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On Equality in Berge's Classical Bound for the Domination Number

by

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On Equality in Berge's Classical Bound for the Domination Number

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Abstract

Let $\gamma(G)$ denote the cardinality of a minimum dominating set of a graph G. A well-known upper bound for $\gamma(G)$, due to Berge (1962), states that for any graph G of order n and maximum degree Δ, $\gamma(G) \leq n - \Delta$. Similarly, Hedetniemi and Laskar (1984) proved $\gamma_c(G) \leq n - \Delta$, where $\gamma_c(G)$ denotes the cardinality of a minimum connected dominating set of G. In this paper, we characterize the regular graphs with $\gamma(G) = n - \Delta$, the regular graphs with $\gamma_c(G) = n - \Delta$ and the triangle-free graphs with $\gamma_c(G) = n - \Delta$. Moreover, we prove that both the problem of deciding whether $\gamma(G) = n - \Delta$ and the problem of deciding whether $\gamma_c(G) = n - \Delta$ are $co-NP$-complete.

1 Introduction

We consider finite, simple graphs $G = (V(G), E(G))$ with maximum degree $\Delta(G)$ and $n(G)$. Given any subset $U \subseteq V(G)$ the induced subgraph on U is denoted by $G[U]$. For any vertex $v \in V(G)$, the *open neighbourhood* $N(v)$ of v is defined by $N(v) = \{u \in V(G) \,|\, uv \in E(G)\}$ and the *closed neighbourhood* $N[v]$ of v is defined by $N[v] = N(v) \cup \{v\}$. For any set $U \subseteq V(G)$, let $N(U) = \cup_{u \in U} N(u)$ and $N[U] = N(U) \cup U$. For sets $U, W \subseteq V(G)$, we say that U *dominates* W if $W \subseteq N[U]$. If $U \subseteq V(G)$ dominates $V(G)$, then U is called a *dominating set* of G. The *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set of G. A *connected dominating set* of G is a dominating set D of G with the additional property that the induced graph $G[D]$ is connected. The *connected domination number* $\gamma_c(G)$ is the cardinality of a minimum connected dominating set of G. For any undefined concept the reader may refer to Diestel

(1997) and Haynes et al. (1998). When no confusion is possible, we may denote any parameter $f(G)$ of G by f.

A classical result by Berge (1962) states that for any graph G, $\gamma(G) \leq n(G) - \Delta(G)$. It seems that Domke et al. (1997) were the first to consider the problem of characterizing the graphs with $\gamma = n - \Delta$. They obtained a characterization of the connected bipartite graphs with $\gamma = n - \Delta$. Favaron and Mynhardt (1997) continued the study of the problem, and gave the following characterization of graphs with $\gamma = n - \Delta$.

Theorem 1.1 (Favaron and Mynhardt, 1997)
Let G be a graph and x a vertex of G with maximum degree Δ. Let $B = N(x)$, $C = V(G) - N[x]$ and $R = B - N(C)$. For each $c \in C$, let $B_c = N(c) \cap B$. Then $\gamma = n - \Delta$ if and only if

(i) *C is independent,*

(ii) *every vertex of B is adjacent to at most one vertex in C and*

(iii) *for every non-empty subset C' of C, the subset $B' = (\cup_{u \in C'} B_u) \cup R$ of B is either empty or not dominated by a set consisting of exactly one vertex of each B_u, $u \in C'$.*

This characterization does not lead to a polynomial algorithm for determining whether $\gamma = n - \Delta$. One of the main results of this paper shows that the general problem of determining whether $\gamma = n - \Delta$ is $co - NP$-complete. However, for some classes of graphs characterizations leading to polynomial algorithms can be found.

If G is a disconnected graph with components $H_1, \ldots, H_k$, $\Delta(G) \geq 1$ and $\gamma(G) = n(G) - \Delta(G)$, then all but one component of G are K_1-components. This shows that it is sufficient to consider the connected graphs with $\gamma = n - \Delta$.

Favaron and Mynhardt (1997) gave the following characterization of connected triangle-free graphs with $\gamma = n - \Delta$.

