Q)$ and $\mathbb{D}(A, \mathscr{N})$ in (1.25) and (2.16) are obtained from the conjunction of (4.8) and (ii) (the latter is redundant for $n = 2$ and $n = 3$). Similarly (2.17) follows from (4.14).

4.3. **Proof of Theorem 4.7.** The following arguments reexploit the L_p-estimates of paradifferential operators in Yamazaki's work [Yam86a]. However, they are only needed in the paramultiplicative setting, where [Joh95a, Thm. 5.1] contains a catalogue, that is used freely below. Since the nature of the proof is well known, the formulation will be brief.

In the following (s_1, p_1, q_1) is arbitrary in $\mathbb{D}(L_u)$, ie together with the given (s_0, p_0, q_0) it belongs to $\mathbb{D}(B)$. Since (4.6) holds, it follows from [Joh95a, Th 5.1] that, if $\frac{1}{p_2} := \frac{1}{p_0} + \frac{1}{p_1}$ and $\frac{1}{q_2} := \frac{1}{q_0} + \frac{1}{q_1}$,

$$\pi_2(\cdot,\cdot)\colon B^{s_0-d_0}_{p_0,q_0} \oplus B^{s_1-d_1}_{p_1,q_1} \to B^{s_0-d_0+s_1-d_1}_{p_2,q_2} \tag{4.22}$$

is bounded. It is therefore seen that

$$\pi_2(P_0(D)\ell_\Omega\cdot,P_1(D)\ell_\Omega\cdot)\colon B^{s_0}_{p_0,q_0}(\overline{\Omega})\oplus B^{s_1}_{p_1,q_1}(\overline{\Omega})\to B^{s_0-d_0+s_1-d_1}_{p_2,q_2} \hookrightarrow B^{s_1-d_1-(\frac{n}{p_0}-s_0+d_0)}_{p_1,q_1}. \tag{4.23}$$

The π_1-term in L_u is straightforward to treat for $s_0-d_0<\frac{n}{p_0}$: under the assumption that the first space has strictly negative smoothness index, which in this case may be obtained by use of the embedding $B^{s_0-d_0}_{p_0,q_0}\hookrightarrow B^{s_0-d_0-\frac{n}{p_0}}_{\infty,\infty}$, there are analogous estimates for π_1, which for $\varepsilon_0=0$ gives

$$\pi_1(P_0(D)\ell_\Omega\cdot,P_1(D)\ell_\Omega\cdot)\colon B^{s_0}_{p_0,q_0}(\overline{\Omega})\oplus B^{s_1}_{p_1,q_1}(\overline{\Omega})\to B^{s_1-d_1-(\frac{n}{p_0}-s_0+d_0)_+-\varepsilon_0}_{p_1,q_1}. \tag{4.24}$$

In the same manner one has for $s_0-d_1<\frac{n}{p_0}$ and $\varepsilon_1=0$ that

$$\pi_3(P_0(D)\ell_\Omega\cdot,P_1(D)\ell_\Omega\cdot)\colon B^{s_1}_{p_1,q_1}(\overline{\Omega})\oplus B^{s_0}_{p_0,q_0}(\overline{\Omega})\to B^{s_1-d_0-(\frac{n}{p_0}-s_0+d_1)_+-\varepsilon_1}_{p_1,q_1}. \tag{4.25}$$

For $s_0-d_0>\frac{n}{p_0}$ one may recall, eg from [Joh95a, Th 5.1], the estimate

$$\pi_1(\cdot,\cdot)\colon L_\infty\oplus B^{s_1}_{p_1,q_1}\to B^{s_1}_{p_1,q_1}; \tag{4.26}$$

it clearly yields the conclusion in (4.24) with $\varepsilon_0=0$. The term with π_3 may be treated analogously for $s_0-d_1>\frac{n}{p_0}$, leading to (4.25) once again. For $s_0-d_j=\frac{n}{p_0}$ one can use (4.24) and (4.25) at the expense of some $\varepsilon_j>0$ fulfilling $0<\varepsilon_1<d_1-d_0$, or $\varepsilon_0=\varepsilon_1$ if $d_1=d_0$. This is unless $q_0\le 1$ for then the embedding into L_∞ applies.

Comparing the three estimates (incl. the ε-modifications), (4.23) is the same as (4.24), except when $\frac{n}{p_0}-s_0+d_0\le 0$, but in this case $B^{s_1-d_1}_{p_1,q_1}$ or $B^{s_1-d_1-\varepsilon_1}_{p_1,q_1}$ in (4.24) clearly contains the space on the right hand side of (4.23). Similarly the co-domain of (4.25) equals the last space in (4.23), except for $\frac{n}{p_0}-s_0+d_1\le 0$, but then the assumption that $d_0\le d_1$ yields that also $\frac{n}{p_0}-s_0+d_0\le 0$ so that there is an embedding into the corresponding space in (4.24). Regardless of whether $(\frac{n}{p_0}-s_0+d_j)_+$ equals 0 for none, one or all j in $\{0,1\}$, it follows that L_u is a bounded linear operator

$$L_u\colon B^{s_1}_{p_1,q_1}\to B^{s_1-\omega}_{p_1,q_1}, \tag{4.27}$$

when ω is as in (4.15) and (s_1,p_1,q_1) fulfils (4.6).

In the Triebel–Lizorkin case the above argument works with minor modifications. First of all, by Lemma 1.3 boundedness of

$$\pi_2(\cdot,\cdot)\colon F^{s_0-d_0}_{p_0,q_0}\oplus F^{s_1-d_1}_{p_1,q_1}\to F^{s_0-d_0+s_1-d_1}_{p_2,t} \tag{4.28}$$

holds for all sufficiently large t, when only $s_0-d_0+s_1-d_1>(\frac{n}{p_2}-n)_+$. Then $F^{s_0-d_0+s_1-d_1}_{p_2,t}\hookrightarrow F^{s_1-d_1-(\frac{n}{p_0}-s_0+d_0)}_{p_1,q_1}$ yields an analogue of (4.23).

Secondly, for $s_0-d_0<\frac{n}{p_0}$, one has for $r<0$

$$\pi_1(\cdot,\cdot)\colon B^{r}_{\infty,\infty}\oplus F^{s_1}_{p_1,q_1}\to F^{s_1+r}_{p_1,q_1}. \tag{4.29}$$

Combining this with $F^{s_0-d_0}_{p_0,q_0}\hookrightarrow B^{s_0-d_0-\frac{n}{p_0}}_{\infty,\infty}$, formula (4.24) is carried over to the Triebel–Lizorkin case. Otherwise one may proceed as in the Besov case. Thereby the theorem is proved.

4.4. **Boundedness in a borderline case.** In the last cases given by (3.16) it is more demanding to estimate L_u. For later reference a first result on such extensions of $\mathbb{D}(L_u)$ is sketched, using techniques from a joint work with W. Farkas and W. Sickel [FJS00], where approximation spaces $A^s_{p,q}$ (that go back to S. M. Nikolskiĭ) were useful for the borderline investigations.

Recall that $A^s_{p,q}(\mathbb{R}^n)$ for $s \ge (\frac{n}{p}-n)_+$, $p,q \in]0,\infty]$ (with $q \le 1$ for $s = \frac{n}{p}-n$), consists of the $u \in \mathscr{S}'(\mathbb{R}^n)$ that have an $\mathscr{S}'$-convergent decomposition $u = \sum_{j=0}^\infty v_j$ fulfilling $v_j \in \mathscr{S}' \cap L_p$, $\operatorname{supp} \hat{v}_j \subset \{\, |\xi| \le 2^{j+1}\,\}$ and

$$(\sum_{j=0}^\infty 2^{sjq} \| v_j \,|L_p\|^q)^{\frac{1}{q}} < \infty. \tag{4.30}$$

The quasi-norm of u in $A^s_{p,q}$ is then the infimum over these numbers, as one runs through the set of all such decompositions.

The idea of [FJS00] is that, while the dyadic ball criterion cannot yield convergence for $s = \frac{n}{p} - n$ (cf the remarks preceeding Lemma 1.6), one can sometimes nevertheless show directly that such a series $\sum v_j$ converges to some u in L_1 or $\mathscr{S}'$; then the finiteness of the above number gives $\sum v_j \in A^s_{p,q}$.

Theorem 4.11. *Let $B = \pi_\Omega(P_0(D)\cdot, P_1(D)\cdot)$ with $d_0 \le d_1$ and let $u \in B^{s_0}_{p_0,q_0}(\overline{\Omega})$ be fixed. For (s,p,q) such that*

$$s_0 + s = d_0 + d_1 + (\tfrac{n}{p_0} + \tfrac{n}{p} - n)_+, \qquad \tfrac{1}{q_2} := \tfrac{1}{q_0} + \tfrac{1}{q} \ge 1 \tag{4.31}$$

the operator L_u is continuous

$$L_u \colon B^s_{p,q}(\overline{\Omega}) \to B^{s-\omega(s,p,q)}_{p,\infty}(\overline{\Omega}). \tag{4.32}$$

provided, in case $\frac{1}{p_2} := \frac{1}{p_0} + \frac{1}{p} > 1$, that either $p_2 \ge q_2$ or $p \ge 1$ holds.

Moreover, $L_u \colon F^{s_0}_{p_0,q_0}(\overline{\Omega}) \to B^{s-\omega(s,p,q)}_{p,\infty}(\overline{\Omega})$ is continuous if $u \in F^{s_0}_{p_0,q_0}(\overline{\Omega})$, provided $[\![q_0 > 1]\!]$ in (4.15) *is replaced by $[\![p_0 > 1]\!]$ (no restrictions for $p_2 < 1$).*

Proof. With notation as in the proof of Theorem 4.7, the assumption $q_2 \le 1$ gives $\ell_{q_2} \hookrightarrow \ell_1$, so for $p_2 \ge 1$ insertion of $1 = 2^{s_0 - d_0 + s_1 - d_1}$ into a double application of Hölder's inequality shows that the series defining $\pi_2(P_0(D)\ell_\Omega\cdot, P_1(D)\ell_\Omega\cdot)$ converges absolutely in L_{p_2}. There is a Sobolev embedding $L_{p_2} \hookrightarrow B^{\tilde{s}}_{p_1,\infty}$ for $\tilde{s} = s_1 - d_1 - (\frac{n}{p_0} - s_0 + d_0)$, since $p_1 \ge p_2$, so the conclusion of (4.23) holds with the modification that the sum-exponent is ∞ in this case.

For $p_2 < 1$ one uses the Nikolskiĭ–Plancherel–Polya inequality to estimate L_1-norms by $2^{\frac{n}{p_2}-n} = 2^{s_0+s_1-d_0-d_1}$ times corresponding L_{p_2}-norms, leading to convergence in L_1. After this convergence has been established, the same estimates also give the strengthened conclusion that

$$\pi_2(P_0(D)\ell_\Omega\cdot, P_1(D)\ell_\Omega\cdot) \colon B^{s_0}_{p_0,q_0} \oplus B^{s_1}_{p_1,q_1} \to A^{\frac{n}{p_2}-n}_{p_2,q_2}, \tag{4.33}$$

for approximation spaces $A^s_{p,q}$ defined on $\mathbb{R}^n$ as in [FJS00]. By [FJS00, Th 6] the conjunction of $r \ge \max(p_2, q_2)$ and $o = \infty$ is equivalent to

$$A^{\frac{n}{p_2}-n}_{p_2,q_2} \hookrightarrow B^{\frac{n}{r}-n}_{r,o}. \tag{4.34}$$

It follows that $\pi^{12}_{2\Omega}(u,\cdot) = r_\Omega \pi_2(P_0(D)\ell_\Omega u, P_1(D)\ell_\Omega\cdot)$ maps $B^{s_1}_{p_1,q_1}(\overline{\Omega})$ continuously into $B^{\frac{n}{p_2}-n}_{p_2,\infty}(\overline{\Omega})$ for $p_2 \ge q_2$, hence into $B^{\tilde{s}}_{p_1,\infty}(\overline{\Omega})$ as desired; for $p_1 \ge 1$ the same conclusion is reached directly from the L_1-estimate above.

