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December 2004 R-2004-33

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URL: www.math.auc.dk/research/reports/reports.htm

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

Largest non-unique subgraphs

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Abstract

The reduction number $r(G)$ of a graph G is the maximum integer $m \leq |E(G)|$ such that the graphs $G - E, E \subseteq E(G), |E| \leq m$, are mutually non-isomorphic, i.e., each graph is unique as a subgraph of G. We prove that $r(G) \leq \frac{|V(G)|}{2} + 1$ and show by probabilistic methods that $r(G)$ can come close to this bound for large orders. By direct construction, we exhibit graphs with large reduction number, although somewhat smaller than the upper bound. We also discuss similarities to a parameter introduced by Erdős and Rényi capturing the degree of asymmetry of a graph, and we consider graphs with few circuits in some detail.

Keywords: Asymmetry, edge deletion, isomorphism, random graph, reduction number, spanning subgraph, symmetry, tree, unicyclic graph, unique subgraph.
AMS Mathematics Subject Classification: 05C75, 05C05.

1 Introduction

From one point of view this paper is a parallel to the paper [8] by Erdős and Rényi, where they define a measure of the degree of asymmetry of a graph and deduce a number of properties of it. From another point of view, it is a continuation of previous work by the authors and others on symmetry properties of subgraphs obtained from a graph by deletion of edge sets. A question of Entringer and Erdős [7] gave this continuation the present direction.

To elaborate on the second point of view, the papers [6], [11], [12] and [13] are all concerned with situations where all connected subgraphs obtained by deleting

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different edge sets of the same (restricted) size from a given graph are isomorphic. The question by Entringer and Erdős suggested interest in the case where deletion of different edge sets of the same size always gives *non-isomorphic* subgraphs. A variant of this was treated in [2]. The Entringer-Erdős line was taken up by Desphande in [5], and it is this line of investigation that we shall carry further in the present paper.

We consider finite graphs without loops and multiple edges. The basic definition is the following.

Definition 1 A subgraph of G is called *unique* if it is isomorphic to no other subgraph of G.

We shall, as mentioned above, be interested in subgraphs obtained by deletion of edge sets from a graph G, i.e., we shall consider spanning subgraphs of G.

Definition 2 Let G be a graph and let m be an integer, $0 \leq m \leq |E(G)|$. Then G is called *m-uniquely deletable* if each spanning subgraph of G obtained by deleting any set of at most m edges from G is unique: if $E_1, E_2 \subsetneqq E(G)$, $|E_1| \leq m$, $|E_2| \leq m$ and $G - E_1 \cong G - E_2$, then $E_1 = E_2$.

It follows from Definition 2 that every graph is 0-uniquely deletable and that a graph G cannot be m-uniquely deletable for any $m \geq |E(G)| - 1$, except if $|E(G)| = 0$ in which case G is 0-uniquely deletable, or if $|E(G)| = 1$ where G is also 1-uniquely deletable, because if $|E(G)| \geq 2$ then deleting distinct sets of $|E(G)| - 1$ edges always results in isomorphic graphs (K_2 and isolated vertices).

Following Desphande [5], we define our main subject, the parameter $r(G)$.

Definition 3 For any graph G, the *reduction number* $r(G)$ is

$$r(G) = \max\{m \mid G \text{ is } m\text{-uniquely deletable }\}.$$

In [3] by Brigham and Dutton, a slightly similar topic is considered, namely whether there exist *disjoint* edge sets E_1 and E_2 of given size such that $G - E_1 \cong G - E_2$.

Going back to the first point of view on the rôle of this paper mentioned in the opening paragraph, Erdős and Rényi defined the parameter $A(G)$ as the smallest number of edge changes needed to make the graph G have a non-trivial automorphism, where an edge change consists of either deleting an edge or adding an edge between two non-neighbouring vertices. This parameter and our $r(G)$ display some similarities, but examples show that either one of them can be large while the other is small. We explore the relation between the two in a forthcoming paper [1]. One common feature between their paper and ours is that we can show that the value of the parameter can be quite big compared to the number of vertices, but that we cannot actually *construct* graphs with such values. Instead, probabilistic methods are used. The result that we show about this can be summarized as follows.

Main Theorem *Any graph G of order n satisfies*

$$r(G) \leq \frac{n}{2} + 1,$$

and for all large n, there exists a graph G of order n with

$$r(G) > \frac{n}{2} - \sqrt{n \log n}.$$

The structure of the paper is as follows. In the next section, Section 2, we record some simple properties of $r(G)$. In Sections 3 and 4, the Main Theorem is proved as Theorem 2 and Theorem 3, respectively. Section 5 is devoted to trees and unicyclic graphs, and finally in Section 6 we present an explicit construction of graphs G with arbitrarily large $r(G)$.

2 Simple properties of $r(G)$

2.1 Extreme values for $r(G)$

Two edges e, e' of a graph G are called *deletion-similar* if $G - e \cong G - e'$, and they are called *similar* if $\pi(e) = e'$ for some automorphism π in $Aut(G)$, the automorphism group of G. Obviously, similar edges are also deletion-similar. Two edges are *pseudosimilar* if they are deletion-similar but not similar.

Graphs with reduction number 0 are described below.

Proposition 1 *Let G be a graph. The $r(G) = 0$ if and only if either*

1. *$|E(G)| = 0$, or*
2. *Some component of G of order at least 3 has a non-identity automorphism, or*
3. *G contains two distinct isomorphic components of order at least 2, or*
4. *G has two pseudosimilar edges.*

Proof. Assume first that $r(G) = 0$ and that G has at least one edge. Then, as $r(G) = 1$ for graphs G with $|E(G)| = 1$, it must have more than one edge and must contain a pair of distinct deletion-similar edges. If one such pair are similar, they must either be in the same component, and 2 occurs, or in distinct components, and 3 occurs. If not, 4 holds.

The if part of the proposition is obvious if 1, 3 or 4 occur. The case of 2 is also clear, as a non-identity automorphism of a graph of order at least 3 must map some edge onto a different edge. □

Corollary 1 *Any graph G with at most one isolated vertex, a non-trivial automorphism and $|E(G)| > 1$ either has $r(G) = 0$ or is disconnected with non-isomorphic components, exactly one of which is a K_2 and none of the others has a non-identity automorphism.* □

We have already remarked in the introduction that a graph G with at least 2 edges has $r(G) \leq |E(G)| - 2$. This observation can in fact be extended in the following way.

We let P_n denote the path and C_n the circuit with n vertices, respectively.

Proposition 2 *For $2 \leq i \leq 5$ we have that if $|E(G)| \geq i$ then $r(G) \leq |E(G)| - i$.*

Proof. Any graph with at least three edges contains one of $K_3, K_{1,3}, P_4, P_3 \cup K_2$, and $3K_2$ as a subgraph and consequently also contains two distinct copies of P_3 or two distinct copies of $2K_2$. Therefore it is not $(|E(G)| - 2)$-uniquely deletable, and so $r(G) \leq |E(G)| - 3$ as desired.