Theorem 1.2 (Favaron and Mynhardt, 1997)
Let G denote a connected triangle-free graph, and let v denote any vertex of G with maximum degree. Then $\gamma(G) = n(G) - \Delta(G)$ if and only if

(i) *G is bipartite with partition sets $N(v)$ and $V(G) - N(v)$,*

(ii) *$|V(G) - N(v)| \leq |N(v)|$,*

(iii) *$\deg(u) \leq 2$ for every $u \in N(v)$, and*

(iv) *If $\deg(u) = 2$ for every $u \in N(v)$, then $\deg(u) \geq 2$ for every $u \in V(G) - N(v)$.*

This characterization gives a polynomial algorithm for recognizing triangle-free graphs with $\gamma = n - \Delta$.

In Section 2 we characterize the regular graphs with $\gamma = n - \Delta$.

We also consider the problem of characterizing the graphs with $\gamma_c = n - \Delta$. Hedetniemi and Laskar (1984) characterized the trees with $\gamma_c = n - \Delta$.

Proposition 1.3 (Hedetniemi and Laskar, 1984)
Let T denote a tree of order $n \geq 2$ and let $l(T)$ denote the number of leaves of T. Then

$$\gamma_c(T) = n - l(T) \leq n - \Delta(T)$$

Furthermore, $\gamma_c(T) = n - \Delta$ if and only if T has at most *one vertex of degree greater than two.*

It follows that $\gamma_c(G) \leq n(G) - \Delta(G)$ for any connected graph G. This seems to be the only work done on the problem of characterizing the graphs with $\gamma_c = n - \Delta$.

In Section 4 we characterize the class of triangle-free graphs with $\gamma_c = n - \Delta$, and in Section 5 we characterize the regular graphs with $\gamma_c = n - \Delta$. In Section 6, we show that, in general, the problem of deciding whether $\gamma_c = n - \Delta$ is $co - NP$-complete.

2 Regular Graphs with $\gamma = n - \Delta$

Theorem 2.1
Let G denote a connected regular graph. Then $\gamma = n - \Delta$ if and only if G is a complete graph, or n is even and G is a complete graph with a perfect matching removed, i.e. $G = K_n - M$, where M is a perfect matching of K_n.

Proof. First, suppose $\gamma(G) = n - \Delta$. Let v be any vertex of G. Since G is regular, the vertex v has maximum degree. If $G - N[v]$ does not contain any vertices, then $\Delta = n - 1$ and so G is complete. Hence assume that $G - N[v]$ contains at least one vertex. By Theorem 1.1, the graph $G - N[v]$ consists of $n - \Delta - 1$ isolated vertices. Hence every vertex of $V(G) - N[v]$ has all its neighbours in $N(v)$, and since G is Δ-regular, each vertex of $V(G) - N[v]$ is adjacent to every vertex of $N(v)$. Suppose that there is more than one vertex in $G - N[v]$. Then there are at least two vertices in $V(G) - N[v]$, say a and b, with a common neighbour, say x, in $N(v)$ and so $(V(G) - (N(v) \cup \{a, b\})) \cup \{x\}$ is a dominating set of G of cardinality $n - \Delta - 1$, a contradiction. Hence $G - N[v]$ contains exactly one vertex. This implies $\Delta = n - 2$, i.e. G is the connected $(n - 2)$-regular graph. It is easy to see that this graph is isomorphic to $K_n - M$, where M is a perfect matching of K_n.

Conversely, if G is complete, then $\gamma(G) = 1 = n - \Delta$, and if $G = K_n - M$, then $\gamma(G) = 2 = n - \Delta$. ■

3 Preliminary Results on Graphs with $\gamma_c = n - \Delta$

For extreme values of Δ the situation is simple.

Observation 3.1
Let G denote a connected graph with $\gamma_c(G) = n - \Delta(G)$.

- *If $\Delta(G) = 1$, then $G = K_2$.*
- *If $\Delta(G) = 2$, then $G \in \{C_n, P_n\}$.*
- *If $\Delta(G) = n - 2$, then G can be any connected graph with $\Delta(G) = n - 2$.*
- *If $\Delta(G) = n - 1$, then G can be any connected graph with $\Delta(G) = n - 1$.*

Proposition 3.2
Let G be a connected graph with $\gamma_c(G) = n(G) - \Delta(G)$. Then the following conditions (i-iii) are satisfied for every vertex v of degree $\Delta(G)$.