Since (4.24) and (4.25) also hold in the present context, and since this implies weaker statements with the sum-exponents equal to ∞ on the right hand sides there, L_u has the property in (4.27) except that the co-domain should be $B^{s_1-\omega}_{p_1,\infty}$.

For the $F^s_{p,q}$-spaces the estimates of $\pi^{12}_{2\Omega}(u,\cdot)$ are derived in the same way, except that the ℓ_{q_2}-norms are calculated pointwisely, before the L_{p_2}-norms. Indeed, for $p_2 \geq 1$, Lemma 1.6 gives (since $q_2 \leq 1$ in this case) that $\pi_2(P_0(D)\ell_\Omega\cdot, P_1(D)\ell_\Omega\cdot)$ maps $F^{s_0}_{p_0,q_0} \oplus F^{s_1}_{p_1,q_1}$ to L_{p_2}: for $p_2 > 1$ this co-domain is embedded via $F^{\tilde{s}}_{p_1,q_1}$ into $B^{\tilde{s}}_{p_1,\infty}$, while $L_{p_2} \hookrightarrow B^0_{1,\infty} \hookrightarrow B^{\tilde{s}}_{p_1,\infty}$ for $p_2 = 1$.

For $p_2 < 1$ one finds by the vector-valued Nikolskiĭ–Plancherel–Polya inequality in Lemma 1.7 that eg (when $f_k := \Phi_k(D)f$ etc on $\mathbb{R}^n$)

$$\| \sum_{k=0}^{\infty} |f_k g_k| \|_1 \leq c \| (\sum_{k=0}^{\infty} 2^{k(\frac{n}{p_2}-n)q_2} |f_k g_k|^{q_2})^{\frac{1}{q_2}} \|_{p_2} \leq c' \| f \,|F^{s_0-d_0}_{p_0,q_0}\| \, \| g \,|F^{s_1-d_1}_{p_1,q_1}\|. \tag{4.35}$$

In this way $\pi^{12}_{2\Omega}(u,\cdot)$ is shown to map $F^{s_1}_{p_1,q_1}$ into $L_1(\Omega)$. Hence into $B^{\tilde{s}}_{p_1,\infty}(\overline{\Omega})$ for $p_1 \geq 1$. In general there is $p_3 \in \,]p_2, p_1[\,$ ($p_0 < \infty$) and the $A^{\frac{n}{p_3}-n}_{p_3,p_3}$-norm of $\pi^{12}_{2\Omega}(u,v)$ is estimated by an $L_{p_2}(\ell_{q_2})$-norm as in the middle of (4.35), for the sum and integral may be excanged and the estimate realised through Lemma 1.7. By (4.34)–(4.35) this means that $\pi^{12}_{2\Omega}(u,\cdot)$ maps $F^{s_1}_{p_1,q_1}$ into $B^{\frac{n}{p_3}-n}_{p_3,\infty} \hookrightarrow B^{\tilde{s}}_{p,\infty}$ for $p_2 < 1$. Comparison with the $F^s_{p,q}$-results for the other terms shows that $L_u \colon F^{s_1}_{p_1,q_1} \to B^{\tilde{s}}_{p_1,\infty}$. $\square$

The above result suffices for the present paper, but it could probably be sharpened in several ways, perhaps with a consistent use of $A^s_{p,q}$ as co-domains.

4.5. **Relations to pseudo-differential operators of type** $1,1$**.** For the local regularity improvements later, it is convenient to express paralinearisations via pseudo-differential operators with symbols in $S^d_{1,1}$. Recall that $a(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ belongs to $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ for $d \in \mathbb{R}$, if it for all multiindices α, β and x, $\xi \in \mathbb{R}^n$ satisfies

$$|D^\beta_x D^\alpha_\xi a(x,\xi)| \leq c_{\alpha\beta} \langle \xi \rangle^{d-|\alpha|+|\beta|}. \tag{4.36}$$

The operator $a(x,D)\varphi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\mathrm{i}x\cdot\xi} a(x,\xi) \hat{\varphi}(\xi)\, d\xi$ is obviously continuous $S^d_{1,1} \times \mathscr{S} \to \mathscr{S}$ with respect to the Frechét topologies. In general $A := a(x,D)$ is a linear operator in $\mathscr{S}'(\mathbb{R}^n)$ defined on some subspace $D(A) \subset \mathscr{S}'(\mathbb{R}^n)$; the definition may be made by a paradifferential splitting in three terms, analogous to (3.13). This was done implicitly in [Mey81, Th 2-3], and a detailed description can be found in [Joh04b, Joh04a].

While it is yet only partially understood what $D(A)$ is, Hörmander [Hör97, Ch 9.3] determined (up to a limit point) the s for which A extends to a continuous map $H^{s+d}_2 \to H^s_2$. Eg continuity for all $s \in \mathbb{R}$ is proved there for $a(x,\xi)$ satisfying the twisted diagonal condition. However, recently it was proved by the author [Joh04b, Joh04a] that there always are bounded extensions, for $1 \leq p < \infty$,

$$F^d_{p,1}(\mathbb{R}^n) \xrightarrow{a(x,D)} L_p(\mathbb{R}^n), \qquad B^d_{\infty,1}(\mathbb{R}^n) \xrightarrow{a(x,D)} L_\infty(\mathbb{R}^n), \tag{4.37}$$

and that, without further knowledge about $a(x,\xi)$, this is optimal within the $B^s_{p,q}$ and $F^s_{p,q}$ scales for $p < \infty$. For $s > (\frac{n}{p} - n)_+$ there is continuity

$$B^{s+d}_{p,q}(\mathbb{R}^n) \xrightarrow{a(x,D)} B^s_{p,q}(\mathbb{R}^n), \qquad F^{s+d}_{p,q}(\mathbb{R}^n) \xrightarrow{a(x,D)} F^s_{p,r}(\mathbb{R}^n) \quad (r \text{ as in (1.39)}). \tag{4.38}$$

This extends to all $s\in\mathbb{R}$ under the twisted diagonal condition; cf [Joh04a, Cor. 6.2]. The reader may consult [Hör97] for a broader account of the $S^d_{1,1}$-theory.

The just mentioned extension results will not be directly used here, but they shed light on the difficulties met in connection with the *pseudo-local* property:

$$\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u, \quad u\in D(A). \tag{4.39}$$

For type $1,1$ operators, this question seems to be unsettled (eg because $D(A)$ is not fully determined). But for now it suffices to have (4.39) for certain u with compact support, so the next quasi-general result will do.

To formulate it, $D(A)$ will be considered with the graph topology of A, ie the unique topology that makes the map $u\mapsto(u,Au)$ a homeomorphism $D(A)\to G(A)$ where $G(A)$ is the graph of A (topologised as a subspace of $\mathscr{S}'\times\mathscr{S}'$).

Proposition 4.12. *A pseudo-differential operator $a(x,D)$ in $\operatorname{OP}(S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n))$ fulfils* (4.39) *for every $u\in D(A)$ having the two properties: u is in the graph topology closure of $\mathscr{S}(\mathbb{R}^n)$ and $\chi u_k\to\chi u$ in $D(A)$ for every $\chi\in C_0^\infty(\mathbb{R}^n)$ and every sequence u_k in $\mathscr{S}(\mathbb{R}^n)$ converging to u in $D(A)$.*

The point is that in $D(A)$ such u are not too far away from $\mathscr{S}(\mathbb{R}^n)$ (while eg $v\equiv 1$ is in $B^d_{\infty,1}\subset D(A)$, cf (4.37), without being a limit point of $\mathscr{S}$ in $B^d_{\infty,1}$). Usually the rules of calculus for $S^d_{\rho,\delta}$, $\delta<\rho$, $\delta<1$, enter the proof of pseudo-locality, cf [Hör85, Ch. 18.1], but this can be avoided under the present assumptions:

Proof. The distribution kernel $K_a(x,y)$ is C^∞ for $x\neq y$. Indeed, by integration $K_a(x,y)=(2\pi)^{-n}\mathscr{F}_{\xi\to z}a(x,\xi)\,\big|_{z=y-x}$, since $S^{-\infty}\subset S^d_{1,1}$ is dense; because the function $|z|^{2N}D^\beta_xD^\alpha_z\mathscr{F}_{\xi\to z}a(x,\xi)=\mathscr{F}_{\xi\to z}(\Delta^N_\xi(\xi^\alpha D^\beta_x a(x,\xi)))$ is continuous for N so large that $d+|\beta|+|\alpha|-2N<-n$, any derivative of K_a is so for $x\neq y$.

Let $\psi,\chi\in C_0^\infty(\mathbb{R}^n)$ have supports disjoint from $\operatorname{sing\,supp} u$ such that $\chi\equiv 1$ on a neighbourhood of $\operatorname{supp}\psi$. Then $\chi u\in C_0^\infty(\mathbb{R}^n)$ so that both χu, $(1-\chi)u$ are in $D(A)$, and

$$\psi Au=\psi A(\chi u)+\psi A(1-\chi)u. \tag{4.40}$$

Here $\psi A(\chi u)\in C^\infty(\mathbb{R}^n)$ since $A\colon\mathscr{S}\to\mathscr{S}$; the last term has distribution kernel

$$K(x,y)=\psi(x)K_a(x,y)(1-\chi(y)), \tag{4.41}$$

which is C^∞ since K_a is so for $x\neq y$. Moreover, $K\in\mathscr{S}(\mathbb{R}^{2n})$ since

$$(1+|y|)^{2N}\le(1+|y-x|)^{2N}(1+|x|)^{2N} \tag{4.42}$$

so the formula with $|z|^{2N}$ yields rapid decay with respect to y ($x\in\operatorname{supp}\psi\Subset\mathbb{R}^n$).

Now $\psi A(1-\chi)u\in C^\infty(\mathbb{R}^n)$ will follow in a standard way, by combining that $u\in\mathscr{S}'(\mathbb{R}^n)$ and $K\in\mathscr{S}(\mathbb{R}^{2n})$ with the formula

$$\psi(x)A(1-\chi)u(x)=\langle\, u,\,K(x,\cdot)\,\rangle. \tag{4.43}$$

But even though both sides make sense, this identity needs justification. With $u_k\in\mathscr{S}$ such that $u_k\to u$ in $D(A)$, the assumption gives $\chi u_k\to\chi u$ in $\mathscr{S}'$ and $A(\chi u_k)\to A(\chi u)$ in $\mathscr{S}'$. Combining this, $A((1-\chi)u)=\lim_k A((1-\chi)u_k)$, whence

$$\psi A(1-\chi)u=\lim_k\int_{\mathbb{R}^n}u_k(y)K(x,y)\,dy=\langle\, u,\,K(x,\cdot)\,\rangle. \tag{4.44}$$

All in all this shows that ψAu is C^∞ on $\mathbb{R}^n\setminus\operatorname{sing\,supp} u$. ∎

The above results give the relation of paralinearisations to pseudo-differential operators of type $1,1$ and that they are pseudo-local. The last property is obtained only when the universal extension maps into $\mathscr{E}'(\mathbb{R}^n)$, but one can always multiply ℓ_Ω by a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on a neighbourhood of $\overline{\Omega}$.

