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The Fermi Golden Rule at Thresholds

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Abstract

Let H be a Schrödinger operator on a Hilbert space $\mathcal{H}$, such that zero is a nondegenerate threshold eigenvalue of H with eigenfunction Ψ_0. Let W be a bounded selfadjoint operator satisfying $\langle \Psi_0, W\Psi_0\rangle > 0$. Assume that the resolvent $(H-z)^{-1}$ has an asymptotic expansion around $z=0$ of the form typical for Schrödinger operators on odd-dimensional spaces. Let $H(\varepsilon) = H + \varepsilon W$ for $\varepsilon > 0$ and small. We show under some additional assumptions that the eigenvalue at zero becomes a resonance for $H(\varepsilon)$, in the time-dependent sense introduced by A. Orth. No analytic continuation is needed. We show that the imaginary part of the resonance has a dependence on ε of the form $\varepsilon^{2+(\nu/2)}$ with the integer $\nu \geq -1$ and odd. This shows how the Fermi Golden Rule has to be modified in the case of perturbation of a threshold eigenvalue. We give a number of explicit examples, where we compute the location of the resonance to leading order in ε.

1 Introduction

In this paper we study the following question. Consider a Schrödinger operator

$$H = -\Delta + V \quad \text{on } L^2(\mathbf{R}^3),$$

where for the moment we assume that $V \in C_0^\infty(\mathbf{R}^3)$. The essential spectrum of H is the half line $[0,\infty)$, and it is well known that this spectrum is purely

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absolutely continuous. H may have a finite number of negative eigenvalues, and there may also be an eigenvalue at the threshold zero. Suppose that zero is a nondegenerate eigenvalue with normalized eigenfunction Ψ_0. Let $W \in C_0^\infty(\mathbf{R}^3)$, and assume that it is nonnegative. Consider for small $\varepsilon > 0$ the family of Hamiltonians

$$H(\varepsilon) = H + \varepsilon W.$$

Since the perturbation is nonnegative, the zero eigenvalue cannot become an isolated negative eigenvalue, and since it is well known that $H(\varepsilon)$ cannot have eigenvalues embedded in $(0, \infty)$, only two possibilities remain. Zero can remain an eigenvalue, or it can disappear. In the latter case one expects that it becomes a resonance. This is the question that we will study.

The existence of the resonance can be verified in several ways, depending upon the "definition" of what a resonance is. One may look at this question in the spectral form. Thus one looks at a meromorphic continuation of the resolvent in some sense, and expects to find a pole close to zero in the complex energy plane. One can also study the question from the time-dependent point of view. Here one looks at the behavior of $\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0 \rangle$, which describes the probability to remain in the state Ψ_0 at time t. The resonance will then manifest itself in the form of a behavior of the type

$$\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0 \rangle = e^{-it\lambda(\varepsilon)} + \delta(\varepsilon, t), \quad t > 0. \tag{1.1}$$

i.e. corresponding to a metastable state. Here $\lambda(\varepsilon) = x_0(\varepsilon) - i\Gamma(\varepsilon)$ with $x_0(\varepsilon) > 0$ and $\Gamma(\varepsilon) > 0$ and, as far as a resonance defined in the spectral sense exists, should coincide with the resonance position. The error term in (1.1) should satisfy $\delta(\varepsilon, t) \to 0$ as $\varepsilon \to 0$.

Our main theorem gives conditions on H and W that lead to such results, with expressions for the leading terms in $x_0(\varepsilon)$ and $\Gamma(\varepsilon)$, as $\varepsilon \to 0$. In the case of an eigenvalue embedded in the interior of the absolutely continuous spectrum, formulae for computing the leading term in $\Gamma(\varepsilon)$ are often referred to as the Fermi Golden Rule. Thus we find versions of the Fermi Golden Rule in the case, where the eigenvalue is embedded at a threshold.

Let us give an outline of the main results, referring to the theorems for precise assumptions and conditions. The results are obtained in a semi-abstract framework. One of the main tools needed is the asymptotic expansion of the resolvent $R(z) = (H - z)^{-1}$ around zero. It is convenient to use the variable $\kappa = -i\sqrt{z}$, $z \in \mathbf{C} \setminus [0, \infty)$. We assume an expansion of the form

$$R(-\kappa^2) = \frac{1}{\kappa^2}P_0 + \sum_{j=-1}^{N-1} \kappa^j G_j + \mathcal{O}(\kappa^N)$$

as $\kappa \to 0$. This type of expansion is known to hold for Schrödinger operators in odd dimensions, with sufficiently rapidly decaying V. The expansion holds in the topology of bounded operators between weighted L^2-spaces. See [10, 9, 17, 11]. For the perturbation W we assume that it decays sufficiently rapidly, and as a crucial condition, we require

$$b = \langle \Psi_0, W\Psi_0 \rangle > 0. \tag{1.2}$$

We do not assume that W is nonnegative. We assume that there exists an odd integer ν, such that

$$g_\nu = \langle \Psi_0, W G_\nu W \Psi_0 \rangle \neq 0, \quad G_j = 0, \quad j = -1, 1, 3, \dots, \nu - 2.$$

Our main abstract result then states that (1.1) holds. Furthermore, we have the estimate

$$|\delta(\varepsilon, t)| \leq C\varepsilon^{p(\nu)} |\ln \varepsilon|^\iota,$$

where $\iota = 1$ for $\nu = -1, 1$, and zero otherwise. Here $p(\nu) = \min\{2, (2+\nu)/2\}$. We have the expansions

$$\Gamma(\varepsilon) = -i^{\nu-1} g_\nu b^{\nu/2} \varepsilon^{2+(\nu/2)} (1 + \mathcal{O}(\varepsilon)),$$
$$x_0(\varepsilon) = b\varepsilon(1 + \mathcal{O}(\varepsilon)),$$

as $\varepsilon \to 0$. The proof of these results is based on the representation

$$\langle \Psi_0, e^{-itH(\varepsilon)} \Psi_0 \rangle = \lim_{\eta \searrow 0} \frac{1}{\pi} \int e^{-itx} \operatorname{Im} \langle \Psi_0, (H(\varepsilon) - x - i\eta)^{-1} \Psi_0 \rangle dx.$$