(i) All components of $G - N[v]$ are paths.

(ii) For every path $P : u_1, u_2, \ldots, u_r$ in $G - N[v]$ and $i \in \{2, \ldots, r - 1\}$,

$$\deg_G(u_i) = 2.$$

(iii) Each vertex of $N(v)$ is adjacent to at most one vertex of $V(G) - N[v]$.

Proof. Assume $\gamma_c(G) = n - \Delta$. Let v denote a vertex of G for which $\deg_G(v) = \Delta$. If $\Delta = n - 1$, then (i-iii) are satisfied. Suppose $\Delta \le n - 2$ and define $G' = G - N_G[v]$.

Let $H_1, H_2, \ldots, H_t$ $(t \ge 1)$ denote the components of G'. Note that in G at least one vertex x_i of H_i is adjacent to at least one vertex y_i in $N(v)$. Amongst all spanning trees of H_i, let T_i be one such that $l(T_i)$ is maximum and note that $l(T_i) \ge \Delta(H_i)$. Some of the trees T_i might be K_1's or K_2's. If there are any such trees, then let the trees be indexed such that $T_1, \ldots, T_s$ are K_1's and K_2's while $T_{s+1}, \ldots, T_t$ all have more than two vertices. If $s = t$, then every component of G' is a K_1 or a K_2 and (i) is satisfied. Hence we shall assume $s < t$.

By Proposition 1.3 we have $\gamma_c(T_i) = n(T_i) - l(T_i)$ for all $i = s + 1, \ldots, t$ (this is not true for K_2). Furthermore, $l(T_i) \ge 2$ for all $i = s+1, \ldots, t$. We shall construct a connected dominating set D of G. Let v be in D. For each $T_i = K_1$ assign the vertex y_i to D. For each $T_i = K_2$ assign the vertices x_i and y_i to D. For each T_i,

$(i > s)$ add y_i, x_i and the vertices of a $\gamma_c(T_i)$-set to D. Now we obtain

$$\begin{aligned}
\gamma_c(G) &\leq 1 + \sum_{i=1}^{s} n(T_i) + \sum_{j=s+1}^{t} (\gamma_c(T_j) + 2) \\
&= 1 + \sum_{i=1}^{s} n(T_i) + \sum_{j=s+1}^{t} (n(T_j) - l(T_j) + 2) \\
&\leq 1 + \sum_{i=1}^{s} n(T_i) + \sum_{j=s+1}^{t} n(T_j) \\
&\leq 1 + (n - \Delta - 1). \qquad (1)
\end{aligned}$$

Observe that if $l(T_i) > 2$ for some $i = s+1, \ldots, t$, then the argument of (1) implies $\gamma_c(G) < n - \Delta$, a contradiction. Hence $l(T_i) = 2$ for all $i = s+1, \ldots, t$ and therefore $\Delta(H_i) \leq 2$ for every $i = 1, \ldots, t$. This implies that H_i is either a path or a cycle.

If some vertex $u \in V(H_i)$ with $\deg_{H_i}(u) = 2$ is adjacent to some vertex of $N(v)$, then there exists a spanning tree T of G, where $\deg_T(v) = \Delta(G)$ and $\deg_T(u) \geq 3$. Now Proposition 1.3 implies $\gamma_c(G) \leq \gamma_c(T) < n - \Delta$, a contradiction. Hence only vertices $u \in V(H_i)$ with $\deg_{H_i}(u) = 1$ can be adjacent to vertices of $N(v)$. Since G is connected, it follows that some vertex $u \in V(H_i)$ with $\deg_{H_i}(u) = 1$ is adjacent to some vertex in $N(v)$. This implies that H_i is a path and so (i-ii) are satisfied.