Theorem 4.13. *Let $B(v,w)$ be of product type and $u \in B^{s_0}_{p_0,q_0}(\overline{\Omega})$ for some arbitrary (s_0,p_0,q_0) and suppose ℓ_Ω has range in $\mathscr{E}'(\mathbb{R}^n)$. Then the exact paralinearisation in* (4.2) *factors through a $P_u \in \operatorname{OP}(S^{\omega}_{1,1}(\mathbb{R}^n\times\mathbb{R}^n))$ with ω as in* (4.15). *That is, for every (s,p,q) in $\mathbb{D}(L_u)$ there is a commutative diagram*

$$\begin{array}{ccc} E^s_{p,q}(\overline{\Omega}) & \xrightarrow{\ \ell_\Omega\ } & E^s_{p,q}(\mathbb{R}^n) \\ {\scriptstyle L_u}\big\downarrow & & \big\downarrow{\scriptstyle P_u} \\ E^{s-\omega}_{p,q}(\overline{\Omega}) & \xleftarrow[\ r_\Omega\ ]{} & E^{s-\omega}_{p,q}(\mathbb{R}^n) \end{array} . \tag{4.45}$$

Moreover, $L_u g$ is pseudo-local on every $g \in E^s_{p,q}(\overline{\Omega})$ when (s,p,q) is in $\mathbb{D}(L_u)$.

Proof. By linearity, it suffices to treat $P_m(D) = D^{\eta_m}$ for $|\eta_m| = d_m$, and $d_0 \le d_1$. Given $u \in B^{s_0}_{p_0,q_0}$, let $\tilde u = \ell_\Omega u$. Then L_u is a composite $L_u = r_\Omega a(x,D)\ell_\Omega$ for a symbol $a(x,\xi)$ satisfying (4.36) for $d=\omega$ with ω as in (4.15), namely

$$a(x,\xi) = -\sum_{j=0}^{\infty}\big(\Psi_{j+1}(D_x)D_x^{\eta_0}\tilde u(x)\xi^{\eta_1} + \Psi_{j-2}(D_x)D_x^{\eta_1}\tilde u(x)\xi^{\eta_0}\big)\Phi_j(\xi) \tag{4.46}$$

Indeed, if L_u is applied to $g \in \mathscr{S}$ the formula for $a(x,\xi)$ follows directly from Definition 4.2. To prove that one may take $P_u = a(x,D)$, note that $a(x,\xi)$ is C^∞ since each ξ is in $\operatorname{supp}\Phi_j$ for at most two values of j, and for these $2^{j-1} \le |\xi| \le 2^{j+1}$, so that $|D^\alpha(\xi^{\eta_m}\Phi_j(\xi))| \le c\langle\xi\rangle^{d_m-|\alpha|}$ holds for all α. Concerning the estimates for $x \in \mathbb{R}^n$, note that when $k = j+1$ and $\varepsilon > 0$ is fixed, the convenient short hand $\varepsilon' := \varepsilon[\![\frac{n}{p_0} - s_0 + d_0 = 0]\!][\![q_0 > 1]\!]$ fulfils $\varepsilon' \ge 0$ and gives

$$|D_x^\beta \Psi_k(D)D^{\eta_0}\tilde u(x)| \le c\langle\xi\rangle^{|\beta|+(\frac{n}{p_0}-s_0+d_0)_+ + \varepsilon'}. \tag{4.47}$$

Indeed, for $q_0 \le 1$ the Nikolskiĭ–Plancherel–Polya inequality yields,

$$\begin{aligned} |\Psi_k(D)D^{\beta+\eta_0}\tilde u(x)| &\le c\sum_{l=0}^{k} 2^{l(s_0-|\beta+\eta_0|)}\,\|\,\Phi_l(D)D^{\beta+\eta_0}\tilde u\,|L_{p_0}\|\\ &\qquad\qquad\qquad\qquad \times 2^{l(|\beta|+(\frac{n}{p_0}-s_0+d_0)_+)} \\ &\le c\,\|\,u\,|B^{s_0}_{p_0,q_0}\|\langle\xi\rangle^{(\frac{n}{p_0}-s_0+d_0)_+ + |\beta|}; \end{aligned} \tag{4.48}$$

for $q_0 > 1$ one can apply Hölder's inequality to the first line in (4.48), after the factor $2^{k(\frac{n}{p_0}-s_0+d_0)_+ + k|\beta|}$ has been taken out in front of the summation (unless $\varepsilon' > 0$, in which case $|\beta|$ should have ε added and subtracted). Terms with $|\Psi_{j-2}(D)D^{\beta+\eta_1}\tilde u(x)|$ are estimated analogously, in the first line of (4.48) the factor $2^{l(s_0-|\beta|-d_0)}$ may be estimated by $2^{l(s_0-|\beta+\eta_1|)}$ (which is absorbed by the Besov norm on u) times $2^{j(d_1-d_0)}$; the latter, together with the estimate of $D^\alpha(\xi^{\eta_1}\Phi_j(\xi))$, gives the estimates in (4.36).

To prove (4.45) also for non-smooth functions, it is noted that the proof of the Paralinearisation Theorem, 4.7, yields that $P_u\colon E^s_{p,q}(\mathbb{R}^n) \to E^{s-\omega}_{p,q}(\mathbb{R}^n)$ is bounded for $(s,p,q) \in \mathbb{D}(L_u)$. This is seen as in (4.27) by keeping one entry in the bilinear expressions equal to $\tilde u$ while the other entry runs through $P_j(D)(E^s_{p,q}(\mathbb{R}^n))$, and by invoking the definition of P_u given after (4.36). From the definition of L_u it is then evident that $L_u = r_\Omega \circ P_u \circ \ell_\Omega$, hence (4.45) holds.

Finally, when g is given as in the theorem, then $x \in \operatorname{sing\,supp} \ell_\Omega g$ implies that $x \in \operatorname{sing\,supp} g \cup \mathbb{R}^n \setminus \Omega$. One can assume $q < \infty$, for s can be replaced by a slightly smaller value. Then the distribution $\ell_\Omega g$ is in the closure of C_0^∞ in the norm of $E^s_{p,q}$. (Since $q < \infty$, the truncated Littlewood–Paley decompositions converge to u in $E^s_{p,q}$, and they are C^∞ so multiplication by a cut-off function gives the claim.) By the continuity of P_u, this implies that $\ell_\Omega g$ is a limit point of $\mathscr{S}$ inside $D(P_u)$, and since multiplication by a test function is continuous in $E^s_{p,q}$, it is also so in $D(P_u)$. Therefore Proposition 4.12 gives that $\operatorname{sing\,supp} \ell_\Omega g$ is not enlarged by P_u, so $r_\Omega P_u \ell_\Omega g$ is C^∞ in the part of Ω where g is so. □

Remark 4.14. As indicated above, the definition, domain and basic continuity properties of type $1,1$ operators still need a further clarification. To avoid any ambiguity, the exact paralinearisations have been defined here without reference to these operators, and the Paralinearisation Theorem was for the same reason proved directly, before the factorisation through type $1,1$ operators was established.

Remark 4.15. One way to attempt a symbolic calculus would be to replace ℓ_Ω by e_Ω, ie by extension by zero outside of Ω. The resulting linearisation $\tilde{L}_u$ would have the form $\tilde{L}_u = r_\Omega P e_\Omega$ where P is in $\mathrm{OP}(S^\omega_{1,1}(\mathbb{R}^n \times \mathbb{R}^n))$ by Theorem 4.13. For $\tilde{L}_u$ to be moderate it would suffice to show that it has order ω in spaces with $s > 0$, so it would for a start be necessary to introduce further conditions in order that the two applications of e_Ω make sense, and secondly it is envisaged that the transmission property would be needed for P. However, transmission *conditions* have been worked out for $S^d_{\rho,\delta}$ with $\delta < 1$, cf [GH91], and there is for $\delta = 1$ a fundamental difficulty because $\mathrm{OP}(S^\omega_{1,1})$ in general, cf (4.37), is defined on H^s_p for $s > \omega > 0$ — whereas the usual induction proof of the continuity of truncated pseudo-differential operators with transmission property effectively requires application to spaces with $s < 0$ (in the induction step, $r_\Omega P$ is applied to distributions supported by $\Gamma \subset \mathbb{R}^n$). Furthermore, also composites like $(R_D L_u)^N$ should be covered, hence the general rules of composition with the operators in the Boutet de Monvel calculus should be established. All in all this is better investigated elsewhere; it would undoubtedly be useful, say in reductions where traces or solution operators of other problems are applied to the parametrix formula.

5. The von Karman equations of non-linear vibration

The preceding sections apply to the equations for a thin, buckling plate, initially filling an open domain $\Omega \subset \mathbb{R}^2$. The following is inspired by [Lio69, Ch. 1.4] and by the thourough treatise of P. G. Ciarlet [Cia97, Ch. 5]. An excerpt of von Karman's work [vK10] is conveniently found in [Cia97, p. lxiii].

In the stationary case the problem is to find two real-valued functions u_1 and u_2 (displacement and stress) defined in Ω and fulfilling

$$\Delta^2 u_1 - [u_1, u_2] = f \quad \text{in } \Omega \tag{5.1a}$$

$$\Delta^2 u_2 + [u_1, u_1] = 0 \quad \text{in } \Omega \tag{5.1b}$$

$$\gamma_k u_1 = 0 \quad \text{on } \Gamma \text{ for } k = 0, 1 \tag{5.1c}$$

$$\gamma_k u_2 = \psi_k \quad \text{on } \Gamma \text{ for } k = 0, 1. \tag{5.1d}$$

Hereby Δ^2 denotes the biharmonic operator, whilst $[\cdot,\cdot]$ as in Example 3.1 stands for the bilinear operator

$$[v,w] = D_1^2 v D_2^2 w + D_2^2 v D_1^2 w - 2D_{12}^2 v D_{12}^2 w. \tag{5.2}$$

For the real-valued case with $\psi_0 = \psi_1 = 0$, it is well known that Brouwer's fixed point theorem implies the existence of solutions with $u_j \in F^2_{2,2}(\overline{\Omega})$ for given data $f \in F^{-2}_{2,2}(\overline{\Omega})$; cf [Lio69, Thm. 4.3]. For $\psi_k \in F^{2-k-1/2}_{2,2}(\Gamma)$ solutions are established by non-linear minimisation in [Cia97, Th 5.8-3]. Concerning the regularity it was eg shown in [Lio69, Thm. 4.4] that if $f \in L_p(\Omega)$ for some $p > 1$, then any of the above solutions of (5.1) fulfils that $u_1 \in F^4_{p,2}(\overline{\Omega})$ while u_2 belongs to $F^4_{q,2}(\overline{\Omega})$ for any $q < \infty$. It was also noted in [Lio69] that reiteration would give more, eg that the problem is hypoelliptic. Corresponding results for non-trivial ψ_0 and ψ_1 may be found in Theorem 5.8-4 (and its proof) in [Cia97].