The idea is first to localize to the interval $I_\varepsilon = (\frac{1}{2}b\varepsilon, \frac{3}{2}b\varepsilon)$, depending on ε, and then, by using the resolvent expansions, replace the term

$$\operatorname{Im} \langle \Psi_0, (H(\varepsilon) - x - i\eta)^{-1} \Psi_0 \rangle$$

in the integrand by a Lorentzian function

$$\frac{\Gamma(\varepsilon)}{(x - x_0(\varepsilon))^2 + \Gamma(\varepsilon)^2}.$$

To obtain this approximation we use the Schur-Livsic-Feshbach formula to localize the essential terms. During all steps one has to control the errors. The detailed computations then lead to the result outlined above.

We apply these semi-abstract results to a number of cases involving Schrödinger operators in three and one dimensions, and on the half line. We consider both the one channel and the two channel case. As an example of the type of results obtained, assume as above that $H = -\Delta + V$ on $L^2(\mathbf{R}^3)$ has zero as a nondegenerate eigenvalue (and no threshold resonance), with Ψ_0 a normalized realvalued eigenfunction. Let

$$X_j = \int_{\mathbf{R}^3} \Psi_0(\mathbf{x})V(\mathbf{x})x_j d\mathbf{x}, \quad j = 1, 2, 3.$$

Assume that at least one $X_j \neq 0$. Then $\nu = -1$, and we have

$$g_{-1} = \frac{b^2}{12\pi}(X_1^2 + X_2^2 + X_3^2).$$

There is a large literature concerned with establishing the spectral form of the Fermi Golden Rule in a rigorous framework. In particular, using dilation-analyticity, it was established in [23], in a large number of cases, including atoms and molecules. See also the discussion in [21]. The case of bound states embedded at a threshold is much less studied. The coupling constant case has been studied by several authors, see for example [20, 5, 6]. To compare with our results, one has to take $W = -V$, in order to satisfy (1.2), since $(-\Delta + V)\Psi_0 = 0$ implies $\langle \Psi_0, V\Psi_0 \rangle = \langle \Psi_0, \Delta\Psi_0 \rangle < 0$. Then the explicit results in [5] correspond to our case $\nu = -1$, and the results agree. The spectral form of the resonance problem has been studied near band edges for periodic Schrödinger operators in the semi-classical limit, in [15]. A general framework for a unified treatment of resonances and eigenvalues near thresholds has been given in [7], using meromorphic continuation of resolvents.

The only work in the spectral form directly related to our study that we are aware of (even at a nonrigorous level), is that of Baumgartner [3], where some simplified two-channel models are considered. In these cases explicit computations can be performed, and one can explain how the usual Fermi Golden Rule has to be modified to be applied in the threshold case.

There are many other cases in recent research on models in quantum field theory, and the study of open systems, where the Fermi Golden Rule has been rigorously established. Since these models are not directly related to our study, we do not elaborate.

The time-dependent approach has been developed much later. Let us remark that here there is no need of an analytic continuation. The time-dependent approach, without analyticity, was initiated in [19] and continued in [13]. In [8] it was investigated how to get a better error term by using the perturbation theory in the spirit of Simon [23] in the dilation-analytic framework. More recently, a number of authors have developed a general time-dependent approach, without analyticity, see for example [14, 25, 24, 16, 4]. As far as we can determine, none (see however the remarks in [24, 4] concerning some examples of Schrödinger operators in high dimensions) of these approaches can be applied directly to the threshold eigenvalue case.

It should be noted that all the time-dependent approaches (except [8]) use the Feshbach projection method in some form, or something equivalent to it. As we already said, we also rely heavily on the Feshbach projection method.

As in [3] one may describe the results stated above as explaining how the usual Fermi Golden Rule has to be modified to be applied in the threshold case. In the case of a resonance arising from the perturbation of an eigenvalue embedded in the continuum, one finds that the imaginary part of the resonance behaves like ε^2 as $\varepsilon \to 0$. We find the behavior $\varepsilon^{2+(\nu/2)}$, $\nu = -1, 1, \ldots$, which is quite different. In particular, for $\nu \geq 1$ the resonance arising from the threshold eigenvalue has a larger lifetime than one arising from an eigenvalue embedded in the continuum, i.e. one has an enhancement of the lifetime. On the contrary, for $\nu = -1$ the lifetime is smaller, i.e. one has an enhancement of the decay. This is clearly seen in the two channel case, when a threshold resonance is present in the open channel, and can be explained heuristically as the effect of the increase, due to the threshold resonance, of the density of states near the threshold.

Let us briefly outline the contents of the paper. In Section 2 we give the Schur-Livsic-Feshbach formula, and introduce the factorization method. In Section 3 we give our semi-abstract results, modelled on Schrödinger operators in odd dimensions. The main result is stated as Theorem 3.7. Then in Section 4 we verify the assumptions in Section 3 in a number of cases: (i) A Schrödinger operator on $L^2(\mathbf{R}^3)$, both in the one channel and the two channel case. (ii) A Schrödinger operator on $L^2(\mathbf{R})$. Here it only makes sense to consider the two channel case, since for rapidly decaying potentials zero cannot be an eigenvalue. (iii) The operator $-d^2/dr^2 + \ell(\ell+1)r^{-2}$ on $L^2(\mathbf{R}_+)$. The results in this section relate directly to [3, Section V]. In all cases we determine the values of ν and compute g_ν explicitly.

2 The Schur-Livsic-Feshbach formula and the factorization method

Let H be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$ and E_0 a nondegenerate eigenvalue of H,

$$H\Psi_0 = E_0\Psi_0, \quad \|\Psi_0\| = 1. \tag{2.1}$$

We can without loss of generality in the sequel take $E_0 = 0$. Suppose now that a perturbation, described by the self-adjoint operator W, is added so the perturbed dynamics is generated by

$$H(\varepsilon) = H + \varepsilon W, \quad \varepsilon > 0. \tag{2.2}$$

Note that we only consider positive values of the parameter ε. For the sake of simplicity we shall assume that W is bounded, but all the results below extend to the case, when W is bounded with respect to H, with bound less than one.