Similarly we see that if G is a graph with four or more edges containing a component with at least four edges, then G contains $K_{1,4}$, or $K_{1,3}$ with one edge subdivided, or K_3 together with a pendent edge, or P_5 or C_4. In each of the five cases we can find two distinct, but isomorphic subgraphs with three edges, proving $r(G) \leq |E(G)| - 4$. If a graph G with at least 4 edges has no component with at least 4 edges, then $r(G) = 0$ by Proposition 1, condition 2 or 3.

If G has at least 5 edges we shall argue that either $r(G) = 0$ or G contains two distinct, isomorphic subgraphs with four edges and therefore $r(G) \leq |E(G)| - 5$: If each component of G contains at most 4 edges, then $r(G) = 0$ by Proposition 1 (a component with 3 or 4 edges must satisfy condition 2, and two K_2 components satisfy condition 3), so assume finally that G has a component with at least 5 edges. Then G as a subgraph contains one of the following graphs: $K_{1,5}, K_{1,4}$ with an edge subdivided, $K_{1,3}$ with two edges subdivided, $K_{1,3}$ with two subdivision vertices on one edge, K_2 together with two pendent edges from each end, P_6, C_5, K_4 with one edge missing, K_3 together with two pendent edges from a vertex, K_3 together with a pendent edge from each of two vertices, K_3 together with a pendent edge which is subdivided, or C_4 with a pendent edge. In each of the twelve cases G contains two distinct, isomorphic subgraphs with four edges. □

Corollary 2 *Any graph G with $|E(G)| \neq 1$ which has at most 5 vertices or at most 5 edges has $r(G) = 0$.*

Proof. The edge part follows directly from Proposition 2 (and the fact that a graph without edges has reduction number zero). For the vertex part, note that a graph with at most five vertices and at least six edges satisfies the assumptions of Corollary 1. □

Proposition 2 cannot be extended to $i = 6$, and Corollary 2 can neither be extended to the case of 6 vertices nor the case of 6 edges. All this is shown by the graph G of Figure 1 which has 6 vertices and edges and $r(G) = 1$.

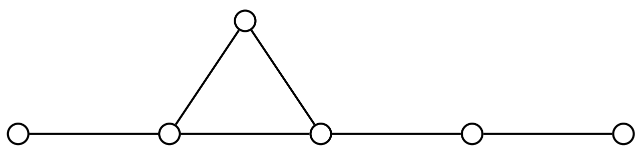

Figure 1: A smallest graph G with $|E(G)| \neq 1$ and $r(G) \neq 0$; it has $r(G) = 1$.

2.2 The reduction number of subgraphs

The following observation is useful.

Proposition 3 *For any graph G and any $F \subseteq E(G)$, $r(G - F) \geq r(G) - |F|$.*

Proof. Clearly $r(G) \leq r(G - F) + |F|$ if $r(G - F) = |E(G - F)|$, and if not the same inequality follows from the fact that if E_1 and E_2 are distinct edge sets of $G - F$ whose deletions give isomorphic subgraphs, then $E_1 \cup F$ and $E_2 \cup F$ are distinct edge sets of G with the same property. □

We note that the inequality of Proposition 3 is sharp in the sense that it can be achieved. There is equality if G is the graph of Figure 1 and F consists of any single edge of G. But it is by no means always attained, and in fact it is quite possible that $r(G - F) > r(G)$. This follows immediately from the fact that both $r(K_n) = 0$ and $r(\overline{K_n}) = 0$, but by successively deleting edges to go from one to the other, graphs with positive reduction number may be passed.

The next proposition is also in [5].

Proposition 4 *Let G be a graph with at least one component of order at least three, and let $G_1, \cdots, G_c$ be those components of G which contain at least two edges. Then $r(G) \leq \min\{r(G_i) \mid 1 \leq i \leq c\}$.*

Proof. For each i we have that $|E(G_i)| \geq 2$ and therefore, by Definition 2, G_i contains two distinct edge subsets E_1, E_2 of cardinality $r(G_i)+1$ whose deletions leave two isomorphic graphs. But E_1, E_2 are also subsets of G so $r(G) \leq r(G_i)$. □

Remarks

1. Strict inequality occurs in Proposition 4 whenever G consists of two isomorphic copies of the same graph with positive reduction number. Then $r(G) = 0$ by Proposition 1 no matter how large the reduction number of each component is.

2. A graph G consisting of two non-isomorphic components G_1, G_2 such that $r(G) < \min\{r(G_1), r(G_2)\}$ can be obtained as follows. Let G_1 be a connected graph with $r(G_1) = m \geq 3$ (we shall see that such graphs exist) and let $F \subsetneqq E(G_1)$ be a set of t edges such that $t < \frac{m}{2}$. Then, by Proposition 3, $G_2 = G_1 - F$ is a graph with $r(G_2) \geq m - t$ and $G = G_1 \cup G_2$ has $r(G) \leq t$ since $G - F = (G_1 - F) \cup G_2 = G_2 \cup G_2$ has $r(G - F) = 0$. Either G_2 is connected, or it has a component G_2' with at least as high reduction number, and $r(G_1 \cup G_2') \leq t$.

2.3 The relation to symmetry

Definition 4 A graph G is called *symmetric* if it has a non-trivial automorphism, i.e., if $\mathrm{Aut}(G) \neq \{\mathrm{id}\}$. Otherwise G is called *asymmetric*.

The Erdős-Rényi measure of asymmetry introduced in [8] naturally has the property that it is zero for a symmetric graph. The reduction number behaves in a similar way, but the details are more subtle.

As can also be seen from Corollary 1, if G is symmetric and neither contains $2K_1$ nor K_2 as components then G contains distinct edges e_1, e_2 such that $G - e_1 \cong G - e_2$ and hence $r(G) = 0$.

Theorem 1 *Assume that G is symmetric and that G has at most one isolated vertex. Then $r(G) > 0$ implies*
(i) precisely one component C of G is a K_2-component with vertices x, y, and
(ii) if $G \neq C$ and $G - C \neq K_1$, then $G - C$ is asymmetric with $r(G - C) > 0$.