These results are generalised in three ways in the present paper, as a consequence of the general investigations: firstly the assumptions on the data and on the solution (u_1,u_2) are considerably weaker, including fully inhomogeneous data; secondly the weak solutions are carried over to a wide range of spaces with $p \neq 2$. Thirdly the non-linear terms are shown to have no influence on the solution's regularity (within the Besov and Triebel–Lizorkin scales).

In the discussion of the von Karman equation, the coupling of the two non-linear equations is a little inconvenient, since the Exact Paralinearisation Theorem, 4.7, needs a modification to this situation. But this can be done easily when u_1 and u_2 are given in the same space, for in the proof of Theorem 4.7 the mapping properties will then remain the same regardless of whether u_1 or u_2 is inserted in the various π_j-expressions. For brevity, it is left for the reader to substantiate this expansion of the theorem. (More general methods will be developed in Section 6 below.)

Because $[v,w]$ is of type $2,2$, the quadratic standard domain in (4.8) is for $Q_0(u) := [u,u]$ given by $s > 2 + (\frac{2}{p} - 1)_+$, and clearly $(s,p,q) = (2,2,2)$ is at the boundary of and therefore outside of $\mathbb{D}(Q_0)$; cf Figure 3. Hence Theorem 4.7 does barely not apply as it stands (cf the formulation below (4.15)).

In order to carry over the weak solutions to other spaces, one can use the more refined paraproduct estimates for the borderlines in Theorem 4.11. In fact the codomain of type $B_{p,\infty}$ is embedded into $E^{s-\omega-\varepsilon}_{p,q}$ for $\varepsilon > 0$, so this gives that $L_{(u_1,u_2)}$ has order $\omega = 3+\varepsilon$ when both (s_0,p_0,q_0) and (s,p,q) equal $(2,2,2)$. For other choices of (s,p,q) the continuity properties of $L_{(u_1,u_2)}$ are given by Theorem 4.7. In addition, the considerations in Lemma 4.3 show that $L_{(u_1,u_2)}$ linearises the non-linear terms, since (3.16) is fulfilled at $(2,2,2)$. In this way Theorem 2.2 can be used for the von Karman problem, when $\mathbb{D}(\mathscr{N})$ is taken as $\mathbb{D}(Q_0) \cup \{\,(2,2,2)\,\}$ and $\mathbb{D}(B_u)$ likewise consists of the union of $\mathbb{D}(L_{(u_1,u_2)})$ and $(2,2,2)$. (The parameter domains were not required to be open in Theorem 2.2.)

One could envisage other problems in which the weak solutions belong to spaces at the borderline of the quadratic standard domain, so that results like Theorem 4.11 would be the only manageable way to apply Theorem 2.2.

For the von Karman problem, however, the symmetry properties of $[v,w]$ make it possible to avoid the rather specialised estimates in Theorem 4.11. Indeed, as recalled in Example 3.1, $[\cdot,\cdot]$ is a restriction of

$$B(v,w) = D_{12}^2(D_1 v D_2 w + D_2 v D_1 w) - D_1^2(D_2 v D_2 w) - D_2^2(D_1 v D_1 w). \tag{5.3}$$

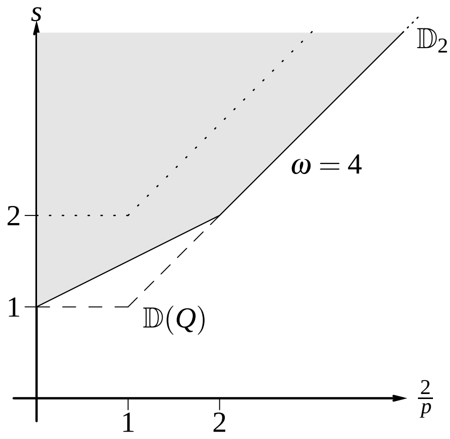

FIGURE 3. The quadratic standard domains of Q and Q_0 (in dots) in relation to $\mathbb{D}_2$.

Since B is of type $1,1,2$, the domain $\mathbb{D}(Q)$ is now given by $s > 1 + (\frac{2}{p} - 1)_+$ according to (4.8). But by (1.36) the appropriate parameter domain for the linear part is $\mathbb{D}_2$, and, cf Figure 3,

$$\mathbb{D}(Q) \cap \mathbb{D}_2 = \mathbb{D}_2. \tag{5.4}$$

On the resulting domain, $\mathbb{D}_2$, the operator Q is Δ^2-moderate in view of Corollary 4.8 ((i) and (iii) hold with $d_A = 4$, $d_2 = 2$ and $d_0 = d_1 = 1$). It is moreover easy to infer from (4.15) that $\omega = 4$ holds on the borderline with $s = 2/p$ (for $p < 1$) of $\mathbb{D}_2$.

This leads to the following result on the fully inhomogeneous problem:

Theorem 5.1. *Let two functions u_1, $u_2 \in B^s_{p,q}(\overline{\Omega})$ with (s,p,q) in $\mathbb{D}_2$ solve*

$$\Delta^2 u_1 - B(u_1,u_2) = f_1 \quad \text{in } \Omega \tag{5.5a}$$

$$\Delta^2 u_2 + B(u_1,u_1) = f_2 \quad \text{in } \Omega \tag{5.5b}$$

$$\gamma_k u_1 = \varphi_k \quad \text{on } \Gamma \text{ for } k = 0,\, 1 \tag{5.5c}$$

$$\gamma_k u_2 = \psi_k \quad \text{on } \Gamma \text{ for } k = 0,\, 1, \tag{5.5d}$$

for data $f_k \in B^{t-4}_{r,o}(\overline{\Omega})$, with $k = 1,\ 2$, together with φ_0, $\psi_0 \in B^{t-\frac{1}{r}}_{r,o}(\Gamma)$ and φ_1, $\psi_1 \in B^{t-1-\frac{1}{r}}_{r,o}(\Gamma)$ whereby $(t,r,o) \in \mathbb{D}_2 \cap \mathbb{D}(L_{(u_1,u_2)})$, that is

$$t > 1 + \tfrac{1}{r} + (\tfrac{1}{r} - 1)_+, \qquad t > 2 - s + (\tfrac{2}{r} + \tfrac{2}{p} - 2)_+. \tag{5.6}$$

Then u_1, u_2 belong to $B^t_{r,o}(\overline{\Omega})$.

If instead $f_k \in F^{t-4}_{r,o}(\overline{\Omega})$, φ_0, $\psi_0 \in B^{t-\frac{1}{r}}_{r,r}(\Gamma)$ and φ_1, $\psi_1 \in B^{t-1-\frac{1}{r}}_{r,r}(\Gamma)$ for some (t,r,o) fulfilling (5.6)*, then it follows that u_1, $u_2 \in F^t_{r,o}(\overline{\Omega})$.*

Since $\mathbb{D}_2$ is open, it is not a loss of generality here to assume for the Triebel–Lizorkin case that u_1 and u_2 are given in a Besov space.

One can prove the theorem directly, as indicated above, but it will follow from the general considerations in Section 6. So instead we note the following consequence on existence of solutions in Besov and Triebel–Lizorkin spaces. In particular it is noteworthy that solutions despite the general function spaces are shown to exist for data with arbitrarily large norms:

Corollary 5.2. *Let there be given data* $f \in B^{s-4}_{p,q}(\overline{\Omega})$ *and* $\psi_k \in B^{s-k-\frac{1}{p}}_{p,q}(\Gamma)$, *for* $k = 0,\ 1$, *for some* (s,p,q) *fulfilling*

$$s > 2 + (\tfrac{2}{p} - 1)_+, \quad or \tag{5.7a}$$

$$s = 2 + (\tfrac{2}{p} - 1)_+ \quad and \quad q \le 2. \tag{5.7b}$$

Then there exists a solution (u_1,u_2) *in* $B^s_{p,q}(\overline{\Omega})^2$ *of the equations in* (5.1).

If $f \in F^{s-4}_{p,q}(\overline{\Omega})$ *and* $\psi_k \in B^{s-k-\frac{1}{p}}_{p,p}(\Gamma)$, *for* $k = 0,\ 1$, *and* (s,p,q) *fulfils either* (5.7a) *or*

$$s = 2 + (\tfrac{2}{p} - 1)_+, \quad and\ q \le 2\ if\ p \ge 2, \tag{5.8}$$

then (5.1) *has a solution* (u_1,u_2) *in* $F^s_{p,q}(\overline{\Omega})^2$.

Proof. Under the assumptions on (s,p,q), the data f and ψ_k belong to $F^{-2}_{2,2}(\overline{\Omega})$ and $B^{2-k-\frac{1}{2}}_{2,2}(\Gamma)$, as seen by the usual embeddings. So by [Cia97, Th 5.8-3] there is a solution $(u_1,u_2) \in F^2_{2,2}(\overline{\Omega})^2$; according to Theorem 5.1 it also belongs to $B^s_{p,q}(\overline{\Omega})^2$ or $F^s_{p,q}(\overline{\Omega})^2$, respectively. □

Corollary 5.2 clearly gives a solvability theory for the sector bounded by the dotted lines in Figure 3.

Example 5.3. Equation (5.1) may be considered with force term $f(x_1,x_2)$ equal to $1(x_1) \otimes \delta_0(x_2)$ and $0 \in \Omega$. Such singular data could model displacements and stresses generated by a heavy rod lying along the x_1-axis on a table, obtained by clamping a wooden plate along its edges to a sturdy metal frame.

This $f \in B^{\frac{1}{p}-1}_{p,\infty}(\overline{\Omega})$ for every $p \in [1,\infty]$, for f may be seen as $r_\Omega(\varphi(x_1) \otimes \delta_0(x_2))$ for some test function φ equal to 1 on a large ball, and $(\varphi,\delta_0) \mapsto \varphi \otimes \delta_0$ is bounded $L_p(\mathbb{R}) \oplus B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}) \to B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^2)$ for $p \ge 1$. (This follows from the dyadic corona criterion, eg by an easy variant of [Joh96, Prop. 2.6], where for $s < 0$ the continuity $B^s_{p,q}(\mathbb{R}) \oplus B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}) \to B^{s+\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^2)$ was proved for $\varphi \otimes \delta_0$.)

By Corollary 5.2, there is for every set of $\psi_k \in B^{3-k}_{p,\infty}(\Gamma)$, $k = 0,1$, with fixed $p \in [1,\infty]$, a solution in $B^{3+\frac{1}{p}}_{p,\infty}(\overline{\Omega})^2$ of (5.1). For $\psi_0 = \psi_1 = 0$ it belongs to this space for every $p \in [1,\infty]$, according to Theorem 5.1.

Remark 5.4. Although the coupling of the two non-linear equations in (5.1), as described, could be handled using that u_1 and u_2 are sought after in the same space, it seems more flexible to stick with the general set-up in Section 2 by developing a theory in which the pair (u_1,u_2) is regarded as the unknown, entering the bilinear form twice. This only requires some projections onto u_1 and u_2, cf the details around (6.19) below. For this purpose it is convenient to generalise product type operators to a framework of vector bundles, as done in the next section.

6. Systems of semi-linear boundary problems

In this section the abstract results of Section 2 and those on paralinearisation of product type operators in Section 3 will be carried over to a general framework for semi-linear elliptic boundary problems. This is formulated in a vector bundle set-up, not just because this may be useful for the handling of non-linearities, as mentioned in Remark 5.4 above, but also because vector bundles are natural for linear elliptic systems of multi-order.