If some vertex of $N(v)$ have more than one neighbour in $V(G) - N[v]$, then G has a spanning tree T with at least two vertices of degree greater than two, and so Proposition 1.3 implies $\gamma_c(G) < n - \Delta$, a contradiction. Thus each vertex of $N(v)$ has at most one neighbour in $V(G) - N[v]$. This establishes (iii). ■

The induced graph $G[N(v)]$ seem to elude characterization. One way to overcome this problem is to require the graph to be triangle-free, since in a triangle-free graph G the induced subgraph $G[N(v)]$ contains no edge.

4 Triangle-free Graphs with $\gamma_c = n - \Delta$

Theorem 4.1
Let G denote a connected triangle-free graph. Then $\gamma_c(G) = n - \Delta$ if and only if the following conditions (i-iii) are satisfied for every vertex v of degree $\Delta(G)$.

(i) All components of $G - N[v]$ are paths.

(ii) For every path $P : u_1, u_2, \ldots, u_r$ in $G - N[v]$ and $i \in \{2, \ldots, r-1\}$,

$$\deg_G(u_i) = 2.$$

(iii) *Each vertex of $N(v)$ is adjacent to at most one vertex of $V(G) - N[v]$.*

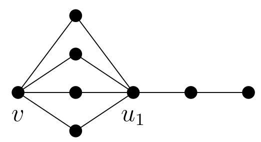
Proof. If $\gamma_c(G) = n - \Delta$, then it follows from Proposition 3.2 that (i-iii) are satisfied for every vertex v of G with $\deg_G(v) = \Delta$.

Now, suppose (i-iii) are satisfied for some vertex v of G with $\deg_G(v) = \Delta$. If $\Delta \leq 1$, then $G \in \{K_1, K_2\}$, and $\gamma_c(G) = n - \Delta$. If $\Delta = 2$, then G is either a path or a cycle. In either case, $\gamma_c(G) = n - \Delta$. Hence we may assume $\Delta \geq 3$. Let D denote a $\gamma_c(G)$-set. Suppose that $G - N[v]$ contains precisely one component, say $P^1 : u_1, \ldots, u_r$. Then D contains at least $n(P^1)$ vertices of $V(P^1) \cup N(u_1) \cup N(u_r)$. We shall consider three cases.

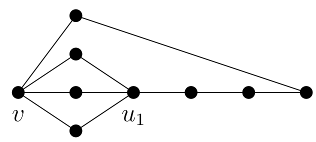
(i) P^1 is a singleton.

(ii) $n(P^1) \geq 2$ and only one end-vertex of P^1 is adjacent to a vertex of $N(v)$ in G.

(iii) $n(P^1) \geq 2$ and both end-vertices of P^1 are adjacent to vertices of $N(v)$ in G.



(a) Since v is not a stem, u_1 must be adjacent to every vertex of $N(v)$.



(b) If D does not contain v, then D contains every vertex of P^1.

Figure 1:

(i) In this case we have $\Delta = n - 2$, and so $\gamma_c(G) > 1$. On the other hand, $\gamma_c(G) \leq n - \Delta = 2$ and so $\gamma_c(G) = n - \Delta$.

(ii) Assume that $n(P^1) \geq 2$ and that, in G, only one end-vertex of P^1 is adjacent to a vertex of $N(v)$. Let u_1 denote the vertex of P^1 which is adjacent to one or more vertices of $N(v)$. Now D contains at least $n(P^1) = n - \Delta - 1$ vertices of $N[P^1]$ in order to dominate P^1. Suppose that these vertices dominate G. Then there are no leaves adjacent to v, and, since $G - N[v]$ contains only one component, $N(v) = N(v) \cap N(u_1)$ (See Figure 1a). But then u_1 has degree $\Delta + 1$ in G, a contradiction.

(iii) Assume that $n(P^1) \geq 2$ and that, in G, both end-vertices of P^1 are adjacent to vertices of $N(v)$. Again, D contains at least $n(P^1) = n - \Delta - 1$ vertices of $N[P^1]$ in order to dominate P^1. If $v \in D$, then $|D| \geq n - \Delta$. Hence we may assume that $v \notin D$. Now in order for $G[D]$ to be connected, the set D

contains every vertex of P^1, and for D to dominate v, D contains at least one vertex of $(N(u_1) \cup N(u_r)) \cap N(v)$ (See Figure 1b). This shows that $|D| \geq n - \Delta$.