Proof. Let G be symmetric and $r(G) > 0$. Then $E(G) \neq \emptyset$ because $r(G) = 0$ for a graph with no edge. For any $\pi \in Aut(G)$ we have $\pi(e) = e$, $\forall e \in E(G)$ since the existence of an edge e such that $\pi(e) \neq e$ would imply $G - e \cong G - \pi(e)$ and hence $r(G) = 0$. Either G has two isolated vertices, which by hypothesis is excluded, or for $\pi \in Aut(G) \setminus \{id\}$ there exists an edge $e = xy$ such that $\pi(x) = y, \pi(y) = x$. If one of x, y, say x, has another neighbour z, the edge xz is mapped onto $y\pi(z) \neq xz$, a contradiction. Thus C with vertices x, y is a K_2-component of G. If G contains two K_2-components, certainly $r(G) = 0$. That proves (i). If $r(G \setminus C) = 0$ we would have $r(G) = 0$, so $r(G \setminus C) > 0$. If $Aut(G \setminus C) \neq \{id\}$ we should have two K_2-components in G, therefore $Aut(G \setminus C) = \{id\}$. That proves (ii). $\square$

Corollary 3 *If G is symmetric, then $r(G) = 0$ unless either G contains two isolated vertices or precisely one component of G is a K_2.*

Remark

Let H be a graph with $r(H) = r$; then $G_1 = pK_1 \cup H$ has $r(G_1) = r$, and $G_2 = K_2 \cup H$ can, depending on H, have a value $r(G_2)$ from 0 up to $r + 1$.

The parameter $A(G)$ of Erdős and Rényi satisfies $A(G) = A(\overline{G})$ for all graphs G. It seems unlikely that there is a simple relationship between $r(G)$ and $r(\overline{G})$.

3 An upper bound for $r(G)$.

In this section we prove the upper bound stated in the main theorem.

Let $\mathcal{H} = \{H_1, H_2, \cdots, H_m\}$ be a family of m sets whose union contains n elements. Let the *symmetric difference* $(H \setminus H') \cup (H' \setminus H)$ of two sets be denoted by $H \triangle H'$. The average size of the symmetric difference of pairs of sets in $\mathcal{H}$ is $\tilde{\triangle} = \frac{\sum_{1 \leq i < j \leq m} |H_i \triangle H_j|}{\binom{m}{2}}$. Formulated for a hypergraph $\mathcal{H}$ with n vertices and m edges H_i we have, with d_k denoting the number of hyperedges containing the vertex v_k:

Lemma 1 *If $\mathcal{H}$ has n vertices and m edges, then $\tilde{\triangle} \leq \frac{mn}{2(m-1)}$.*

Proof. By definition,

$$\begin{aligned} \binom{m}{2} \tilde{\triangle} &= \sum_{i<j} (|H_i| + |H_j| - 2|H_i \cap H_j|) \\ &= \sum_{i=1}^{m} (m-1)|H_i| \;-\; \sum_{1 \leq i < j \leq m} 2|H_i \cap H_j| \\ &= \sum_{k=1}^{n} (m-1) d_k - \sum_{k=1}^{n} 2\binom{d_k}{2} \\ &= \sum_{k=1}^{n} d_k (m - d_k) \leq n \frac{m^2}{4}. \end{aligned}$$

The last inequality comes from the fact that each of n parabolas $-d_k^2 + m d_k$ attain their maximum for $d_k = \frac{m}{2}$. $\square$

Theorem 2 *A graph G with n vertices has $r(G) \leq \frac{n+2}{2}$.*

Proof. Assume that the maximum valency $\Delta(G) \le \frac{n+6}{4}$. If G contains no pair of vertices with a common neighbour, all components are K_1's or K_2's and hence $r(G) \le 1 \le \frac{n+2}{2}$. Otherwise G contains vertices u, v whose neighbourhoods satisfy $N(u) \cap N(v) \neq \emptyset$. We have $|N(u) \triangle N(v)| \le |N(u)| + |N(v)| - 2 \le \frac{n+2}{2}$. This implies $r(G) \le \frac{n+2}{2}$ since G has two distinct edges sets, each of size at most $\frac{n+4}{2}$, whose deletions leave isomorphic graphs, namely all edges from $\{u, v\}$ to $N(u) \triangle N(v)$ together with one further edge between a vertex of $N(u) \cap N(v)$ and respectively u or v.

Assume that G contains a vertex u with valency $d(u) \ge \frac{n+7}{4}$. Consider the family $\{N(v) - u \mid v \in N(u)\}$ of subsets in $V(G) - u$. There are $d(u) \ge \frac{n+7}{4}$ sets in the family, we shall consider only $m' = \lfloor \frac{n+7}{4} \rfloor \ge \frac{n+4}{4}$ of these sets. Their union contains n' vertices, $n' \le n - 1$. Applying Lemma 1 to this family we obtain $\tilde{\triangle} \le \frac{m'n'}{2(m'-1)} \le \frac{\frac{n+4}{4}(n-1)}{2\frac{n}{4}} = \frac{n}{2} + \frac{3n-4}{2n} < \frac{n+3}{2}$. I.e., we have proven existence of v_1, v_2 in $N(u)$ such that $N(v_1) \cap N(v_2)$ is nonempty, as it contains u, and $|N(v_1) \triangle N(v_2)| \le \frac{n+2}{2}$.

Deleting all edges from $\{v_1, v_2\}$ to $N(v_1) \triangle N(v_2)$ and one further edge from respectively v_1 or v_2 to u we obtain two distinct sets of at most $\frac{n+4}{2}$ edges whose deletions leave isomorphic graphs, therefore $r(G) \le \frac{n+2}{2}$. □

4 Existence of graphs G with $r(G)$ nearly $n/2$

Complementing the results of the previous section, and thereby proving the second half of our main theorem, here we prove that for large n there exist graphs G of order n whose $r(G)$ is very close to $n/2$. This is the second part of the main theorem stated in the introduction. We denote by $\mathbf{G}_n$ the random graph on n vertices, with edge probability 1/2.

Theorem 3 *For n large, the random graph $\mathbf{G}_n$ satisfies the inequality*

$$r(\mathbf{G}_n) > n/2 - \sqrt{n \log n}$$

with positive probability. In particular, for every sufficiently large n there exists a graph G of order n and $r(G) > n/2 - \sqrt{n \log n}$.

First, let us summarize the structure of the proof. Assuming that the graphs $G' := G - E'$ and $G'' := G - E''$ ($E' \neq E''$) are isomorphic, there exists a permutation $\pi : V \to V$, other than the identity, that transforms G' to G''. We may view this situation from the other side, asking for a lower bound on the smallest possible number of edges, $m(\pi) = m(\pi, G)$, for which there exist edge sets E', E'' such that $|E'| = |E''| = m(\pi)$ and $\pi(G - E') = G - E''$. (In this latter question we may even allow $E' = E''$. If the minimum of $m(\pi)$ over all permutations π is attained only with identical E', E'' then $r(G) \ge \min_\pi m(\pi)$, and otherwise $r(G) = \min_\pi m(\pi) - 1$.) Denoting by $k(\pi)$ the number of vertices fixed by π (hence $0 \le k(\pi) \le n-2$), we are going to prove the following three asymptotic estimates as $n \to \infty$:

A : For the permutations π with $k(\pi) = n-2$, the sum of the probabilities of $m(\pi) \le n/2 - \sqrt{n \log n} + 1$ is at most $1 - 1/n$.