At the heuristic level, it is argued that due to the perturbation, for ε small enough, 0 turns into a resonance having an (approximate) exponential decay law,

$$\langle \Psi_0, e^{-iH(\varepsilon)t}\Psi_0\rangle = e^{-i\lambda(\varepsilon)t} + \delta(\varepsilon, t), \tag{2.3}$$

with $\delta(\varepsilon, t) \to 0$, as $\varepsilon \to 0$, and

$$\lambda(\varepsilon) = E(\varepsilon) - i\Gamma(\varepsilon), \quad E(\varepsilon) = 0 + \varepsilon\langle \Psi_0, W\Psi_0\rangle + \mathcal{O}(\varepsilon^2). \tag{2.4}$$

The goal is to compute $\lambda(\varepsilon)$, and to obtain bounds on $|\delta(\varepsilon, t)|$. Again at the heuristic level, it is argued that the main contribution to the left hand side of (2.3) is given by energies near to $E(\varepsilon)$, so one first considers (see [8])

$$A_{g_\varepsilon}(t) = \langle \Psi_0, e^{-iH(\varepsilon)t} g_\varepsilon(H(\varepsilon))\Psi_0\rangle, \tag{2.5}$$

where g_ε is the (possibly smoothed) characteristic function of a closed interval I_ε, containing the relevant energies. Let us remark that usually (see e.g. [8, 4]) I_ε is chosen to be a neighborhood of E_0, independent of ε. One of the key points of our approach is to make an appropriate ε-dependent choice of I_ε. Also, since we are mainly interested in uniform (with respect to t) estimates on $\delta(\varepsilon, t)$, we shall take g_ε to be the characteristic function of I_ε. Taking a smoothed characteristic function of I_ε allows one to obtain a power like decrease in t of $\delta(\varepsilon, t)$, but the control of the coefficient is poor. In fact, if the length of I_ε goes to zero as $\varepsilon \to 0$, this coefficient blows up.

From Stone's formula (suppose that the end points of I_ε are not eigenvalues of $H(\varepsilon)$) one gets

$$A_{g_\varepsilon}(t) = \lim_{\eta \searrow 0} \frac{1}{\pi} \int_{I_\varepsilon} dx\, e^{-ixt}\, \mathrm{Im}\langle \Psi_0, (H(\varepsilon) - x - i\eta)^{-1}\Psi_0\rangle. \tag{2.6}$$

As in [19], in order to compute the integrand in (2.6), we use the well known Schur-Livsic-Feshbach formula. More precisely, if P_0 is the orthogonal projection on Ψ_0, $Q_0 = 1 - P_0$, and $R_{0,\varepsilon}(z)$ is the resolvent of $Q_0 H(\varepsilon) Q_0$ as an operator in $Q_0\mathcal{H}$, then in operator matrix form in $\mathcal{H} = P_0\mathcal{H} \oplus Q_0\mathcal{H}$, we have for $R_\varepsilon(z) = (H(\varepsilon) - z)^{-1}$ the representation

$$R_\varepsilon(z) = \begin{bmatrix} R_{\rm eff}(z) & -\varepsilon R_{\rm eff}(z) P_0 W Q_0 R_{0,\varepsilon}(z) \\ -\varepsilon R_{0,\varepsilon}(z) Q_0 W P_0 R_{\rm eff}(z) & R_{22} \end{bmatrix}, \tag{2.7}$$

with

$$R_{22} = R_{0,\varepsilon}(z) + \varepsilon^2 R_{0,\varepsilon}(z) Q_0 W P_0 R_{\rm eff}(z) P_0 W Q_0 R_{0,\varepsilon}(z),$$

where, with a slight abuse notation, we write $R_{\rm eff}(z) = (H_{\rm eff}(z) - z)^{-1}$, and furthermore (remember that we assume $\mathrm{Rank}\, P_0 = 1$)

$$\begin{aligned} P_0(H_{\rm eff}(z) - z)P_0 &= F(z,\varepsilon)P_0 \\ &= \left(\varepsilon\langle\Psi_0, W\Psi_0\rangle - z - \varepsilon^2\langle\Psi_0, W R_{0,\varepsilon}(z) W\Psi_0\rangle\right) P_0. \end{aligned} \tag{2.8}$$

Using (2.8), (2.7), and (2.6), one obtains

$$A_{g_\varepsilon}(t) = \lim_{\eta \searrow 0} \frac{1}{2\pi i} \int_{I_\varepsilon} dx\, e^{-ixt} \Big(\frac{1}{F(x + i\eta, \varepsilon)} - \frac{1}{F(x - i\eta, \varepsilon)} \Big). \tag{2.9}$$

The whole problem is to have a "nice" formula for $F(z, \varepsilon)$, so that the integral in (2.9) can be estimated. For that purpose we need some information on $R_{0,\varepsilon}(z)$. Let

$$W = A^* D A \tag{2.10}$$

be a factorization of W with D a self-adjoint involution. An example of such a factorization is the polar decomposition of W,

$$W = |W|^{1/2} D |W|^{1/2}, \tag{2.11}$$

where we take D to be unitary by defining it to be the identity on $\mathrm{Ker}\, W$.