6.1. **General linear elliptic systems.** Because the parametrix construction relies on a linear theory with the properties in (I)–(II) of Section 2, it is natural to utilise the Boutet de Monvel calculus [BdM71]. The L_p-results for this are reviewed briefly below (building on [Joh96], that extends L_p-results of G. Grubb [Gru90] and J. Franke [Fra85, Fra86a]). Introductions to the calculus may be found in [Gru97, Gru91] or [JR97, Sect. 4.1], and a thourough account in [Gru96].

Recall that $\Omega \subset \mathbb{R}^n$ denotes a smooth, open, bounded set with $\partial\Omega = \Gamma$. The main object is then a multiorder Green operator, designated by $\mathscr{A}$, i.e.,

$$\mathscr{A} = \begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix} \tag{6.1}$$

where $P = (P_{ij})$ and $G = (G_{ij})$, $K = (K_{ij})$, $T = (T_{ij})$ and $S = (S_{ij})$. Here $i \in I_1 := \{1, 2, \dots, i_1\}$ and $i \in I_2 := \{i_1 + 1, \dots, i_2\}$, respectively, in the two rows of the block matrix $\mathscr{A}$. Similarly $j \in J_1 := \{1, 2, \dots, j_1\}$ and $j \in J_2 := \{j_1 + 1, \dots, j_2\}$, respectively, hold in the two columns of $\mathscr{A}$; that is, $\mathscr{A}$ is an $i_2 \times j_2$ matrix operator with indices belonging to $I \times J$, when $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$.

Each P_{ij}, G_{ij}, K_{ij}, T_{ij} and S_{ij} belongs to the poly-homogeneous calculus of pseudo-differential boundary problems. More precisely, P is a pseudo-differential operator satisfying the uniform two-sided transmission condition (at Γ), G is a singular Green operator, K a Poisson and T a trace operator, while S is an ordinary pseudo-differential operator on Γ. (The exact requirements on the symbols and symbol kernels may be found in the above references.) The operator in the ij^{th} entry of $\mathscr{A}$ is taken to be of order $d + b_i + a_j$, where $d \in \mathbb{Z}$, $\mathbf{a} = (a_j) \in \mathbb{Z}^{j_2}$ and $\mathbf{b} = (b_i) \in \mathbb{Z}^{i_2}$; for each j, both $P_{ij,\Omega} + G_{ij}$ and T_{ij} is supposed to be of class $\kappa + a_j$ for some fixed $\kappa \in \mathbb{Z}$. For short $\mathscr{A}$ is then said to be of order d and class κ (or to be of order d and class κ relatively to $(\mathbf{a}, \mathbf{b})$, more precisely).

Recall that the transmission condition ensures that $P_\Omega := r_\Omega P e_\Omega$ has the same order on all spaces on which it is defined. More explicitly this means that each $P_{ij,\Omega}$ has order $d + a_j + b_i$ on every $B^s_{p,q}$ and $F^s_{p,q}$ with arbitrarily high $s > \kappa + a_j + 1 - \frac{1}{p}$; implying, say that $C^\infty(\overline{\Omega})$ is mapped into $C^\infty(\overline{\Omega})$, without blow-up at Γ. (Thus P_Ω has the transmission *property*.)

The operators are supposed to act on spaces of sections of vector bundles E_j over Ω and F_j over Γ, with j running in J_1 and J_2, respectively, and to map into sections of other such bundles E'_i and F'_i. The fibres of E_j, F_j have dimension M_j, N_j, while $\dim E'_i = M'_i$ and $\dim F'_i = N'_i$. Letting

$$V = (E_1 \oplus \dots \oplus E_{j_1}) \cup (F_{j_1+1} \oplus \dots \oplus F_{j_2}) \tag{6.2}$$

$$V' = (E'_1 \oplus \dots \oplus E'_{i_1}) \cup (F'_{i_1+1} \oplus \dots \oplus F'_{i_2}), \tag{6.3}$$

$\mathscr{A}$ is a map $C^\infty(V) \to C^\infty(V')$. For these spaces of sections, one may regard $C^\infty(V)$ as an abbreviation for $C^\infty(E_1) \oplus \dots \oplus C^\infty(F_{j_2})$ or, alternatively, V as a vector bundle with base manifold $\Omega\cup\Gamma$ in the sense of [Lan72]. The dimension of the base manifold as well as of the fibres over its points x depends on whether $x\in\Omega$ or $x\in\Gamma$. Similar remarks apply to V'.

The following spaces are adapted to the orders and classes of $\mathscr{A}$,

$$B^{s+\mathbf{a}}_{p,q}(V)=(\bigoplus_{j\le j_1}B^{s+a_j}_{p,q}(E_j))\oplus(\bigoplus_{j_1<j}B^{s+a_j-\frac{1}{p}}_{p,q}(F_j)) \tag{6.4}$$

$$B^{s-\mathbf{b}}_{p,q}(V')=(\bigoplus_{i\le i_1}B^{s-b_i}_{p,q}(E'_i))\oplus(\bigoplus_{i_1<i}B^{s-b_i-\frac{1}{p}}_{p,q}(F'_i)). \tag{6.5}$$

The spaces $F^{s+\mathbf{a}}_{p,q}(V)$ and $F^{s-\mathbf{b}}_{p,q}(V')$ are defined analogously (with $p<\infty$), except that $q=p$ in the summands over Γ; as usual $F^s_{p,p}(F_j)=B^s_{p,p}(F_j)$ etc. For convenience

$$\|v\,|B^{s+\mathbf{a}}_{p,q}\|=\big(\|v_1\,|B^{s+a_1}_{p,q}(E_1)\|^q+\dots+\|v_{j_2}\,|B^{s+a_{j_2}-\frac{1}{p}}_{p,q}(F_{j_2})\|^q\big)^{\frac{1}{q}} \tag{6.6}$$

$$\|v\,|F^{s+\mathbf{a}}_{p,q}\|=\big(\|v_1\,|F^{s+a_1}_{p,q}(E_1)\|^p+\dots+\|v_{j_2}\,|F^{s+a_{j_2}-\frac{1}{p}}_{p,p}(F_{j_2})\|^p\big)^{\frac{1}{p}}, \tag{6.7}$$

with similar conventions for $B^{s-\mathbf{b}}_{p,q}$ and $F^{s-\mathbf{b}}_{p,q}$.

With respect to the defined spaces, $\mathscr{A}$ is *continuous*

$$\mathscr{A}\colon B^{s+\mathbf{a}}_{p,q}(V)\to B^{s-d-\mathbf{b}}_{p,q}(V'),\quad \mathscr{A}\colon F^{s+\mathbf{a}}_{p,q}(V)\to F^{s-d-\mathbf{b}}_{p,q}(V'), \tag{6.8}$$

for each $(s,p,q)\in\mathbb{D}_\kappa$, when $p<\infty$ in the Triebel–Lizorkin spaces.

Ellipticity for multi-order Green operators is similar to this notion for single-order operators, except that the principal symbol $p^0(x,\xi)$ is a matrix with p^0_{ij} equal to the principal symbol of P_{ij} *relatively* to the order $d+b_i+a_j$ of P_{ij}; invertibility of $p^0(x,\xi)$ should hold for all $x\in\Omega$ and $|\xi|\ge1$. The principal boundary operator $a^0(x',\xi',D_n)$ is similarly defined and should be invertible as an operator from $\mathscr{S}(\overline{\mathbb{R}}_+)^M\times\mathbb{C}^N$ to $\mathscr{S}(\overline{\mathbb{R}}_+)^{M'}\times\mathbb{C}^{N'}$ with $M:=M_1+\dots+M_{j_1}$ and analogous definitions of N, M' and N'.

For the mapping properties of elliptic systems $\mathscr{A}$ and their parametrices one has the next theorem, which is an anbridged version of [Joh96, Thm 5.2].

Theorem 6.1. *Let $\mathscr{A}$ denote a multi-order Green operator going from V to V', and of order d and class κ relatively to $(\mathbf{a},\mathbf{b})$ as described above. If $\mathscr{A}$ is injectively or surjectively elliptic, then $\mathscr{A}$ has, respectively, a left- or right-parametrix $\widetilde{\mathscr{A}}$ in the calculus. $\widetilde{\mathscr{A}}$ can be taken of order $-d$ and class $\kappa-d$, and then $\widetilde{\mathscr{A}}$ is bounded in the opposite direction in* (6.8) *for all the parameters $(s,p,q)\in\mathbb{D}_\kappa$. The corresponding is true for $F^{s+\mathbf{a}}_{p,q}(V)$ and $F^{s-d-\mathbf{b}}_{p,q}(V')$.*

In the elliptic case, all these properties hold for $\mathscr{A}$, and the parametrices are two-sided.

The above theorem deliberately focuses on the necessary mapping properties, so it may provide a false impression of what is known about elliptic systems. For one thing, (6.8) is sharp with respect to (s,p,q) ((6.8) can only hold outside $\overline{\mathbb{D}}_\kappa$ if the class is effectively lower than κ). Secondly the Fredholm properties have not been mentioned at all; the kernel of $\mathscr{A}$ is a finite-dimensional space in $C^\infty(V)$, which is independent of (s,p,q) and of the choice of function space, and the range is closed with complements that can be chosen to have similar properties. The reader is referred to [Gru90, Joh96] for this. In particular the (s,p,q)-invariance of the range complements implies that the compatibility conditions on the data are fulfilled for all (s,p,q), if they are so for one parameter. Hence these conditions can be ignored in the following regularity investigations.

For the inverse regularity properties of a, say injectively elliptic system $\mathscr{A}$, note that, by the above theorem, the left-parametrix $\widetilde{\mathscr{A}}$ may be chosen so that

$$\mathscr{R} := I - \widetilde{\mathscr{A}}\mathscr{A} \tag{6.9}$$

has class κ and order $-\infty$, hence is continuous

$$\mathscr{R}\colon B^{s+\mathbf{a}}_{p,q}(V) \to C^\infty(V) \quad \text{for every} \quad (s,p,q) \in \mathbb{D}_\kappa. \tag{6.10}$$

So if $\mathscr{A}u = f$ for some $u \in B^{s_1+\mathbf{a}}_{p_1,q_1}(V)$ and data $f \in B^{s_0-d-\mathbf{b}}_{p_0,q_0}(V')$, and if (s_j,p_j,q_j) belongs to $\mathbb{D}_\kappa$ for $j=0$ and 1, then the identity (6.9) applied to $\mathscr{A}u = f$ yields that

$$u = \widetilde{\mathscr{A}}f + \mathscr{R}u \in B^{s_0+\mathbf{a}}_{p_0,q_0}(V). \tag{6.11}$$

Cf the detailed argument given for $\mathscr{A} = \left(\begin{smallmatrix}-\Delta\\ \gamma_0\end{smallmatrix}\right)$ in (1.8)–(1.10) ff.