B : For the permutations π with $n - 2\sqrt{n} \le k(\pi) \le n-3$, the sum of the probabilities of $m(\pi) < n/2$ is $o(1/n)$.

C : For the permutations π with $k(\pi) < n - 2\sqrt{n}$, the sum of the probabilities of $m(\pi) < n/2$ is $o(1/n)$.

Summing up for the three cases, we obtain that the existence of two distinct sets of at most $n/2 - \sqrt{n \log n} + 1$ edges whose removals yield isomorphic subgraphs has probability at most $1 - 1/n + o(1/n) < 1$ as n gets large; and then there exist graphs G of order n in which $r(G) > n/2 - \sqrt{n \log n}$ holds, indeed.

We shall use the following terminology. A *moving vertex* (under the permutation π) is a vertex v such that $\pi(v) \ne v$. Similarly, a *moving pair* uv is an unordered pair of vertices u, v whose image under π is different from itself; i.e., uv is *not* a moving pair if and only if either both $\pi(u) = u$ and $\pi(v) = v$ (both vertices are non-moving), or $\pi(u) = v$ and $\pi(v) = u$ (the vertex pair forms a 2-element orbit in V, and it is a fixed element under the permutation induced by π on the unordered vertex pairs). In this notation, uv and vu mean the same pair.

The following observation is clear by definition.

Fact 1 *If $\pi(G - E') = G - E''$ and uv is a moving pair such that $uv \in E(G)$ but $\pi(uv) \notin E(G)$, then $uv \in E'$.* □

A quantitative relationship between moving vertices and moving pairs is established next.

Fact 2 *If precisely ℓ vertices are moving under π, then the number of moving pairs is at least*

$$\frac{\ell(2n - \ell - 2)}{2}.$$

Proof. Let us denote by $k = n - \ell$ the number of vertices fixed by π. Among all the $\binom{n}{2}$ unordered pairs, the only non-moving ones are those $\binom{k}{2}$ induced by the k non-moving vertices, plus the vertex pairs which are 2-element orbits under π.

Since the latter pairs form a matching inside the set of the $n-k$ moving vertices, their number cannot exceed $\frac{1}{2}(n-k)$, and therefore at least

$$\frac{n(n-1)-k(k-1)-(n-k)}{2}=\frac{(n^2-k^2)-2(n-k)}{2}=\frac{\ell(2n-\ell-2)}{2}$$

pairs are moving. □

For the probabilistic analysis, the following well-known property of binomial distributions will be needed. (Cf. e.g. IX.(6.4), VII.(1.6) and VII.(1.8) in [9].)

Fact 3 *If ξ is a random variable with binomial distribution, making N independent trials ξ_i each with probability* $\mathsf{Prob}\,(\xi_i=1)=p$ *and* $\mathsf{Prob}\,(\xi_i=0)=q=1-p$*, then $\xi:=\sum\xi_i$ has expectation pN, and moreover it satisfies the inequality*

$$\mathsf{Prob}\,\left(|\xi-pN|>\alpha\sqrt{pqN}\right)<e^{-\alpha^2/2} \tag{1}$$

for any $\alpha>1$. □

In some cases it suffices to apply the simpler formula that if ξ is a random variable with binomial distribution and expectation $M(\xi)$, then

$$\mathsf{Prob}\,\left(|\xi-M(\xi)|>\alpha\sqrt{M(\xi)}\right)<e^{-\alpha^2/2} \tag{2}$$

for every $\alpha>0$.

We also recall an elementary property of random graphs. For a vertex v, denote by $N(v)$ the (open) *neighbourhood* of v, i.e., the set of vertices adjacent to v; and by $N[v]:=N(v)\cup\{v\}$ the *closed neighbourhood* of v.

Fact 4 *It has probability at least $1/n$ that in the random graph $\mathbf{G}_n$ of order n and edge probability $1/2$, the inequality*

$$|N(u)\setminus N[v]|+|N(v)\setminus N[u]|>\frac{1}{2}\,n-\sqrt{n\log n}+1 \tag{3}$$

holds for all pairs u,v of vertices.

Proof. For each vertex $z\notin\{u,v\}$ we define $\xi_z=1$ to hold if and only if precisely one of uz and vz is an edge (i.e., if z is in the symmetric difference of the neighbourhoods of u and v); and $\xi_z=0$ otherwise. Then $\xi_{uv}:=\sum_{z\neq u,v}\xi_z$ has a binomial distribution with $N=n-2$ trials, probability $p=1/2$, and expectation $n/2-1$. We apply the inequality (1) with the value $\alpha=\sqrt{4\log n-2\log 2}$. This will make the right-hand side $e^{-\alpha^2/2}=2/n^2$. Moreover, one can check that for n

sufficiently large, say if $n \geq \frac{64}{(\log 2)^2} \log n$, we also have $\alpha\sqrt{(n-2)/4} < \sqrt{n \log n} - 2$. Thus, the probability that a fixed vertex pair violates (3) is smaller than $2/n^2$. Since we have just $\binom{n}{2}$ pairs of vertices, the probability that no pair u, v violates (3) is greater than $1 - (n^2 - n)/n^2 = 1/n$. $\square$

Now we are in a position to prove the main result of this section.

Proof of Theorem 3. Suppose that the edge-deleted subgraphs $\mathbf{G}_n - E'$ and $\mathbf{G}_n - E''$ are isomorphic, and this isomorphism is established by some (nontrivial) permutation $\pi : V \to V$.

Case A : $k(\pi) = n - 2$

By Fact 1, if π is a permutation with just one 2-element orbit (u, v), then E' has to contain all edges joining $\{u, v\}$ with the symmetric difference of the neighbourhoods of u and v (with u and v excluded). Thus, Fact 4 implies that $|E'| > n/2 - \sqrt{n \log n} + 1$ holds with probability at least $1/n$.