Take $\mathrm{Im}\, z \to \infty$, and use regular perturbation theory to obtain

$$\begin{aligned} Q_0 R_{0,\varepsilon}(z) Q_0 &= Q_0(H - z)^{-1}Q_0 \\ &\quad - \varepsilon Q_0(H - z)^{-1} Q_0 W Q_0 (H - z)^{-1} Q_0 + \cdots \\ &= Q_0(H - z)^{-1}Q_0 - \varepsilon Q_0(H - z)^{-1}Q_0 \\ &\quad \times A^*[D + \varepsilon A Q_0 (H - z)^{-1} Q_0 A^*]^{-1} A Q_0 (H - z)^{-1} Q_0. \end{aligned} \tag{2.12}$$

With the notation

$$G(z) = A Q_0 (H - z)^{-1} Q_0 A^*, \tag{2.13}$$

one has for $\operatorname{Im} z \to \infty$

$$\begin{aligned} F(z,\varepsilon) = \varepsilon\langle\Psi_0, W\Psi_0\rangle - z \\ - \varepsilon^2\langle\Psi_0, A^*D\{G(z) - \varepsilon G(z)[D+\varepsilon G(z)]^{-1}G(z)\}DA\Psi_0\rangle. \end{aligned} \quad (2.14)$$

Since $F(z,\varepsilon)$ is analytic in z, the equality (2.14) holds true for all z, for which either the right hand side, or the left hand side, exists. In particular, (2.14) holds true for $\operatorname{Im} z \neq 0$.

The formulae (2.9) and (2.14) are the starting formulae of our approach. The next main ingredient is the expansion of $G(z)$ around 0. For the examples considered here, the corresponding expansions are provided by the results or methods in [10, 9, 17, 11].

3 The case of odd dimensions

As already said, the main ingredient of our approach is the asymptotic expansion for $G(z)$. In this section we shall use this expansion in a somewhat abstract setting, having in mind Schrödinger and Dirac operators in odd dimensions. More precisely we assume H and W to satisfy the following conditions (A1)–(A5). Here $\rho(H)$ denotes the resolvent set, and $\sigma(H)$ the spectrum, with standard notation for the components of the spectrum. We have taken $E_0 = 0$ in the sequel.

Assumption 3.1. (A1) *There exists $a > 0$, such that $(-a, 0) \subset \rho(H)$ and $[0, a] \subset \sigma_{\mathrm{ess}}(H)$.*

(A2) *Assume that zero is a nondegenerate eigenvalue of H: $H\Psi_0 = 0$, with $\|\Psi_0\| = 1$, and there are no other eigenvalues in $[0, a]$. Let $P_0 = |\Psi_0\rangle\langle\Psi_0|$ be the orthogonal projection onto the one-dimensional eigenspace.*

(A3) *Assume*

$$\langle\Psi_0, W\Psi_0\rangle = b > 0. \quad (3.1)$$

(A4) *For $\operatorname{Re}\kappa \geq 0$ and $z \in \mathbf{C} \setminus [0,\infty)$ we let*

$$\kappa = -i\sqrt{z}, \quad z = -\kappa^2. \quad (3.2)$$

There exist $N \in \mathbf{N}$ and $\delta_0 > 0$, such that for $\kappa \in \{\kappa \in \mathbf{C} \,|\, 0 < |\kappa| < \delta_0, \operatorname{Re}\kappa \geq 0\}$ we have

$$A(H+\kappa^2)^{-1}A^* = \frac{1}{\kappa^2}\widetilde{P}_0 + \sum_{j=-1}^{N} \widetilde{G}_j\kappa^j + \kappa^{N+1}\widetilde{G}_N(\kappa), \quad (3.3)$$

where

$$\widetilde{P}_0 = AP_0A^*, \tag{3.4}$$

$$\widetilde{G}_j \quad \text{are bounded and self-adjoint,} \tag{3.5}$$

$$\widetilde{G}_{-1} \quad \text{is of finite rank and self-adjoint,} \tag{3.6}$$

$$\widetilde{G}_N(\kappa) \quad \text{is uniformly bounded in } \kappa. \tag{3.7}$$

Taking into account that (remember that $Q_0 = 1 - P_0$)

$$(H + \kappa^2)^{-1} = \frac{1}{\kappa^2}P_0 + Q_0(H + \kappa^2)^{-1}Q_0, \tag{3.8}$$

one has from (3.3), (3.4), and (3.8) that

$$G(z) = \sum_{j=-1}^{N} \widetilde{G}_j\kappa^j + \kappa^{N+1}\widetilde{G}_N(\kappa). \tag{3.9}$$

From (3.9) we get

$$\langle \Psi_0, A^*DG(z)DA\Psi_0\rangle = \sum_{j=-1}^{N} g_j\kappa^j + \kappa^{N+1}g_N(\kappa), \tag{3.10}$$

where

$$g_j = \langle \Psi_0, A^*D\widetilde{G}_jDA\Psi_0\rangle, \tag{3.11}$$

$$g_N(\kappa) = \langle \Psi_0, A^*D\widetilde{G}_N(\kappa)DA\Psi_0\rangle. \tag{3.12}$$

Notice that due to (3.5) we have

$$g_j = \overline{g}_j. \tag{3.13}$$

Finally, we need one further assumption.

Assumption 3.2. (A5) *There exists an odd integer, $-1 \leq \nu \leq N$, such that*

$$g_\nu \neq 0, \quad \widetilde{G}_j = 0 \text{ for } j = -1, 1, \ldots, \nu - 2. \tag{3.14}$$

A few remarks about the above assumptions. (A1) is nothing but the fact that we consider the perturbation of eigenvalues lying at a threshold, and that the threshold is not embedded in the essential spectrum. Assumptions (A2) is a simplifying "nondegeneracy" condition. There are many interesting cases from a physical point of view, where these two assumptions do not hold.