Moreover, it may now be explicated how this elliptic framework fits with the conditions (I)–(II) of Section 2: for each fixed $q \in]0,\infty]$ let

$$\mathbb{S} = \{\,(s,p) \mid s \in \mathbb{R},\ 0 < p \le \infty\,\} \tag{6.12}$$

$$X^s_p = B^{s+\mathbf{a}}_{p,q}(V), \qquad Y^s_p = B^{s-\mathbf{b}}_{p,q}(V'), \tag{6.13}$$

and note that (I) holds. Moreover, concerning (II) it is possible, when $\widetilde{\mathscr{A}}$ is chosen of class $\kappa - d$, to take

$$A_{(s,p)} = \mathscr{A}|_{B^{s+\mathbf{a}}_{p,q}(V)}, \qquad \mathbb{D}(A) = \mathbb{D}_\kappa, \qquad \widetilde{A} = \widetilde{\mathscr{A}}. \tag{6.14}$$

For corresponding spaces $X^s_p = F^{s+\mathbf{a}}_{p,q}(V)$ and $Y^s_p = F^{s-\mathbf{b}}_{p,q}(V')$, however, one needs a little precaution because the sum and integral exponents in (6.7) are equal in the spaces over the boundary bundles F_j. Indeed, (2.3) is then not a direct consequence of (1.33) ff, but for $p > r$,

$$F^{s+a_j-\frac{1}{p}}_{p,p}(F_j) \hookrightarrow F^{s+a_j-\frac{1}{p}}_{r,p}(F_j) \hookrightarrow F^{s+a_j-\frac{1}{r}}_{r,r}(F_j). \tag{6.15}$$

In this way (I) and (II) holds also for these spaces.

Example 6.2. For the biharmonic Dirichlét problem, which enters the von Karman equations, it is natural to let

$$\mathscr{A} = \begin{pmatrix} \Delta^2 & 0 \\ 0 & \Delta^2 \\ \gamma_0 & 0 \\ \gamma_1 & 0 \\ 0 & \gamma_0 \\ 0 & \gamma_1 \end{pmatrix}, \tag{6.16}$$

whereby $d = 4$, $\kappa = 2$, $\mathbf{a} = (0,0)$ and $\mathbf{b} = (0,0,-4,-3,-4,-3)$. The choice in (6.13) amounts to

$$X^s_p = B^s_{p,q}(\overline{\Omega})^2 \tag{6.17}$$

$$Y^{s-4}_p = B^{s-4}_{p,q}(\overline{\Omega})^2 \oplus (B^{s-\frac{1}{p}}_{p,q}(\Gamma) \oplus B^{s-1-\frac{1}{p}}_{p,q}(\Gamma))^2; \tag{6.18}$$

this is clear since one can use the trivial bundles $V = \Omega \times \mathbb{C}^2$ and $V' = (\Omega \times \mathbb{C}^2) \cup (\Gamma \times \mathbb{C})^4$ for this problem.

6.2. **General product type operators.** Together with the Green operator $\mathscr{A}$ in (6.16) above, a treatment of the von Karman equation may conveniently use the bilinear operator $\tilde{B}$ given on $v = (v_1, v_2)$ and $w = (w_1, w_2)$ by

$$\tilde{B}(v,w) = \begin{pmatrix} -[v_1,w_2] & [v_1,w_1] & 0 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}. \tag{6.19}$$

Indeed, in the set-up of the previous section, a solution u of (5.1) is a section of the trivial bundle $\Omega \times \mathbb{C}^2$, of which the two canonical projections u_1 and u_2 enter directly into the expressions in (5.1). The same projections enter for $v = w = u$ in (6.19) above, and this is taken as the guiding principle in a generalisation of product type operators to vector bundles.

To this end, let $V \xrightarrow{\beta} \Omega$ and $V' \xrightarrow{\beta'} \Omega$ be vector bundles over Ω. When V is covered by a system of local trivialisations

$$\tau_l \colon \beta^{-1}(U_l) \to U_l \times \mathbb{C}^N \tag{6.20}$$

there are associated projections $\mathrm{pr}_{lk} \colon U_l \times \mathbb{C}^N \to \mathbb{C}$ mapping (x,t) to t_k, the k^{th} canonical coordinate in $\mathbb{C}^N$. In addition, when $\tau_l' \colon \beta'^{-1}(U_l) \to U_l \times \mathbb{C}^{N'}$ is a trivialisation of V', then the projection pr'_{lk} onto the k^{th} coordinate of $\mathbb{C}^{N'}$ is defined analogously.

Definition 6.3. When B maps pairs (v,w) of sections of V bilinearly to sections of V', then B is said to be of *product type* if for each l the composite

$$\mathrm{pr}'_{lk} \tau_l'(B(v,w)) \tag{6.21}$$

is a map only depending on two projections $\mathrm{pr}_{lk_0} \tau_l(v)$ and $\mathrm{pr}_{l\bar{k}_0} \tau_l(w)$ and if, as such, it is of product type on U_l (in the sense of Definition 4.1).

Finite sums of such operators are also said to be of product type.

In (6.19) above one clearly has this structure since eg v_1, w_2 may be read as the projections onto $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$ of two sections v, w of the bundle $\Omega \times \mathbb{C}^2$.

In relation to a given elliptic system $\mathscr{A}$ of order d and class κ with respect to a fixed set of numbers $(\mathbf{a}, \mathbf{b})$, it is useful to introduce a set of product type operators with compatible mapping properties.

Since the non-linearities typically send sections over Ω to other such sections (and do not involve sections over Γ), the following framework should suffice for most applications:

Given bundles over Ω as in (6.2)–(6.3), there are bundles

$$W = E_1 \oplus \cdots \oplus E_{j_1}, \qquad W' = E_1' \oplus \cdots \oplus E_{i_1}', \tag{6.22}$$

$$\beta_j \colon E_j \to \Omega, \qquad \beta_i' \colon E_i' \to \Omega \tag{6.23}$$

in which sections w and w', respectively, may naturally be regarded as j_1- and i_1-tuples of sections (by means of projections pr_j and pr_i')

$$w = (w_1, \dots, w_{j_1}), \qquad w' = (w_1', \dots, w_{i_1}'). \tag{6.24}$$

There is also a covering $\Omega = \bigcup U_l$ of local coordinate systems $U_l \mapsto \kappa_l(U_l)$, having associated trivialisations τ_{jl} and τ_{il}', for each j, i and l, together with associated projections pr_{jlk} and pr'_{ilk} onto the k^{th} coordinate of $\mathbb{C}^{M_j}$ and $\mathbb{C}^{M_i'}$, respectively:

$$\beta_j^{-1}(U_l) \xrightarrow{\tau_{jl}} U_l \times \mathbb{C}^{M_j} \xrightarrow{\mathrm{pr}_{jlk}} \mathbb{C}, \qquad \beta'^{-1}_i(U_l) \xrightarrow{\tau_{il}'} U_l \times \mathbb{C}^{M_i'} \xrightarrow{\mathrm{pr}'_{ilk}} \mathbb{C}. \tag{6.25}$$

For short, $\tau_{jlk} := \mathrm{pr}_{jlk} \circ \tau_{jl} \circ \mathrm{pr}_j$ in the following. τ_{ilk}' will be similarly defined.

Definition 6.4. Given integers $(\mathbf{a},\mathbf{b})$ as in (6.4)–(6.5) ff, an operator B of product type from W to W' is said to be of product type (d_0,d_1) *compatible* with $(\mathbf{a},\mathbf{b})$ if the following holds:

(i) For each $i\in\{1,\dots,i_1\}$ and each τ'_{ilk}, the map

$$(v,w)\mapsto \tau'_{ilk}B_i(v,w) \tag{6.26}$$

depends only on two projections $\tau_{j_0lk_0}(v)$ and $\tau_{\bar{j}_0l\bar{k}_0}(w)$, where $1\le k_0\le M_{j_0}$ and $1\le \bar{k}_0\le M'_{\bar{j}_0}$, so that for some B_{ilk},

$$\tau'_{ilk}B_i(v,w)=B_{ilk}(\tau_{j_0lk_0}(v),\tau_{\bar{j}_0l\bar{k}_0}(w)). \tag{6.27}$$

(ii) Each B_{ilk} is a product type operator on the open set $\kappa_l(U_l)$ of $\mathbb{R}^n$ with linear operators $P_0(D)$, $P_1(D)$ and L, as in (3.1), of orders

$$d_0+a_{j_0},\quad d_1+a_{\bar{j}_0}\quad\text{and}\quad b_i, \tag{6.28}$$

respectively.

The definition is extended to those B for which each map in (6.26) is a finite sum of terms, that each have the properties in (i) and (ii). So, by introducing trivial terms, (6.27) could have had a sum over all (j_0,l,k_0) and $(\bar{j}_0,l,\bar{k}_0)$ on the right hand side.

Finally, such operators $B(\cdot,\cdot)$ may be lifted to operators from $V\oplus V$ to V' by sending sections over Γ to zero and by having trivial values over $F'_{i_1+1}\oplus\dots\oplus F'_{i_2}$. Such liftings will also be denoted by B (and are tacitly understood).

In the next result pseudo-local operators are defined as usual to be those that decrease or preserve singular supports; the singular support of eg a section v of W is the complement in Ω of the x for which $\tau_{jlk}\circ v$ is C^∞ from a neighbourhood of x to $\mathbb{C}$, for all $U_l\ni x$ and all j and k.

Theorem 6.5. *Let $B(\cdot,\cdot)$ be of product type (d_0,d_1) compatible with $(\mathbf{a},\mathbf{b})$ and with $d_0\le d_1$; and let Besov and Triebel–Lizorkin spaces be defined as in* (6.4)–(6.5) *ff above, with the unified notation $E^{s+\mathbf{a}}_{p,q}(V)$ and $E^{s-\mathbf{b}}_{p,q}(V')$. Then $u\mapsto Q(v):=B(v,v)$ is bounded*

$$E^{s+\mathbf{a}}_{p,q}(V)\to E^{s-\sigma(s,p,q)-\mathbf{b}}_{p,q}(V')\quad\textit{for every } (s,p,q)\in\mathbb{D}(Q), \tag{6.29}$$

whereby $\mathbb{D}(Q)$ and $\sigma(s,p,q)$ are given by (4.8) *and* (4.10)*, respectively.*

Moreover, for each $u\in E^{s_0+\mathbf{a}}_{p_0,q_0}(V)$ there is a moderate linearisation L_u (that is, $-L_uu=Q(u)$ if $(s_0,p_0,q_0)\in\mathbb{D}(Q)$), which with ω as in (4.15) *is bounded*

$$L_u\colon E^{s_1+\mathbf{a}}_{p_1,q_1}(V)\to E^{s_1-\omega-\mathbf{b}}_{p_1,q_1}(V') \tag{6.30}$$

whenever (s_1,p_1,q_1) belongs to the parameter domain given by (4.14)*. Moreover, L_u is pseudo-local on every such $E^{s_1+\mathbf{a}}_{p_1,q_1}(V)$.*

Proof. Let (s_0,p_0,q_0) and (s_1,p_1,q_1) be given such that (4.6) holds. When U_l and i, k are fixed, there is for each pair of projections $\tau_{j_0lk_0}(u)$ and $\tau_{\bar{j}_0l\bar{k}_0}(v)$, by Theorem 4.7, a linear operator $L^{j_0lk_0}_{ilk}$ such that

$$L^{j_0lk_0}_{ilk}(\tau'_{\bar{j}_0l\bar{k}_0}(v))=B^{j_0lk_0,\bar{j}_0l\bar{k}_0}_{ilk}(\tau_{j_0lk_0}(u),\tau_{\bar{j}_0l\bar{k}_0}(v)), \tag{6.31}$$

and such that $L^{j_0lk_0}_{ilk}$ sends $B^{s_1+a_{\bar{j}_0}}_{p_1,q_1}(U_l)$ into $B^{s_1-\omega-b_i}_{p_1,q_1}(U_l)$ with ω as in (4.15); the last fact is due to (6.28) and to cancellation of the numbers $a_{\bar{j}_0}$.