Case B : $n - 2\sqrt{n} \leq k(\pi) \leq n - 3$

Consider any fixed permutation π in the given range, and denote by Y and Z the set of moving and non-moving vertices, respectively. Here $|Z| \geq n - 2\sqrt{n}$. We fix a subset $Z' \subseteq Z$ of cardinality $N := n - \lfloor 2\sqrt{n} \rfloor$. The general idea of the proof is to introduce a set of 0–1 random variables whose sum, ξ_π is a lower bound on the number of pairs zy ($z \in Z'$, $y \in Y$) such that z is adjacent to y but not to $\pi(y)$; and then we know by Fact 1 that ξ_π is a lower bound on $|E'|$, too. The random variables to be defined will be independent and identically distributed. More explicitly, we shall have N or $2N$ variables that take value 1 with probability $3/4$ or $1/2$, respectively. Then their sum ξ_π has a binomial distribution with expectation $M(\xi_\pi) = 3N/4$ or $M(\xi_\pi) = N$. We then get, from (2), in the first case

$$
\begin{aligned}
\mathsf{Prob}\,(|E'| < n/2) &\leq \mathsf{Prob}\,(\xi_\pi < n/2) \\
&\leq \mathsf{Prob}\left(|\xi_\pi - M(\xi_\pi)| > M(\xi_\pi) - \frac{n}{2}\right) \\
&= \mathsf{Prob}\left(|\xi_\pi - M(\xi_\pi)| > (\sqrt{M(\xi_\pi)} - \frac{n}{2\sqrt{M(\xi_\pi)}})\sqrt{M(\xi_\pi)}\right) \\
&\leq e^{-\frac{1}{2}(M(\xi_\pi) + \frac{n^2}{4M(\xi_\pi)} - n)} \\
&= e^{-\frac{1}{2}(\frac{3}{4}n - \frac{3}{4}\lfloor 2\sqrt{n} \rfloor + \frac{n^2}{3n - 3\lfloor 2\sqrt{n} \rfloor} - n)} \\
&< e^{-\frac{1}{2}(-\frac{n}{4} - \frac{3}{4}\lfloor 2\sqrt{n} \rfloor + \frac{n}{3})} \\
&= e^{-\frac{1}{2}(\frac{n}{12} - \frac{3}{4}\lfloor 2\sqrt{n} \rfloor)} \\
&< e^{-\frac{1}{25}n}
\end{aligned}
$$

for large n. Similarly for the second case, where the corresponding estimates give a bound of $e^{-\frac{1}{9}n}$.

On the other hand, there are fewer than

$$\binom{n}{\lfloor 2\sqrt{n}\rfloor}\lfloor 2\sqrt{n}\rfloor! < n^{2\sqrt{n}} = e^{2\sqrt{n}\log n}$$

permutations with at most $2\sqrt{n}$ moving vertices. Thus, summing up for all permutations belonging to Case B, the total probability that $|E'| < n/2$ is not higher than $e^{2\sqrt{n}\log n - \frac{n}{25}} < e^{-\frac{n}{26}}$ for n large, i.e., this probability is exponentially small.

To define the random variables generating ξ_π, let us recall that at least three vertices are moving.

Case B.1 : At most one orbit has size 2

Then some orbit $(y_1, \ldots, y_s) \subseteq Y$ has size $s \geq 3$. Let $\xi_z = 0$ if either all of zy_1, zy_2, zy_3 are edges, or all of them are non-edges ; and let $\xi_z = 1$ otherwise. These are N random variables with $\mathsf{Prob}\ (\xi_z = 1) = 3/4$ for all $z \in Z'$. If $\xi_z = 1$, then the smallest subscript i can be identified such that zy_i is an edge and zy_{i+1} is a non-edge (for $i = s$ we mean $y_{s+1} = y_1$). Thus, $|E'| \geq \xi_\pi$ holds for $\xi_\pi = \sum_{z\in Z'} \xi_z$.

Case B.2 : There exist two orbits of size 2

For each $z \in Z'$ and each of these two orbits, say o_i $(i = 1, 2)$, a random variable $\xi_{z,i}$ is defined, such that $\xi_{z,i} = 1$ if there is precisely one edge from z to o_i, and $\xi_{z,i} = 0$ otherwise. Then the sum of these $2N$ random variables $\xi_\pi := \sum_{z\in Z'} \xi_{z,1} + \sum_{z\in Z'} \xi_{z,2}$ is a lower bound on $|E'|$.

Case C : $k(\pi) < n - 2\sqrt{n}$

Since $|Z|$ may be small, in this case we may need to find an increasing number of edges for E' in each orbit. It will now be convenient to view orbits on the set of *vertex pairs* rather than just orbits of vertices.

As more than $2\sqrt{n}$ vertices are moving, then we obtain by Fact 2 that there exist more than $\frac{3}{2}n^{3/2}$ moving pairs, for the expression of Fact 2 is an increasing function of ℓ in the interval considered, and $2n^{3/2} - 2n - 2\sqrt{n} > \frac{3}{2}n^{3/2}$ for n large. We are going to define $\lceil \frac{1}{2}n^{3/2}\rceil$ random variables as follows. Consider any orbit of moving pairs, say $(f_1, f_2, \ldots, f_s)$. The i-th random variable for this orbit is defined to have value 1 if f_{2i-1} is an edge *and* f_{2i} is a non-edge ; and 0 otherwise. In this way, to an orbit of s moving pairs, there are $\lfloor s/2\rfloor \geq s/3$ random variables associated, so that their total number is at least $\frac{1}{2}n^{3/2}$ indeed. We keep just $t := \lceil \frac{1}{2}n^{3/2}\rceil$ of them, say $\xi_1, \ldots, \xi_t$.

It is clear that $\mathsf{Prob}\,(\xi_i = 1) = 1/4$ for each i, and if $\xi_i = 1$ then it provides an edge for E'. That is, $\xi_\pi := \sum_{i=1}^{t} \xi_i$ is a lower bound on $|E'|$. Moreover, by the definition of $\mathbf{G}_n$, the ξ_i are totally independent, thus ξ_π has binomial distribution. Certainly $M(\xi_\pi) = t/4 = \frac{1}{4}\lceil \frac{1}{2}n^{3/2}\rceil$, therefore the inequality (2) yields

$$\begin{aligned}
\mathsf{Prob}\,(|E'| < n/2) &\leq \mathsf{Prob}\,(\xi_\pi < n/2) \\
&= \mathsf{Prob}\left(|\xi_\pi - M(\xi_\pi)| > (\sqrt{M(\xi_\pi)} - \frac{n}{2\sqrt{M(\xi_\pi)}})\sqrt{M(\xi_\pi)}\right) \\
&\leq e^{-(\frac{1}{16} - o(1))n^{3/2}} \\
&\leq e^{-\frac{1}{17}n^{3/2}}
\end{aligned}$$

for each π if n is large. On the other hand, the total number of permutations is $n! < e^{n \log n}$, implying that the sum of these probabilities taken over all π is still exponentially small.

Combining the above three cases, the proof of Theorem 3 is complete. □□□

Remarks

1. Analyzing in the proof of Fact 4, how much smaller $\alpha\sqrt{(n-2)/4}$ is than $\sqrt{n \log n}$, one can see that the righthand side of (3) can be improved to show that the inequality

$$r(\mathbf{G}_n) > n/2 - \sqrt{n \log n} + \tfrac{\log 2}{4}\sqrt{n/\log n}$$

also holds with positive probability if n is sufficiently large.

2. It would be of interest to construct an infinite sequence of graphs G_n on n vertices such that $r(G_n) = n/2 - o(n)$ as $n \to \infty$. See Section 6 for a weaker construction.