The assumption (A3) is essential. It assures that the perturbation "pushes" the eigenvalue into the positive continuum at a rate of order ε, while (A5) implies (see below) that the "width" $\Gamma(\varepsilon)$ behaves as $\varepsilon^{2+(\nu/2)}$, as $\varepsilon \to 0$. If (A3) does not hold, and the perturbation pushes the eigenvalue into the continuum at a rate as say ε^2, then an exponential decay law may not exist. Let us note that there are examples of Schrödinger operators, where (A5) holds for any odd $\nu \geq -1$. See Section 4.3. We shall consider the problem of relaxing these assumptions in subsequent work. Assumption (A4) is our main tool. In particular, it implies that on $(0, \delta_0^2]$ the spectrum of H is absolutely continuous. We also notice that from (2.13), (3.10), and the first resolvent equation, it follows that

$$i^{\nu-1} g_\nu < 0. \tag{3.15}$$

By the heuristics of naive perturbation theory, one expects that the perturbation turns the zero eigenvalue into a "resonance", whose real part, up to errors of order ε^2, equals $b\varepsilon$. (Note that if (3.1) does not hold, then the eigenvalue may turn into an isolated eigenvalue of $H(\varepsilon)$.) This suggests to take the interval of "relevant energies" to be

$$I_\varepsilon = (\tfrac{1}{2} b\varepsilon, \tfrac{3}{2} b\varepsilon). \tag{3.16}$$

Now the idea of the proof that $A_{g_\varepsilon}(t)$ has the form of the right hand side of (2.3), is very simple: On the interval I_ε the function $F(x + i0, \varepsilon)$ can be approximated by a Lorentzian function, whose parameters give $\lambda(\varepsilon)$.

In what follows *for ε sufficiently small* is a shorthand expression for *there exists ε_0 such that for* $0 < \varepsilon \leq \varepsilon_0$ the given statement holds. All the constants appearing below are finite and strictly positive. Consider

$$D_\varepsilon = \{ z = x + i\eta \,|\, x \in I_\varepsilon,\ 0 < |\eta| < (\varepsilon b)^{2 + \frac{\nu+1}{2}} \}. \tag{3.17}$$

Lemma 3.3. *Let*

$$p(\nu) = \min\{2, \frac{2+\nu}{2}\}. \tag{3.18}$$

Then for ε sufficiently small, and for $z \in D_\varepsilon$, we have

$$F(z, \varepsilon) = H(z, \varepsilon) + r(z, \varepsilon), \tag{3.19}$$

with

$$\sup_{\substack{0<\varepsilon<\varepsilon_0 \\ z \in D_\varepsilon}} \varepsilon^{-(2+p(\nu)+\frac{\nu}{2})} |r(z, \varepsilon)| < \infty, \tag{3.20}$$

and for $\nu = -1$

$$H(z, \varepsilon) = \varepsilon b - z - \varepsilon^2 g_{-1} \kappa^{-1}, \tag{3.21}$$

while for $\nu \geq 1$

$$H(z,\varepsilon) = \varepsilon b - z - \varepsilon^2\Big[a_\nu(\varepsilon)\kappa^\nu + g_{\nu+2}\kappa^{\nu+2} + \sum_{j=0}^{\frac{\nu+3}{2}} f_j(\varepsilon)\kappa^{2j}\Big]. \tag{3.22}$$

Here (*see* (3.11) *for* g_j)

$$a_\nu(\varepsilon) = g_\nu - \varepsilon\langle\Psi_0, A^* D(\widetilde{G}_\nu\widetilde{G}_0 + \widetilde{G}_0\widetilde{G}_\nu)DA\Psi_0\rangle, \tag{3.23}$$

the $f_j(\varepsilon)$ *are polynomials with real coefficients of degree at most* $2+\frac{\nu-1}{2}$, *and*

$$f_0(\varepsilon) = g_0 + \mathcal{O}(\varepsilon). \tag{3.24}$$

Proof. The crucial point is that (and this is the reason for our choice of I_ε), since D_ε is "far" from the origin,

$$\sup_{z\in D_\varepsilon} \varepsilon\|G(z)\| \leq \begin{cases} C\varepsilon^{1/2} & \text{for } \nu = -1, \\ C\varepsilon & \text{for } \nu > -1. \end{cases}$$

Accordingly, for sufficiently small ε we have

$$\sup_{z\in D_\varepsilon} \|(D+\varepsilon G(z))^{-1}\| \leq 2, \tag{3.25}$$

and then from (2.14)

$$F(z,\varepsilon) = \varepsilon b - z - \varepsilon^2\Big\langle\Psi_0, A^*\Big(\sum_{k=0}^{m}(-\varepsilon)^k(DG(z)D)^{k+1}\Big)A\Psi_0\Big\rangle + q_m(z,\varepsilon). \tag{3.26}$$

By choosing $m=0$ for $\nu=-1$, $m=1$ for $\nu=1$, and $m = 1+\frac{\nu+3}{2}$ for $\nu>1$, $q_m(z,\varepsilon)$ satisfies (3.20), i.e.

$$\sup_{\substack{0<\varepsilon<\varepsilon_0 \\ z\in D_\varepsilon}} \varepsilon^{-(2+p(\nu)+\frac{\nu}{2})}|q_m(z,\varepsilon)| < \infty. \tag{3.27}$$

Plug the expansion (3.10) into (3.26) with $N=-1$ for $\nu=-1$, $N=3$ for $\nu=1$, and $N=\nu+5$ for $\nu>1$, and then keep in $H(z,\varepsilon)$ all the terms, which do not satisfy (3.27). □

Consider now the function $H(z,\varepsilon)$. From the definition of D_ε, it follows that for $z\in D_\varepsilon$ one has $|\operatorname{Im}\kappa^\nu| \leq C\varepsilon^{\nu/2}$, $|\operatorname{Im}\kappa^2| \geq C\varepsilon^{2+\frac{\nu}{2}}$. Since all the coefficients appearing in the definition of $H(z,\varepsilon)$ are real, it follows that for sufficiently small ε we have

$$\inf_{z\in D_\varepsilon} |\operatorname{Im} H(z,\varepsilon)| \geq C\varepsilon^{2+\frac{\nu}{2}}. \tag{3.28}$$

Obviously, $H(z,\varepsilon)$ has limits as $\eta \to 0$,

$$H_{\pm}(x,\varepsilon) = \lim_{\eta \searrow 0} H(x \pm i\eta, \varepsilon). \tag{3.29}$$