Summation over all j_0, k_0 and $\overline{j_0}$, $\overline{k_0}$, by including possible zero-terms, gives

$$\begin{aligned}\tau'_{ilk}B_i(u,v) &= \sum B_{ilk}^{j_0lk_0,\overline{j_0}l\overline{k_0}}(\tau_{j_0lk_0}(u),\tau_{\overline{j_0}l\overline{k_0}}(v))\\ &= \sum L_{ilk}^{j_0lk_0}(\tau'_{\overline{j_0}l\overline{k_0}}(v)).\end{aligned} \tag{6.32}$$

Omitting pr'_{ilk} from the left hand side gives a linear operator $L_{il,u}$ such that

$$\tau'_{il}B_i(u,v) = L_{il,u}(v) \tag{6.33}$$

holds as sections of $U_l\times\mathbb{C}^{M'_i}$. By construction, $L_{il,u}(v)$ belongs to the space $B^{s_1-\omega-b_i}_{p_1,q_1}(U_l)^{M'_i}$ when $v\in B^{s_1+\mathbf{a}}_{p_1,q_1}(V)$.

It is now possible to define a linear operator L_u between the spaces $B^{s_1+\mathbf{a}}_{p_1,q_1}(V)$ and $B^{s_1-\omega-\mathbf{b}}_{p_1,q_1}$ by the formula

$$L_u(v)_i = \sum_l (\tau'_{il})^{-1}\circ L_{il,u}(\psi_l v), \quad \text{for} \quad i\in I_1, \tag{6.34}$$

when $1=\sum_l\psi_l$ is a partition of unity subordinate to the patches U_l. Because the class of pseudo-local operators is closed under addition, it follows from the construction of $L_{il,u}$ and Proposition 4.12 that each $L_{il,u}$ is pseudo-local; and so is L_u.

When (s_0,p_0,q_0) belongs to the domain $\mathbb{D}(Q)$ given by (4.8), then $v=u$ is possible and

$$L_u(u)_i = \sum_l(\tau'_{il})^{-1}\circ\tau'_{il}B_i(u,\psi_l u) = B_i(u,\sum_l\psi_l u) = B_i(u,u). \tag{6.35}$$

Because $(s_0,p_0,q_0)=(s_1,p_1,q_1)$ renders $\omega(s,p,q)$ equal to $\sigma(s,p,q)$, the first part of the theorem is also proved.

Finally the Triebel–Lizorkin spaces can be treated analogously. □

Remark 6.6. Definition 6.4 is inevitably lengthy, because of the non-linearities one meets in practice. Indeed, for the von Karman bracket in (6.19) the choice $k=1$ in (i) of Definition 6.4 leads to $k_0=1$, $\overline{k_0}=2$, while $k=2$ gives $k_0=\overline{k_0}=1\neq k$.

In addition the extension to finite sums is natural in connection with the Navier–Stokes equation, where $W=W'=(\Omega\times\mathbb{C}^n)\oplus(\Omega\times\mathbb{C})$ and for $i=1$ each k gives rise to the sum $\sum_{k_0=1}^n v_{k_0}\partial_{k_0}w_k$, where any $k_0\in\{1,\dots,n\}$ occurs and $\overline{k_0}=k$. (For $i=2$ the zero-operator appears.)

6.3. **Semi-linear elliptic systems.** It is now easy to establish the below Theorem 6.7, which is an adaptation of Theorem 2.2 to the framework of Section 6.

For generality's sake it is observed that it suffices, by (II), to take the linear part $\mathscr{A}$ injectively elliptic, ie with a left parametrix $\widetilde{\mathscr{A}}$ and regularising operator $\mathscr{R}:=I-\widetilde{\mathscr{A}}\mathscr{A}$. Recall that for a product type operator B, the linearisation L_u of $Q(u):=B(u,u)$ furnished by Theorem 6.5 enters the parametrix

$$P^{(N)} = I+\widetilde{\mathscr{A}}L_u+\dots+(\widetilde{\mathscr{A}}L_u)^{N-1}. \tag{6.36}$$

As previously, the domain where Q is $\mathscr{A}$-moderate is written

$$\mathbb{D}(\mathscr{A},Q) = \{\,(s,p,q)\in\mathbb{D}_\kappa\cap\mathbb{D}(Q) \mid \sigma(s,p,q)<d\,\}. \tag{6.37}$$

Using these ingredients, one has the following main result for semi-linear systems:

Theorem 6.7. *Let $\mathscr{A}$ be an injectively elliptic Green operator of order d and class κ relatively to $(\mathbf{a},\mathbf{b})$, and assume that $B(\cdot,\cdot)$ is an operator of product type (d_0,d_1) with $d_0\le d_1$ and compatible with $(\mathbf{a},\mathbf{b})$ so that Q has order function $\sigma(s,p,q)$ on $\mathbb{D}(Q)$ and moderate linearisations L_u, according to Theorem 6.5.*

For a section u of $B^{s_0+\mathbf{a}}_{p_0,q_0}(V)$ with $(s_0,p_0,q_0)\in\mathbb{D}(\mathscr{A},Q)$, and any choice $\widetilde{\mathscr{A}}$ of a left parametrix of $\mathscr{A}$ of class $\kappa-d$, the parametrices $P^{(N)}$ in (6.36) *are bounded endomorphisms on $B^{s+\mathbf{a}}_{p,q}(V)$ for every (s,p,q) in $\mathbb{D}_\kappa\cap\mathbb{D}(L_u)$. And for all (s_1,p_1,q_1) and (s_2,p_2,q_2) in $\mathbb{D}_\kappa\cap\mathbb{D}(L_u)$ the linear operator $(\widetilde{\mathscr{A}}L_u)^N$ maps $B^{s_1+\mathbf{a}}_{p_1,q_1}(V)$ to $B^{s_2+\mathbf{a}}_{p_2,q_2}(V)$ for all sufficiently large N.*

If such a section u solves the equation

$$\mathscr{A}u+Q(u)=f \tag{6.38}$$

for data $f\in B^{t-d-\mathbf{b}}_{r,o}(V')$ with $(t,r,o)\in\mathbb{D}(\mathscr{A})\cap\mathbb{D}(L_u)$, then

$$u=P^{(N)}(\widetilde{\mathscr{A}}f+\mathscr{R}u)+(\widetilde{\mathscr{A}}L_u)^N u. \tag{6.39}$$

and it also holds that $u\in B^{t+\mathbf{a}}_{r,o}(V)$.

The analogous results are valid for the scales $F^{s+\mathbf{a}}_{p,q}(V)$ and $F^{s-\mathbf{b}}_{p,q}(V')$.

Proof. As observed in (6.12)–(6.14), the choice $X^s_p=B^{s+\mathbf{a}}_{p,q}(V)$ and $Y^s_p=B^{s-\mathbf{b}}_{p,q}(V')$ makes conditions (I) and (II) satisfied.

As the B_u in (III) one can take L_u, for its construction via paramultiplication implies that it is unambigously defined on intersections of the form $X^s_p\cap X^{s'}_{p'}$. Similarly there is commutative diagrams for $\mathscr{A}$ and $\widetilde{\mathscr{A}}$ by the general constructions in the Boutet de Monvel calculus and the results in Section 3.3.

Moreover, $\mathbb{D}(\mathscr{A},Q)$ is connected and $\delta=d-\omega(s,p,q)$ is constant with respect to (s,p) and positive; hence (IV) and (V) hold. The claims on $P^{(N)}$ may now be read off from Theorem 2.2. For $(\widetilde{\mathscr{A}}L_u)^N$ the sum exponents should also be handled, but one can assume $q_1=q_2$, for $\mathbb{D}_\kappa\cap\mathbb{D}(L_u)$ is open, hence contains $(s_1-\varepsilon,p_1,q_2)$ for $\varepsilon>0$, so that the larger space $B^{s_1-\varepsilon+\mathbf{a}}_{p_1,q_2}(V)$ is mapped into $B^{s_2+\mathbf{a}}_{p_2,q_2}$ for all sufficiently large N, according to Theorem 2.2.

Finally, since $(s_0-\varepsilon,p_0,q_0)$ also belongs to $\mathbb{D}_\kappa\cap\mathbb{D}(L_u)$ for sufficiently small $\varepsilon>0$, one can assume $q_0=o$. So according to Theorem 2.2 the section u belongs to $X^t_r=B^{t+\mathbf{a}}_{r,o}(V)$ as stated. □

It should be mentioned that while the abstract framework in Theorem 2.2 was formulated with only s and p as parameters, for convenience, the third parameter q was easily handled in the proof above by simple embeddings.

From the given examples it is clear that Theorem 5.1 on the von Karman problem is just a special case of the above result. One also has

Corollary 6.8. *For operators $\mathscr{A}$ and B as in Theorem 6.7, the equation*

$$\mathscr{A}u+Q(u)=f \tag{6.40}$$

is hypoelliptic, ie for f in $C^\infty(V')$ any solution u belongs to $C^\infty(V)$.

Finally it is shown that this corollary has a much sharper *local* version. This is derived directly from the parametrix formula (6.39) and from the obvious fact that the class of pseudo-local maps is stable under composition. In particular $\widetilde{\mathscr{A}}L_u$ is pseudo-local. (This really only involves the $P_\Omega+G$-part of $\widetilde{\mathscr{A}}$, since L_u goes from W to W'. And the pseudo-differential part clearly inherits pseudo-locality from the operators on $\mathbb{R}^n$, since $P_\Omega = r_\Omega P e_\Omega$. For the singular Green part one can extend [Gru96, Cor. 2.4.7] by means of Rem. 2.4.9 there on (x_n, y_n)-dependent singular Green operators to get the pseudo-local property.)

For this purpose, let $\Xi \subset \overline{\Omega}$ be a subregion open relatively to $\overline{\Omega}$ (ie $\Xi = \overline{\Omega} \cap O$ for an open set $O \subset \mathbb{R}^n$), so that $\Xi \cap \Omega$ possibly adheres to a part $\Gamma_0 \subset \Gamma$ of the boundary. Then, if f in (6.38) in addition fulfils $f \in B^{t_1-d-\mathbf{b}}_{r_1,o_1}(V'_{|\Xi};\mathrm{loc})$ the idea is to show for a solution u in (6.38) that $u \in B^{t_1+\mathbf{a}}_{r_1,o_1}(V_{|\Xi},\mathrm{loc})$.

More precisely, $f \in B^{t_1-d-\mathbf{b}}_{r_1,o_1}(V'_{|\Xi};\mathrm{loc})$ means that φf is in $B^{t_1-d-\mathbf{b}}_{r_1,o_1}(V')$ for every φ in $C^\infty(\overline{\Omega})$ with compact support contained in Ξ. Hereby φf is calculated fibre-wisely for the components of f, both in the bundles E'_i over Ω, for $i \le i_1$, and in the F'_i over Γ, for $i_1 < i \le i_2$. That $u \in B^{t_1+\mathbf{a}}_{r_1,o_1}(V_{|\Xi},\mathrm{loc})$ is defined similarly.