3. Improving results of Entringer and Erdős [7], and Harary and Schwenk [10], Brouwer proved [4] that if $f(n)$ denotes the largest number of unique subgraphs any graph on n vertices can have, then $\log f(n) = \frac{n^2}{2} - n \log n + O(n)$. Though the present section is also concerned with graphs having many unique subgraphs, the obvious difference from [4] to our problem is that in [4] all edge subsets, small or large, whose deletions give unique subgraphs are considered and counted, while we only consider edge sets up to size $r(G)$. Therefore, if we count the unique subgraphs considered here, we get a much smaller number, namely $\sum_{i=0}^{r} \binom{\binom{|E(G)|}{2}}{i}$, where $r \sim \frac{n}{2}$; i.e., its logarithm is $O(n \log n)$.

5 Trees and unicyclic graphs.

In this section we consider graphs with few circuits.

Theorem 4 *([8], [5]) If G contains no circuit, then* $0 \leq r(G) \leq 1$.

Proof. Let $P = x_0x_1x_2 \dots x_k$ be a longest path in G. We may assume $k \geq 3$, because otherwise each component is a K_1, a K_2 or a star $K_{1,s}, s \geq 2$, and the theorem holds. We have $d_G(x_k) = 1$ since G contains no circuit. If e_1, e_2 are distinct, pendent edges from x_{k-1} then $G - e_1 \cong G - e_2$ and $r(G) = 0$, hence we may assume that $d_G(x_{k-1})$=2.

We may assume that $d_G(x_{k-2})$=2, because $d_G(x_{k-2}) \geq 3$ implies that $x_{k-2}y \in E(G)$ for some y not in P. If $d_G(y) = 1$ then $E_1 = \{x_{k-2}y, x_{k-1}x_k\}$ and $E_2 = \{x_{k-2}x_{k-1}, x_{k-1}x_k\}$ are distinct edge sets with $G - E_1 \cong G - E_2$ giving $r(G) \leq 1$. Otherwise $d_G(y) \geq 2$ and $x_{k-2}y, yz \in E(G)$ for some z not in P. We may assume $d_G(y) = 2$. But then $G - x_{k-1}x_k \cong G - yz$ and $r(G)$=0.

With $d_G(x_{k-2}) = d_G(x_{k-1}) = 2, d_G(x_k) = 1$ we see that $G \setminus \{x_{k-3}x_{k-2}, x_{k-2}x_{k-1}\}$ $\cong G \setminus \{x_{k-3}x_{k-2}, x_{k-1}x_k\}$, so that $r(G) \leq 1$. □

Theorem 4 is generalized below.

Theorem 5 *Let G be a graph with n vertices, none of which are isolated. Let k be a nonnegative integer. If G has $n - 1 + k$ edges and no K_2-component, then $r(G) \leq k + 1$. In particular, a tree T has $r(T) \leq 1$ and a unicyclic graph U has $r(U) \leq 2$.*

Proof. We may assume $n \geq 3$. If G is connected, remove a set F of k edges to obtain a spanning tree T. If $r(T) = 1$ there are distinct edge sets $E'_i = \{e_{1i}, e_{2i}\}, i = 1, 2$, such that $T - E'_1 \cong T - E'_2$ and $E_i = F \cup E'_i, i = 1, 2$, are distinct edge sets with $k + 2$ edges such that $G - E_1 \cong G - E_2$. Analogously if $r(T) = 0$. This proves $r(G) \leq k + 1$ for G connected. If G is disconnected, G has $\omega \geq 2$ components and $(n - \omega) + (k + \omega - 1)$ edges and by the pigeon hole principle there exists a component with $n_i \geq 3$ vertices and at most $n_i - 1 + k$ edges. Hence the previous argument and Proposition 4 complete the proof. □

If a vertex in a graph G has two pendent edges, then $r(G) = 0$. Erdős and Rényi ([8], Theorem 6) proved that if a tree T is chosen at random from the set of all possible trees on n labelled vertices then the probability that T has a vertex with two pendent edges tends to one as n grows to infinity. That implies Theorem 6 below.

Theorem 6 $r(G) = 0$ *for almost all trees.* □

It is unknown which trees T have $r(T) = 0$ and which have $r(T) = 1$. By Proposition 1, this is a question about the existence of pseudosimilar edges in trees with trivial automorphism group.

A path $P_n = v_1, v_2, \dots, v_n$, $n \geq 6$, with a pendent edge attached to v_3 is an example of a tree T with $r(T) = 1$.

Desphande ([5]) characterized graphs with reduction number 1 for a small subclass of unicyclic graphs, namely for $\mathcal{G}$ consisting of circuits with precisely two of its vertices having valency greater than two, one of them having a pendent edge, the other a pendent path with two edges. The remaining graphs in this class have reduction number 0.

Theorem 7 *Let $U \in \mathcal{G}$. Then $r(U) = 1$ if and only if*

1. *the circuit of U is a K_3, or*
2. *U is a pentagon with two neighbouring vertices having respectively a pendent edge and a subdivided pendent edge, or*
3. *U is a circuit of odd length $\neq 5$ with any two of its vertices having respectively a pendent and a subdivided pendent edge, or*
4. *U is an even length circuit with a pendent and a subdivided pendent edge attached to two vertices not diametrically opposite on C.* □

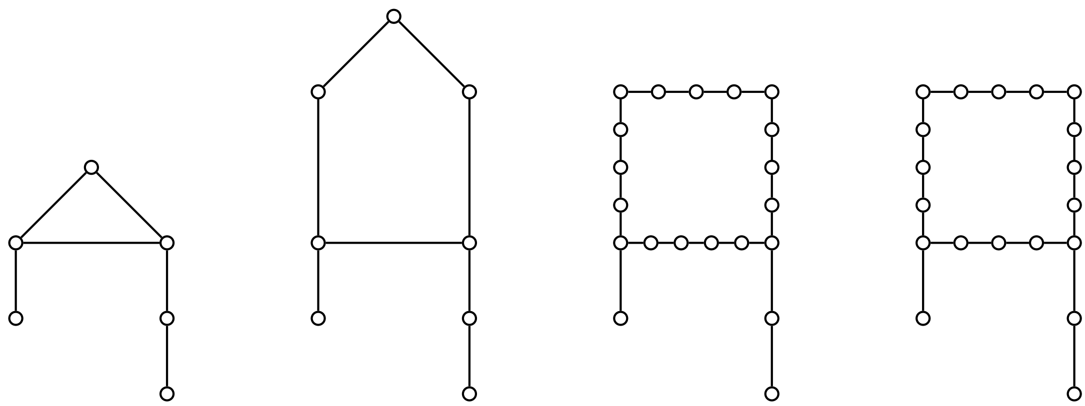

Figure 2: Graphs in $\mathcal{G}$ with $r(U) = 1$

Desphande gave ([5]) for each $m \geq 2$ an example of an infinite tree T_m with $r(T_m) = m$ (with the same definition for the infinite case). He asked — surprisingly — whether any finite graph G with $r(G) \geq 2$ exists. We give an affirmative answer to this several times in the present paper, and in Figure 3 below we include an example of a unicyclic graph U with $r(U) = 2$.