Suppose that $G - N[v]$ contains more than one component. Then D contains v, since otherwise $G[D]$ would be disconnected. Let $P^1, \ldots, P^r$ denote the components of $G - N[v]$. In order to be a connected dominating set, D contains at least $n(P^i)$ vertices of $N[P^i]$ for the domination of P^i, and, since $N[P^1], \ldots, N[P^r]$ are all disjoint, $|D| \geq 1 + \sum_{i=1}^{r} n(P^i) = 1 + (n - \Delta - 1)$. Hence $\gamma_c(G) = n - \Delta$. ■

Using Theorem 4.1 it is easy to design a polynomial algorithm for recognition of triangle-free graphs with $\gamma = n - \Delta$.

5 Regular Graphs with $\gamma_c = n - \Delta$

Theorem 5.1
Let G denote a connected regular graph. Then $\gamma_c(G) = n - \Delta$ if and only if G is one of the following: C_n, K_n, or $K_n - M$, where M is a perfect matching in K_n.

Proof. First, suppose that G is a Δ-regular connected graph. Let v denote any vertex of G. If $V(G) - N[v] = \varnothing$, then $\Delta = n - 1$ and G is a complete graph. Suppose $V(G) - N[v] \neq \varnothing$. If $\Delta = 2$, then $G = C_n$, so we may assume $\Delta \geq 3$. Let u_1 denote a vertex of $V(G) - N[v]$ which is adjacent to a vertex of $N(v)$. Proposition 3.2 (ii), $\Delta \geq 3$ and the regularity of G implies $V(G) - N[v]$ contains at most two vertices. Suppose that u_1 is adjacent to a vertex u_2 in $V(G) - N[v]$. Now u_1 and u_2 are adjacent to precisely $\Delta - 1$ vertices of $N(v)$. Since u_1 and u_2 do not have a common neighbour in $N(v)$, we obtain $(\Delta - 1) + (\Delta - 1) \leq |N(v)| = \Delta$, which implies $\Delta \leq 2$, a contradiction. Hence u_1 has all its neighbours in $N(v)$, i.e. u_1 is adjacent to every vertex of $N(v)$. Then it follows from Proposition 3.2 that $V(G) - N[v]$ only contains this one vertex u_1. Hence $\Delta = n - 2$, and, since G is regular, $G \simeq K_n - M$, where M is a perfect matching in K_n. ■

6 Complexity Results

In this section, we prove one of the main results of this paper, namely that the problem of deciding whether $\gamma = n - \Delta$ is $co - NP$-complete.

Decision Problem 6.1 (MDS ($n - \Delta$))
MINIMUM DOMINATING SET OF CARDINALITY $n - \Delta$
INSTANCE*: A graph G.*
QUESTION*: Does G have a minimum dominating set of cardinality $n(G) - \Delta(G)$?*

In order to prove the $co-NP$-completeness of the above problem, we prove that the **3-SAT** problem can be reduced to the problem of deciding whether $\gamma \leq n - \Delta - 1$.

Decision Problem 6.2 (DS ($n - \Delta - 1$))
DOMINATING SET OF CARDINALITY $\leq n - \Delta - 1$
INSTANCE: *A graph G.*
QUESTION: *Does G have a dominating set of cardinality $\leq n(G) - \Delta(G) - 1$?*

For any boolean variable u, let $\overline{u}$ denote the negation of u. Given a set of independent boolean variables $U = \{u_1, u_2, \ldots, u_p\}$ (independent in the sense that truth values can be assigned completely arbitrarily to the variables of U), we define a clause C of U to be a 3-element set $\{x_1, x_2, x_3\}$, where either $x_i \in U$ or $\overline{x}_i \in U$ for each $i \in \{1, 2, 3\}$.