Also, from (2.8) it follows that $\eta \operatorname{Im} F(x+i\eta,\varepsilon) < 0$, and then from (3.28) and Lemma 3.3, it follows that

$$\eta \operatorname{Im} H(x+i\eta,\varepsilon) < 0. \tag{3.30}$$

Notice also that

$$\overline{H_{+}(x,\varepsilon)} = H_{-}(x,\varepsilon), \tag{3.31}$$

and on I_ε we have

$$|H_{\pm}(x,\varepsilon)| \geq |\operatorname{Im} H_{\pm}(x,\varepsilon)| \geq C\varepsilon^{2+\frac{\nu}{2}}. \tag{3.32}$$

Let $R(x,\varepsilon)$ and $I(x,\varepsilon)$ be the real and the imaginary parts of $H_{+}(x,\varepsilon)$, respectively, such that

$$H_{\pm}(x,\varepsilon) = R(x,\varepsilon) \pm iI(x,\varepsilon). \tag{3.33}$$

From (3.22) one has

$$R(x,\varepsilon) = \varepsilon b - x - \varepsilon^2 \sum_{j=0}^{\frac{\nu+3}{2}} (-x)^j f_j(\varepsilon) \tag{3.34}$$

(for $\nu = -1$ the sum in the right hand side of (3.34) is zero, see (3.21)). For ε sufficiently small we have $R(\frac{\varepsilon b}{2},\varepsilon) > 0$ and $R(\frac{3\varepsilon b}{2},\varepsilon) < 0$, and for $x \in [\frac{\varepsilon b}{2}, \frac{3\varepsilon b}{2}]$,

$$-\frac{3}{2} < \frac{d}{dx} R(x,\varepsilon) < -\frac{1}{2}. \tag{3.35}$$

This implies that for sufficiently small ε the equation $R(x,\varepsilon) = 0$ has a unique solution $x_0(\varepsilon)$, i.e.

$$R(x_0(\varepsilon),\varepsilon) = 0. \tag{3.36}$$

In addition,

$$x_0(\varepsilon) = \varepsilon b + \mathcal{O}(\varepsilon^2). \tag{3.37}$$

The next lemma estimates the error, when $F(z,\varepsilon)$ is replaced with $H(z,\varepsilon)$.

Lemma 3.4. *For sufficiently small ε we have*

$$\left| A_{g_\varepsilon}(t) - \frac{1}{2\pi i} \int_{I_\varepsilon} e^{-ixt} \left[\frac{1}{H_{+}(x,\varepsilon)} - \frac{1}{H_{-}(x,\varepsilon)} \right] dx \right| \leq C\varepsilon^{p(\nu)}. \tag{3.38}$$

Proof. For sufficiently small ε, and $z \in D_\varepsilon$, we have

$$|F(z,\varepsilon)| \geq \tfrac{1}{2}|H(z,\varepsilon)|. \tag{3.39}$$

Indeed, from Lemma 3.3 follows

$$|F(z,\varepsilon)| \geq |H(z,\varepsilon)| - C\varepsilon^{2+p(\nu)+\frac{\nu}{2}},$$

which together with (3.28), and the fact that $p(\nu) \geq \frac{1}{2}$, implies (3.39). Furthermore, from Lemma 3.3 and (3.39) follows

$$\Big|\frac{1}{2\pi i}\int_{I_\varepsilon} e^{-ixt}\left[\frac{1}{F(x+i\eta,\varepsilon)} - \frac{1}{H(x+i\eta,\varepsilon)}\right]dx\Big| \leq C\varepsilon^{2+p(\nu)+\frac{\nu}{2}}\int_{I_\varepsilon}\frac{1}{|H(x+i\eta,\varepsilon)|^2}dx.$$

The estimate (3.28) implies that $\frac{1}{|H(x+i\eta,\varepsilon)|^2}$ is uniformly bounded for a fixed ε. Take the limit $\eta \searrow 0$ to get

$$\Big|\lim_{\eta\searrow 0}\frac{1}{2\pi i}\int_{I_\varepsilon} e^{-ixt}\left[\frac{1}{F(x\pm i\eta,\varepsilon)} - \frac{1}{H(x\pm i\eta,\varepsilon)}\right]dx\Big| \leq C\varepsilon^{2+p(\nu)+\frac{\nu}{2}}\int_{I_\varepsilon}\frac{1}{|H_\pm(x,\varepsilon)|^2}dx. \tag{3.40}$$

Now due to (3.32), (3.35), and (3.36) we have

$$|H_\pm(x,\varepsilon)| \geq C\sqrt{(x-x_0(\varepsilon))^2+\varepsilon^{4+\nu}}, \tag{3.41}$$

and then

$$\varepsilon^{2+p(\nu)+\frac{\nu}{2}}\int_{I_\varepsilon}\frac{1}{|H(x+i\eta,\varepsilon)|^2}dx \leq \varepsilon^{p(\nu)}\int_{-\infty}^{\infty}\frac{\varepsilon^{\frac{4+\nu}{2}}}{(x-x_0(\varepsilon))^2+\varepsilon^{4+\nu}}dx = \varepsilon^{p(\nu)}\int_{-\infty}^{\infty}\frac{1}{x^2+1}dx, \tag{3.42}$$

and the proof of the lemma is finished. □

Let now (see (3.33))

$$\Gamma(\varepsilon) = -I(x_0(\varepsilon),\varepsilon). \tag{3.43}$$

Notice that for sufficiently small ε (see (3.30) and (3.15)) we have $\Gamma(\varepsilon) > 0$.

We want to replace $H_\pm(x,\varepsilon)$ with the following function

$$L_\pm(x,\varepsilon) = -(x-x_0(\varepsilon)) \pm iI(x_0(\varepsilon),\varepsilon). \tag{3.44}$$

Inserted into the expressions above it leads to a Lorentzian function. The next lemma estimates the error, when $H_\pm(x,\varepsilon)$ is replaced by $L_\pm(x,\varepsilon)$.