Theorem 8 *Let U be a unicyclic graph with circuit C and with $r(U) = 2$. Then each pendent tree from C is a path of length 0 or 2, and no two adjacent vertices on C can both have pendent 0-paths.*

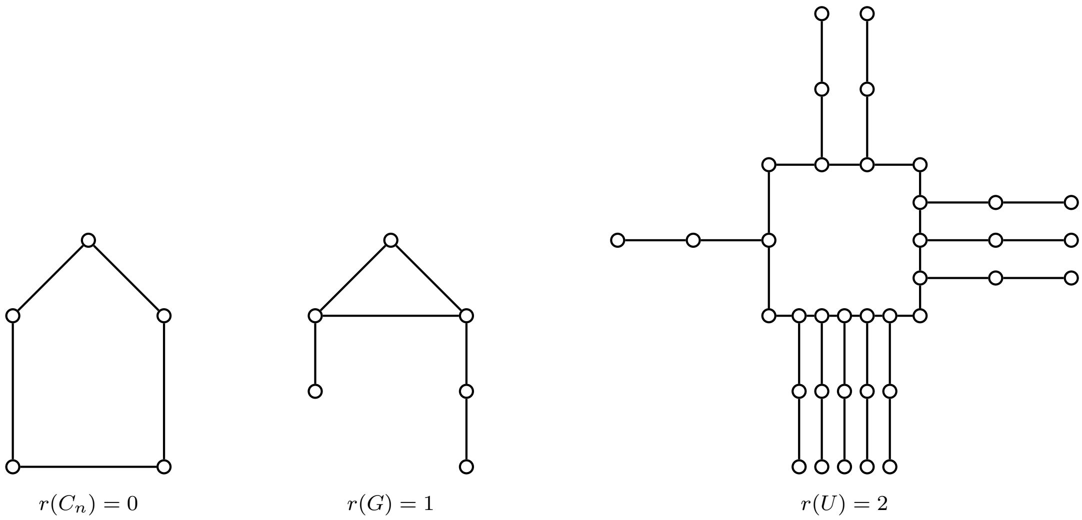

Figure 3: Unicyclic graphs having reduction number respectively 0, 1 and 2.

Proof. Let U be a unicyclic graph with circuit $C = x_1x_2\cdots x_n$ and let T_i denote the tree pendent from x_i, i.e., that component of $U - \{x_{i-1}x_i, x_ix_{i+1}\}$ which contains x_i, and let $r(U) = 2$. We shall prove that for each $i, 1 \le i \le n$, the height of T_i is at most two. Let $x_iy_1y_2\cdots y_s$ be a longest path in T_i from x_i. Assume $s \ge 3$. If y_{s-1} besides y_{s-2}, y_s has a third neighbour y'_s then its valency $d(y'_s) = 1$ which gives $U - y_{s-1}y_s \cong U - y_{s-1}y'_s$, but that would imply $r(U) = 0$, hence $d(y_{s-1}) = 2$. If $d(y_{s-2}) = 2$ we would have $U - \{y_{s-3}y_{s-2}, y_{s-2}y_{s-1}\} \cong U - \{y_{s-3}y_{s-2}, y_{s-1}y_s\}$ implying $r(U) \le 1$. Therefore, y_{s-2} besides y_{s-3}, y_{s-1} has a neighbour y'_{s-1}. If $d(y'_{s-1}) > 1$, either two edges pendent from y'_{s-1} or edges pendent from y'_{s-1}, y_{s-1} are similar, so that $r(U) = 0$. Hence $d(y'_{s-1}) = 1$, and we may assume $d(y_{s-2}) = 3$. But now $U - \{y_{s-3}y_{s-2}, y_{s-2}y'_{s-1}\} \cong U - \{y_{s-3}y_{s-2}, y_{s-1}y_s\}$ implying $r(U) \le 1$. This proves $s \le 2$, i.e., every tree $T_i, 1 \le i \le n$, pendent from C has height at most 2. If T_i has height precisely 2, then T_i is a path on 3 vertices; for assume otherwise that $x_iy_1y_2$ is properly contained in T_i, then $d(y_1) = 2$ since U cannot have two edges pendent from the same vertex. Analogously, if $x_iy'_1$ is an edge pendent from x_i, then $U - y_1y_2$ has two pendent edges from x_i giving $r(U) \le 1$. Thus $d(y'_1) \ge 2$, and $x_iy'_1y'_2$ is a path in T_i. In fact, $d(y'_1) = 2$ and the edges $y_1y_2, y'_1y'_2$ are similar, so that $r(U) = 0$. This contradiction proves that the only tree of height 2 pendent from C possible is P_3. Considering adjacent vertices on C we find below that C has no pendent edge.

$T_i = x_i$, $T_{i+1} = x_{i+1}y$ is impossible since $U - x_{i-1}x_i$ would have two pendent edges from x_{i+1} giving $r(U) \le 1$. Also $T_i = x_iy$, $T_{i+1} = x_{i+1}z_1z_2$ is impossible. For then $U - x_{i-1}x_i$ would have two similar edges x_iy and z_1z_2 giving $r(U) \le 1$. It follows that if $T_i = x_iy_i$ holds for one value of i, then T_i, for every value of i, must consist of a pendent edge to C and obviously $G - x_ix_{i+1} \cong G - x_{i+1}x_{i+2}$ giving $r(U) = 0$. It follows that C can have no pendent edge.

Finally, the case $T_i = x_i$, $T_{i+1} = x_{i+1}$ is impossible. For $r(U) = 2$ implies that some vertex on C has valency greater than 2, and we may choose notation such that $d(x_{i+2}) > 2$, i.e., $T_{i+2} = P_3 = x_{i+2}y_1y_2$. But then $U - x_{i-1}x_i$ has two similar edges, namely x_ix_{i+1} and y_1y_2, yielding $r(U) \leq 1$. This proves Theorem 8. $\square$

The unicyclic graph U in Figure 3 with $r(U) = 2$ consists of a circuit with pendent P_3s from series of vertices separated by valency 2 vertices. U can be described by (1,2,3,5) which indicates that round the circuit appears in that order one vertex with a pendent P_3, a valency 2 vertex, two vertices with pendent P_3s, a valency 2 vertex, three vertices with pendent P_3s, a valency 2 vertex, five vertices with pendent P_3s and finally a valency 2 vertex. The graph indicated by (2,4,8,16) as well as a graph of type $(k, 2k+1, 4k+3, 8k+7, \cdots, \cdot)$, $k \geq 1$, also have reduction number 2. It is an open problem which sequences produce $r(U) = 2$.