Theorem 6.9. *Under hypothesis as in Theorem 6.7, suppose* $f \in B^{t_1-d-\mathbf{b}}_{r_1,o_1}(V'_{|\Xi};\mathrm{loc})$ *holds in addition to* (6.38) *for some* (t_1, r_1, o_1) *in* $\mathbb{D}_\kappa \cap \mathbb{D}(L_u)$. *Then u is also a section of* $B^{t_1+\mathbf{a}}_{r_1,o_1}(V_{|\Xi};\mathrm{loc})$.

The corresponding result holds for the $F^s_{p,q}$*-scale too.*

Proof. Let ψ, χ_0 and $\chi_1 \in C^\infty(\overline{\Omega})$ be chosen so that $\operatorname{supp}\chi_1 \subset \Xi$ and

$$\chi_0 + \chi_1 \equiv 1, \qquad \chi_j \equiv j \text{ on a neighbourhood of } \operatorname{supp}\psi. \tag{6.41}$$

By the parametrix formula (6.39),

$$\psi u = \psi P^{(N)}\big(\widetilde{\mathscr{A}}(\chi_1 f) + \mathscr{R}u\big) + \psi P^{(N)}\widetilde{\mathscr{A}}(\chi_0 f) + \psi(\widetilde{\mathscr{A}L_u})^N u \tag{6.42}$$

and here the last term belongs to $B^{t_1+\mathbf{a}}_{r_1,o_1}(V)$ for a sufficiently large N, according to the first part of Theorem 6.7. Since $\widetilde{\mathscr{A}L_u}$ is pseudo-local so is $P^{(N)}$, and the inclusion $\operatorname{sing\,supp}\widetilde{\mathscr{A}}(\chi_0 f) \subset \operatorname{supp}\chi_0$ therefore implies that $\psi P^{(N)}\widetilde{\mathscr{A}}(\chi_0 f)$ is in $C^\infty(V)$. And because $\widetilde{\mathscr{A}}(\chi_1 f) + \mathscr{R}u$ is in $B^{t_1+\mathbf{a}}_{r_1,o_1}(V)$, the fact that $P^{(N)}$ has order zero gives that also the first term on the right hand side of (6.42) is in $B^{t_1+\mathbf{a}}_{r_1,o_1}(V)$. $\square$

7. Final remarks

The theory above establishes Theorem 6.7 on parametrices of systems of semi-linear elliptic boundary problems. This gives satisfactory and general inverse regularity properties, including hypoellipticity and local properties in subregions $\Xi \subset \overline{\Omega}$ possibly adhering to the boundary; cf Theorem 6.9.

To elucidate advantages of the present methods, one could note that boot-strap procedures applied to general semi-linear problems create troublesome difficulites (for general data) at least when the number of normal derivatives in the boundary condition exceeds the mean order $(d_0 + d_1)/2$ associated with the product type operator $B(v, w)$ (more precisely whenever $\mathbb{D}(\mathscr{A}) \not\supset \mathbb{D}(\mathscr{N}, \delta)$). Eg this would be the case if (1.2) were considered with the Neumann condition instead; cf [Joh93, Thm. 5.5.3] or [Joh95b]. Using Theorem 6.7, or Theorem 2.2, these technicalities do not show up at all.

The parametrix formulae also give structural information about solutions (exploited here in the regularity analysis in subregions $\Xi \subset \overline{\Omega}$), and for regularity questions they imply that improved a priori knowledge of the solutions will allow weaker assumptions on the data.

Furthermore one could wonder whether this more flexible framework may give *more* regularity properties than boot-strap methods. And in high dimensions, eg

$n \geq 5$, it is indeed possible to find affirmative examples, like the polyharmonic operator with the Dirichlét condition, perturbed by the square $Q(u) = u^2$:

$$(-\Delta)^m u + u^2 = f \quad \text{in } \Omega \subset \mathbb{R}^n \tag{7.1}$$

$$\gamma_m u = 0 \quad \text{on } \Gamma. \tag{7.2}$$

Since $(-\Delta)^m\colon H_0^m(\overline{\Omega}) \to H^{-m}(\overline{\Omega})$ is a bijection by Lax–Milgram's lemma, the semilinear problem is by Proposition 2.4 solvable for small data $f \in H^{-m}(\overline{\Omega})$, when $Q(u)$ is of order $\leq 2m$ om H^m, ie for $m \geq n/6$ by Proposition 4.4. But if f also belongs to $B^t_{1,\infty}(\overline{\Omega})$ for some $t > 1-m$, it would be an improvement to conclude that $u \in B^{t+2m}_{1,\infty}$, for via standard embeddings such t yield $B^{t+2m}_{1,\infty} \hookrightarrow W_1^{m+1}$, whereas the a priori information only gives $u \in H^m \subset W_1^m$.

In order that this Besov regularity cannot be obtained by boot-strap methods, $(t+2m, 1)$ should be outside of the domain $\mathbb{D}((-\Delta)^m, Q)$, which by Corollary 4.8 or (4.21) is the case if $t+2m < \frac{n}{1} - 2m$ (provided $f \notin B^{t+\varepsilon}_{1,\infty}$ for $\varepsilon > 0$). This amounts to $m < (n-t)/4$.

For the purpose of the example, it is assumed that $0 \in \Omega \subset \mathbb{R}^n$ for $n \geq 2$. With $x = (x', x'')$ for $x' = (x_1, x_2)$, data are conveniently taken as

$$f(x) = c|x'|^{-3/2} = c(x_1^2 + x_2^2)^{-3/4}. \tag{7.3}$$

Then $f \in B^{-1/2}_{2,\infty}(\overline{\Omega}) \cap B^{1/2}_{1,\infty}(\overline{\Omega})$ This follows from Example 1.2 by tensor product techniques as in Example 5.3; one has in analogy with [Fra86a, Lem. 2.7.1] that $B^s_{p,q}(\mathbb{R}^{n_1}) \otimes B^s_{p,q}(\mathbb{R}^{n_2}) \subset B^s_{p,q}(\mathbb{R}^{n_1+n_2})$ for $s > 0$. Hence $f \in B^{1/2}_{1,\infty}$ and $t = 1/2$.

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be smooth open and bounded, $0 \in \Omega$. Then*

$$(-\Delta)^m u(x) + u(x)^2 = c|x'|^{-3/2} \tag{7.4}$$

has a solution $u \in H_0^m(\overline{\Omega})$ for sufficiently small $c > 0$ when $m \geq n/6$. If moreover $m \in]\frac{n}{6}, \frac{2n-1}{8}[$ every solution in $H^m(\overline{\Omega})$ also belongs to $B^{2m+\frac{1}{2}}_{1,\infty}(\overline{\Omega}) \subset W_1^{2m}(\overline{\Omega})$, so that $D^\alpha u(x)$ is an integrable function for $|\alpha| \leq 2m$. Specifically this entails:

- *For $n = 5$ and $m = 1$, every solution $u \in H^1(\overline{\Omega})$ also lies in $B^{5/2}_{1,\infty} \subset W_1^2$.*
- *For $9 \leq n \leq 11$ and $m = 2$, every solution $u \in H^2(\overline{\Omega})$ is in $B^{9/2}_{1,\infty} \subset W_1^4$.*

In general for $n \geq 13$ there is always some $m \geq 3$ in $]\frac{n}{6}, \frac{2n-1}{8}[$ such that solutions exist in $H^m \cap W_1^{2m}$. In all the cases $B^{2m+\frac{1}{2}}_{1,\infty}$ cannot be reached by reiteration.

Proof. In the following (n,m) should be chosen with $m < (2n-1)/8$, so that $W_1^{2m} \supset B^{2m+\frac{1}{2}}_{1,\infty}$ is outside of $\mathbb{D}((-\Delta)^m, Q)$, cf the above.

To apply the parametrices $H^m(\overline{\Omega})$ must lie in $\mathbb{D}((-\Delta)^m, Q)$. By (i) and (ii) of Corollary 4.8 the square operator is $(-\Delta)^m$-moderate on H^m if $\frac{n}{2} - m + d_0 \geq 0$ and $2m > \frac{n}{2} - m + \sum d_j$, ie if $\frac{n}{6} < m \leq \frac{n}{2}$. Clearly $(2m, 2)$ belongs to the parameter domain $\mathbb{D}_m$ of $((-\Delta)^m, \gamma_m)$; cf (1.36). In addition $(2m + \frac{1}{2}, 1)$ lies there, so it remains to check that it is in $\mathbb{D}(L_u)$. By (4.14) this is the case if $2m + \frac{1}{2} > d_0 + d_1 - m + (\frac{n}{2} + \frac{n}{1} - n)_+$, that is if $3m > \frac{n}{2} - \frac{1}{2}$, or $m > (n-1)/6$; this is redundant in view of the above condition $m > n/6$.

It suffices to pick (n,m) such that $m \in]\frac{n}{6}, \frac{2n-1}{8}[$. This interval has length greater than 1, and a fortiori contains an integer, for $n \geq 14$. It also contains $m = 1$, 2 and 3 in the mentioned dimensions. In all these cases Theorem 6.7 yields that $u \in B^{2m+\frac{1}{2}}_{1,\infty}(\overline{\Omega})$ as desired. □

Post festum, the theorem is somewhat more striking than stated, because for $m < \frac{n}{4}$ the map $u \mapsto u^2$ does not even make sense on W^{2m}_1 in general (cf the counterexample in Remark 3.5, that actually deals with squares of functions). In addition to the technical condition that $B^{2m+t}_{1,\infty}$ should be outside of a certain parameter domain, this observation gives a more fundamental reason that the relation $u \in W^{2m}_1$ cannot be shown via reiteration. It is by use of exact paralinearisation, in which u enters once and in a suitable (ie moderate) way, that $u \in W^{2m}_1$ is obtained.

In the opposite direction, such additional regularity properties cannot be obtained for every semi-linear boundary problem, for it would clearly be necessary that the linear domain $\mathbb{D}(\mathscr{A})$ would contain parameters outside of the domain $\mathbb{D}(Q,\delta)$ associated with $Q(u)$. (If $\mathbb{D}(\mathscr{A}) \setminus \mathbb{D}(Q,\delta) = \emptyset$, it holds that $\mathbb{D}(\mathscr{A}) \cap \mathbb{D}(L_u) = \mathbb{D}(\mathscr{A},Q)$, whence the extended boot-strap method of [Joh95b] applies.) As an example of this, the von Karman equations give a problem in which $\mathbb{D}(\mathscr{A}) \setminus \mathbb{D}(Q,\delta)$ is empty; cf Figure 3. Moreover, in a wider context with non-smooth coefficients, S. I. Pohozaev [Poh93] has given explicit examples of solutions to semi-linear problems in which boot-strap methods give optimal regularity results.

All in all there are legion examples of regularity properties beyond those obtainable by bootstrap methods. These are of importance for the general theory of partial differential equations, albeit at some distance from the most common boundary problems of mathematical physics. The general theory is applied to the stationary von Karman problem in Section 5 and 6, with consequences both for the solvability of this problem and for the regularity of its solutions. This application also illustrates that operators of product type (d_0,d_1,d_2) should preferably be written with d_2 as high as possible in order to enlarge the parameter domains.

Perhaps it also deserves to be emphasised that the consistent use of the parameter domains, as a notion, has paved the way for the qualitative, but consise discussion of boundary problems, not least in this final section.

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