Lemma 3.5. *For sufficiently small ε we have*

$$\Big|\int_{I_\varepsilon} e^{-ixt}\left[\frac{1}{H_\pm(x,\varepsilon)} - \frac{1}{L_\pm(x,\varepsilon)}\right]dx\Big| \le C\max\{\varepsilon^{1+\frac{\nu}{2}}|\ln\varepsilon|, \varepsilon^3\}. \tag{3.45}$$

Proof. As in the proof of Lemma 3.3 we have to estimate $H_\pm(x,\varepsilon)-L_\pm(x,\varepsilon)$. From (2.2) and (3.29) we have

$$\sup_{x\in I_\varepsilon}\left\{\Big|\frac{d^2}{dx^2}R(x,\varepsilon)\Big| + \Big|\frac{d}{dx}R(x,\varepsilon)+1\Big|\right\} \le C\varepsilon^2 \tag{3.46}$$

and

$$\sup_{x\in I_\varepsilon}\Big|\frac{d}{dx}I(x,\varepsilon)\Big| \le C\varepsilon^{1+\frac{\nu}{2}}. \tag{3.47}$$

Then from the Taylor expansion (with remainder) we get

$$|H_\pm(x,\varepsilon) - L_\pm(x,\varepsilon)| \le C(\varepsilon^{1+\frac{\nu}{2}}|x-x_0(\varepsilon)| + \varepsilon^2|x-x_0(\varepsilon)|^2). \tag{3.48}$$

Using (3.41) and (3.48) one obtains that

$$\begin{aligned}
&\Big|\int_{I_\varepsilon} e^{-ixt}\left[\frac{1}{H_\pm(x,\varepsilon)} - \frac{1}{L_\pm(x,\varepsilon)}\right]dx\Big| \\
&\quad\le C\varepsilon^{1+\frac{\nu}{2}}\int_{I_\varepsilon}\frac{|x-x_0(\varepsilon)|}{|x-x_0(\varepsilon)|^2+\varepsilon^{4+\nu}}dx + C\varepsilon^2\int_{I_\varepsilon}\frac{|x-x_0(\varepsilon)|^2}{|x-x_0(\varepsilon)|^2+\varepsilon^{4+\nu}}dx \\
&\qquad\le C(\varepsilon^{1+\frac{\nu}{2}}|\ln\varepsilon| + \varepsilon^3) \le C\max\{\varepsilon^{1+\frac{\nu}{2}}|\ln\varepsilon|, \varepsilon^3\},
\end{aligned}$$

which finishes the proof of the lemma. □

We can now evaluate $A_{g_\varepsilon}(t)$.

Lemma 3.6. *For sufficiently small ε we have*

$$|A_{g_\varepsilon}(t) - e^{-it(x_0(\varepsilon)-i\Gamma(\varepsilon))}| \le C\varepsilon^{p(\nu)}|\ln\varepsilon|^\iota, \tag{3.49}$$

where $\iota = 1$ for $\nu = -1, 1$, and zero otherwise.

Proof. By direct computation

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{I_\varepsilon} e^{-ixt}\left[\frac{1}{L_+(x,\varepsilon)} - \frac{1}{L_-(x,\varepsilon)}\right]dx \\
&\qquad = \frac{1}{\pi}\int_{I_\varepsilon} e^{-ixt}\frac{\Gamma(\varepsilon)}{(x-x_0(\varepsilon))^2+\Gamma(\varepsilon)^2}dx.
\end{aligned} \tag{3.50}$$

Due to (3.37), (3.23), and (2.2) we have

$$\Gamma(\varepsilon) = -i^{\nu-1} g_\nu b^{\frac{\nu}{2}} \varepsilon^{2+\frac{\nu}{2}} + \mathcal{O}(\varepsilon^{3+\frac{\nu}{2}}) \tag{3.51}$$

which together with (3.37) implies

$$\Big|\Big(\int_{\mathbf{R}} - \int_{I_\varepsilon}\Big) e^{-ixt} \frac{\Gamma(\varepsilon)}{(x - x_0(\varepsilon))^2 + \Gamma(\varepsilon)^2} dx\Big| \\ \leq C \int_{\frac{\varepsilon b}{2}}^{\infty} \frac{\Gamma(\varepsilon)}{(x - x_0(\varepsilon))^2 + \Gamma(\varepsilon)^2} dx \leq C\varepsilon^{1+\frac{\nu}{2}}. \tag{3.52}$$

Since by the residue theorem

$$\frac{1}{\pi} \int_{\mathbf{R}} e^{-ixt} \frac{\Gamma(\varepsilon)}{(x - x_0(\varepsilon))^2 + \Gamma(\varepsilon)^2} dx = e^{-it(x_0(\varepsilon) - i\Gamma(\varepsilon))}, \tag{3.53}$$

(3.49) follows from Lemmas 3.4, 3.5, and the results (3.50) and (3.52). □

We are now in a position to formulate the main result.

Theorem 3.7. *Suppose* (A1)–(A5) *hold true. Then for sufficiently small* ε *we have*

$$|\langle \Psi_0, e^{-itH(\varepsilon)} \Psi_0 \rangle - e^{-it(x_0(\varepsilon) - i\Gamma(\varepsilon))}| \leq C\varepsilon^{p(\nu)} |\ln \varepsilon|^\iota, \tag{3.54}$$

where $\iota = 1$ *for* $\nu = -1, 1$, *and zero otherwise. Here*

$$\Gamma(\varepsilon) = -i^{\nu-1} g_\nu b^{\nu/2} \varepsilon^{2+\nu/2} (1 + \mathcal{O}(\varepsilon)), \tag{3.55}$$

$$x_0(\varepsilon) = b\varepsilon(1 + \mathcal{O}(\varepsilon)). \tag{3.56}$$

Proof. The theorem follows from Lemma 3.6 by an argument due to Hunziker [8]. For completeness we reproduce it. Taking $t = 0$ in (3.49) one gets

$$|\langle \Psi_0, g_\varepsilon(H(\varepsilon)) \Psi_0 \rangle - 1| \leq C\varepsilon^{p(\nu)} |\ln \varepsilon|^\iota,$$

which gives (recall that $0 \leq g_\varepsilon(x) \leq 1$)

$$\|(1 - g_\varepsilon(H(\varepsilon)))^{\frac{1}{2}} \Psi_0\|^2 \leq C\varepsilon^{p(\nu)} |\ln \varepsilon|^\iota. \tag{3.57}$$