6 Construction of graphs with prescribed value for $r(G)$.

In Section 4 we proved by probabilistic methods that graphs with arbitrarily large reduction number exist. Here we present an explicit construction.

Theorem 9 *For any positive integer m there exists a graph G with $r(G) = m$.*

Proof. To give an example of a graph G with $r(G) \geq m$ we shall construct G such that its vertices have distinct labels and such that they preserve this label after deletion of at most m edges from G. Thus, if E_1, E_2 are subsets of $E(G)$ with $\max\{|E_1|, |E_2|\} \leq m$ then $G - E_1 \cong G - E_2$ implies $E_1 = E_2$ and hence $r(G) \geq m$.

Let $m \geq 3$ be a fixed integer. For k odd, $n = \binom{k-1}{\frac{k-1}{2}}$ is an even integer, and for k large enough we have $(k-1)(m+1) \leq n-1$. **We choose and fix an odd k such that this inequality holds**. It is well known that if n is an even, positive integer, there exists for all s in the range $0 \leq s \leq n-1$ an s-regular graph with n vertices. So with the two conditions above satisfied we can choose graphs $G_1, G_2, \cdots, G_k$ such that G_α is an $(\alpha-1)(m+1)$-regular graph on n vertices for each $\alpha, 1 \leq \alpha \leq k$.

We can label the vertices of G_α with n distinct, unordered sets of integers $\{\beta_1, \beta_2, \cdots, \beta_{\frac{k-1}{2}}\}$ by selecting $\frac{k-1}{2}$ indices from the set $\{1, 2, \ldots, \alpha-1, \alpha+1, \ldots, k\}$ of indices for the other $k-1$ graphs $G_1, \cdots, G_{\alpha-1}$, $G_{\alpha+1}, \cdots, G_k$.

Let G be the graph obtained from $G_1 \cup G_2 \cup \cdots \cup G_k$ by joining each $v \in V(G_\alpha)$ with label $\{\beta_1, \beta_2 \cdots, \beta_{\frac{k-1}{2}}\}$ to every vertex of the $\frac{k-1}{2}$ graphs $G_{\beta_1}, \cdots, G_{\beta_{\frac{k-1}{2}}}$ for $\alpha = 1, 2, \cdots, k$.

Let $v \in V(G)$ be labelled $(\alpha, \{\beta_1, \beta_2, \cdots, \beta_{\frac{k-1}{2}}\})$, where α indicates that $v \in V(G_\alpha)$, and $\{\beta_1, \beta_2 \cdots, \beta_{\frac{k-1}{2}}\}$ indicates that v is the unique vertex in $V(G_\alpha)$ which is joined to all vertices of $G_{\beta_1}, \cdots, G_{\beta_{\frac{k-1}{2}}}$. Further, we have for $\alpha = 1, 2, \cdots, k$ that v has valency

$$(*)\ d_G(v) = (\alpha - 1)(m + 1) + \tfrac{k-1}{2} \cdot n + (k - \tfrac{k+1}{2}) \binom{k-2}{\frac{k-3}{2}}$$
$$= (\alpha - 1)(m + 1) + \tfrac{3(k-1)}{4} \cdot \binom{k-1}{\frac{k-1}{2}},$$

because v is joined to $(\alpha - 1)(m + 1)$ vertices in G_α, to n vertices in each of $G_{\beta_1}, \cdots, G_{\beta_{\frac{k-1}{2}}}$ and v is in each of the $k - \frac{k-1}{2} - 1$ remaining graphs G_σ joined to $\binom{k-2}{\frac{k-3}{2}}$ vertices having a label $(\sigma, \{\alpha, \tau_2, \tau_3, \cdots, \tau_{\frac{k-1}{2}}\})$ containing α.

Let E be a set of at most m edges from G. We shall prove that each vertex can be distinguished by the same label in $G - E$ as in G so that $G - E_1 \cong G - E_2$ implies $E_1 = E_2$.

For increasing values of α the value of $d_G(v)$ in (*) jumps by $m + 1$. So consider in $G - E$ the vertex v, to obtain $d_G(v)$ we choose that α which gives the smallest number of the form (*) and not less than $d_{G-E}(v)$, i.e.

$$\alpha - 1 = \left\lceil \frac{d_{G-E}(v) - \frac{3(k-1)}{4}\binom{k-1}{\frac{k-1}{2}}}{m+1} \right\rceil$$

and

$$d_G(v) = \left\lceil \frac{d_{G-E}(v) - \frac{3(k-1)}{4}\binom{k-1}{\frac{k-1}{2}}}{m+1} \right\rceil (m + 1) + \frac{3(k-1)}{4} \binom{k-1}{\frac{k-1}{2}}.$$

We have now found the first coordinate α of the label $(\alpha, \{\beta_1, \beta_2, \cdots, \beta_{\frac{k-1}{2}}\})$ for v. Doing this for each vertex we can in $G - E$ identify the k vertex sets $V(G_1), V(G_2), \cdots, V(G_k)$. To identify $\{\beta_1, \beta_2, \cdots, \beta_{\frac{k-1}{2}}\}$ we note that v is in $G - E$ joined to at least $n - m$ vertices in each of $G_{\beta_1}, \cdots, G_{\beta_{\frac{k-1}{2}}}$ but to at most $\binom{k-2}{\frac{k-3}{2}}$ vertices in any $G_\sigma, \sigma \notin \{\alpha, \beta_1, \beta_2 \cdots, \beta_{\frac{k-1}{2}}\}$. So we can recognize the set of indices $\{\beta_1, \beta_2 \cdots, \beta_{\frac{k-1}{2}}\}$ if $\binom{k-2}{\frac{k-3}{2}} < n - m$. But this is equivalent to $m < \frac{n}{2}$, which follows from the previous condition $(k - 1)(m + 1) \leq n - 1$ if $k \geq 3$. So far, we have constructed a graph G with $r(G) \geq m$. If $r(G) > m$, say $r(G) = m + t$, then delete any t and maybe some more suitably chosen edges from G to obtain a graph G' with $r(G') = m$. □

Remark

By application of Stirling's formula $n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$ we obtain for $n \to \infty$ asymptotically that in the proof above $n \sim \binom{k}{\frac{k}{2}} \sim \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{2^k}{\sqrt{k}}$, and from this we derive that $k \sim \log_2 n$.

The number of vertices in G is $N = kn$, and we have asymptotically that $N \sim n\log_2 n,\ \log_2 N \sim k$. Our conditions can be satisfied with m close to $\frac{n}{k}$, which in terms of N is $m \sim \frac{N}{k^2}$ or $m \sim \frac{N}{(\log_2 N)^2}$.

So not only can $r(G)$ be arbitrarily large, but note that for $0 < a < 1$ we have $\frac{N}{(\log_2 N)^2} > N^a$ for N large with respect to α, so comparing m and N, this method constructs graphs with rather large $r(G)$ relative to $|V(G)|$.

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