Decision Problem 6.3 (3-SAT)
3-SATISFIABILITY
INSTANCE: *A set $U = \{u_1, u_2, \ldots, u_p\}$ of variables, and a set $\mathcal{C} = \{C_1, C_2, \ldots, C_q\}$ of clauses.*
QUESTION: *Does $\mathcal{C}$ have a satisfying truth assignment, i.e. an assignment of True and False to the variables in U such that at least one variable in each clause C_i of $\mathcal{C}$ is assigned the value True ?*

Theorem 6.4
*The decision problem **DS**$(n - \Delta - 1)$ is NP-complete.*

The proof uses a construction which, to the best of my knowledge, was first introduced by Garey and Johnson (1979).

Proof. The decision problem **DS**$(n - \Delta - 1)$ is in NP, since if $S \subseteq V(G)$ is a dominating set of cardinality $\leq n(G) - \Delta(G) - 1$, then it can be verified in polynomial time that S is a dominating set.

Next, we show that **3-SAT** is reducible to **DS**$(n - \Delta - 1)$. Given any nontrivial instance $\mathcal{C}$ of **3-SAT**, we construct an instance $G_{\mathcal{C}}$ of **DS**$(n - \Delta - 1)$ as follows. For each variable u_i, construct a triangle with vertices labelled $u_i, \overline{u}_i, v_i$. For each clause $C_j = \{u_i, u_k, u_l\}$ add a vertex C_j, and edges u_iC_j, u_kC_j, u_lC_j. Finally, add a vertex x, and join x to every vertex of $V(G_{\mathcal{C}}) - (\{x\} \cup \{v_1, v_2, \ldots, v_p\})$ (See Figure 2).

Notice that the construction of $G_{\mathcal{C}}$ from $\mathcal{C}$ is done in polynomial time.

Claim 6.5
The vertex x is a vertex of maximum degree in $G_{\mathcal{C}}$ and $\Delta(G_{\mathcal{C}}) = n(G_{\mathcal{C}}) - p - 1$.

Argument. Since $\mathcal{C}$ is a nontrivial instance, we have $|U| = p \geq 2$. Every vertex C_i has degree four; every vertex v_i has degree two; every vertex u_i and $\overline{u}_i$ has degree

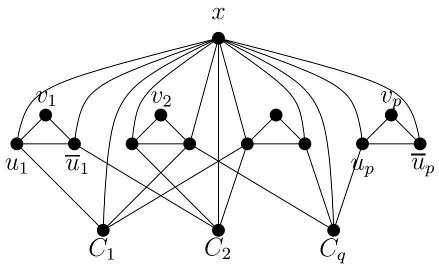


Figure 2: Construction of the graph $G_{\mathcal{C}}$ from an instance $\mathcal{C}$ of the **3SAT** problem.

at most $q+3$. The vertex x has degree $n(G_{\mathcal{C}}) - p - 1 = 2p + q \geq 4 + q$. Hence x is a vertex of maximum degree, and $\Delta(G_{\mathcal{C}}) = n(G_{\mathcal{C}}) - p - 1$. $\diamond$

Hence we have $p = n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$.

Claim 6.6
*The instance $\mathcal{C}$ of **3-SAT** has a satisfying truth assignment if and only if the graph $G_{\mathcal{C}}$ has a dominating set of cardinality $\leq n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$.*

Argument. First, suppose that $\mathcal{C}$ has a satisfying truth assignment. We construct a dominating set S as follows: If u_i is True, then assign u_i to S, else assign $\overline{u}_i$ to S. The set S is a dominating set, since (i) each triangle contains a vertex of S, (ii) each C_j is adjacent to a vertex in S, and (iii) certainly x is dominated by S. Since S contains exactly one vertex from each triangle $u_i\overline{u}_iv_i$, and no more vertices, we obtain $|S| = p = n(G_c) - \Delta(G_{\mathcal{C}}) - 1$. This shows that $G_{\mathcal{C}}$ has a dominating set of cardinality $\leq n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$.

Conversely, suppose that $G_{\mathcal{C}}$ has a dominating set S of cardinality $\leq n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$. Then S contains at least one vertex from each triangle $u_i\overline{u}_iv_i$, and so $|S| \geq p = n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$. It follows that S contains exactly one vertex from each triangle $u_i\overline{u}_iv_i$, and no other vertices. We may assume wlog. that $S \subseteq \{u_1, \overline{u}_1, \ldots, u_p, \overline{u}_p\}$. Now $y \in S$ if and only if $\overline{y} \notin S$, and so we obtain a correct assignment of truth values by letting $y \in \{u_1, \overline{u}_1, \ldots, u_p, \overline{u}_p\}$ be assigned the value True if and only if $y \in S$. Every C_j is dominated by S in G. Suppose $w \in S$ is a vertex dominating C_j. Now w was assigned the value True and the construction of $G_{\mathcal{C}}$ implies w is a variable in the clause C_j. Hence the clause C_j is True and so $\mathcal{C}$ has a satisfying truth assignment. $\diamond$

$\blacksquare$

Corollary 6.7
*The decision problem **MDS**$(n - \Delta)$ is $co - NP$-complete.*

Proof. Given any instance G, we find that the answer to the problem **MDS**$(n-\Delta)$ is YES if and only if the answer to to **DS**$(n - \Delta - 1)$ is NO. Hence **MDS**$(n - \Delta)$

and **DS**$(n-\Delta-1)$ are complementary problems, and so **DS**$(n-\Delta-1) \in NPC$ implies **MDS**$(n-\Delta) \in co-NPC$. ∎

Decision Problem 6.8 (MCDS ($n-\Delta$))
MINIMUM CONNECTED DOMINATING SET OF CARDINALITY $n-\Delta$
INSTANCE*: A graph G.*
QUESTION*: Does G have a minimum connected dominating set of cardinality $n(G)-\Delta(G)$?*

Decision Problem 6.9 (CDS ($n-\Delta-1$))
CONNECTED DOMINATING SET OF CARDINALITY $\leq n-\Delta-1$
INSTANCE*: A graph G.*
QUESTION*: Does G have a connected dominating set of cardinality $\leq n(G)-\Delta(G)-1$?*

Theorem 6.10
*The decision problem **CDS**$(n-\Delta-1)$ is NP-complete.*

The proof is similar to the proof of Theorem 6.4, and so we only present a sketch.

Sketch of proof. The decision problem **CDS**$(n-\Delta-1)$ is obviously in NP. The next step is to show that **3-SAT** is reducible to **CDS**$(n-\Delta-1)$. Given any instance $\mathcal{C}$ of **3-SAT**, let $G_{\mathcal{C}}$ denote the corresponding instance of **CDS**$(n-\Delta-1)$. If $\cap_{i=1}^{q} C_i \neq \varnothing$, then let $G_{\mathcal{C}} = K_3$. If $\cap_{i=1}^{q} C_i = \varnothing$, then construct $G_{\mathcal{C}}$ as follows. For each variable u_i, construct a triangle with vertices labelled $u_i, \overline{u}_i, v_i$. For each clause $C_j = \{u_i, u_k, u_l\}$ add a vertex C_j, and edges u_iC_j, u_kC_j, u_lC_j. Add edges such that the induced subgraph on $\{u_1, \overline{u}_1, \ldots, u_p, \overline{u}_p\}$ is a complete graph. Finally, add a vertex x, and join x to every vertex of $V(G_{\mathcal{C}})-(\{x\}\cup\{v_1, v_2, \ldots, v_p\})$. Now the theorem follows by establishing the two following claims.

Claim 6.11
If $\cap_{i=1}^{q} C_i = \varnothing$, then x is a vertex of maximum degree in $G_{\mathcal{C}}$ and $\Delta(G_{\mathcal{C}}) = n(G_{\mathcal{C}}) - p - 1$.

Claim 6.12
*The instance $\mathcal{C}$ of **3-SAT** has a satisfying truth assignment if and only if the graph $G_{\mathcal{C}}$ has a connected dominating set of cardinality $\leq n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$.*

The details are omitted. ∎

Corollary 6.13
*The decision problem **MCDS**$(n-\Delta)$ is $co-NP$-complete.*

Proof. **MCDS**$(n-\Delta)$ and **CDS**$(n-\Delta-1)$ are complementary problems, and so, since **CDS**$(n-\Delta-1) \in NPC$, we obtain **MCDS**$(n-\Delta) \in co-NPC$. ∎

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