Now

$$|\langle \Psi_0, e^{-itH(\varepsilon)} \Psi_0 \rangle - A_{g_\varepsilon}(t)| \\ = |\langle (1 - g_\varepsilon(H(\varepsilon)))^{\frac{1}{2}} \Psi_0, e^{-itH(\varepsilon)} (1 - g_\varepsilon(H(\varepsilon)))^{\frac{1}{2}} \Psi_0 \rangle| \\ \leq \|(1 - g_\varepsilon(H(\varepsilon)))^{\frac{1}{2}} \Psi_0\|^2,$$

which together with Lemma 3.6 and (3.57) finishes the proof. □

4 Examples

As examples we consider one and two channel Schrödinger operators in odd dimensions. We shall restrict ourselves to the “physical” dimensions one and three. In the three dimensional case we consider various cases for both one and two channel Schrödinger operators. In the one dimensional case with local potentials we only consider the two channel case. We obtain explicit examples with $g_\nu \neq 0$ for ν arbitrarily large. In each case we find ν and g_ν, which gives the leading term in ε of $\Gamma(\varepsilon)$ (see (3.55)).

In the one channel case

$$H = -\Delta + V(\mathbf{x}), \tag{4.1}$$

$$(Wf)(\mathbf{x}) = W(\mathbf{x})f(\mathbf{x}), \tag{4.2}$$

in $L^2(\mathbf{R}^m)$, $m = 1, 3$, with V, W satisfying

$$\langle\cdot\rangle^\beta V \in L^\infty(\mathbf{R}^m), \tag{4.3}$$

$$\langle\cdot\rangle^\gamma W \in L^\infty(\mathbf{R}^m), \tag{4.4}$$

and β, γ are sufficiently large, in order to obtain the expansions below (see [11]), and we suppose that the singularity of $(H + \kappa^2)^{-1}$ at $\kappa = 0$ is coming only from the existence of a nondegenerate eigenvalue at the threshold. Note that we can allow singularities in V and W, but we have decided to omit the technicalities involved in dealing with such singularities.

In the two channel case we consider examples of a nondegenerate bound state of zero energy in the “closed” channel decaying due to the interaction with an odd dimensional Schrödinger operator in the open channel. Since only the bound state in the closed channel is relevant in the forthcoming discussion, we shall take $\mathbf{C}$ as the Hilbert space representing the closed channel, i.e. $\mathcal{H} = L^2(\mathbf{R}^m) \oplus \mathbf{C}$. As the unperturbed Hamiltonian we take

$$H = \begin{bmatrix} -\Delta + V & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.5}$$

where V satisfies (4.3), and as the perturbation we take

$$W = \begin{bmatrix} W_{11} & |W_{12}\rangle\langle 1| \\ |1\rangle\langle W_{12}| & b \end{bmatrix}, \tag{4.6}$$

which is a shorthand for

$$W \begin{bmatrix} f(\mathbf{x}) \\ \xi \end{bmatrix} = \begin{bmatrix} W_{11}(\mathbf{x})f(\mathbf{x}) + W_{12}(\mathbf{x})\xi \\ \int \overline{W_{12}(\mathbf{x})} f(\mathbf{x}) + b\xi \end{bmatrix}. \tag{4.7}$$

Here we assume

$$\langle\cdot\rangle^{\gamma}W_{11} \in L^{\infty}(\mathbf{R}^m), \quad \langle\cdot\rangle^{\gamma/2}W_{12} \in L^{\infty}(\mathbf{R}^m), \tag{4.8}$$

and furthermore that W_{11} is realvalued. In order to satisfy (3.1) we assume $b > 0$ in (4.6).

We use the following factorization of W. To simplify the notation below we introduce the weight function

$$\rho_{\gamma} = \langle\cdot\rangle^{-\gamma/2}. \tag{4.9}$$

In the one channel case we write

$$W = \rho_{\gamma} C \rho_{\gamma}, \tag{4.10}$$

i.e. C is the bounded operator of multiplication with $\langle\mathbf{x}\rangle^{\gamma}W(\mathbf{x})$. Writing the polar decomposition for C (with a self-adjoint D satisfying $D^2 = I$) as

$$C = |C|^{1/2} D |C|^{1/2}, \tag{4.11}$$

we have in this case

$$A = |C|^{1/2}\rho_{\gamma}. \tag{4.12}$$

In the two channel case let

$$B = \begin{bmatrix} \rho_{-\gamma} & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.13}$$

and

$$C = BWB = |C|^{1/2} D |C|^{1/2}, \tag{4.14}$$

where D is defined to be the identity on $\operatorname{Ker} C$, such that D is self-adjoint with $D^2 = I$. The operator C is bounded and self-adjoint, and we take

$$A = |C|^{1/2} B^{-1}, \tag{4.15}$$

i.e.

$$W = B^{-1}|C|^{1/2} D |C|^{1/2} B^{-1}. \tag{4.16}$$

Now, since $|C|^{1/2}$ is bounded, it is clear from (3.3) and (4.15) that we need the expansion of

$$B^{-1}(H+\kappa^2)^{-1}B^{-1} = \begin{bmatrix} \rho_{\gamma}(-\Delta + V + \kappa^2)^{-1}\rho_{\gamma} & 0 \\ 0 & \dfrac{1}{\kappa^2} \end{bmatrix}, \tag{4.17}$$

which, together with the fact that in our case

$$P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.18}$$

reduces the problem of writing down (3.3) to the expansion of the resolvent in the scalar case. Summing up, in all cases the needed expansion of $(H+\kappa^2)^{-1}$ follows at once from the expansion of $\rho_\gamma(-\Delta+V+\kappa^2)^{-1}\rho_\gamma$. The expansions of $(-\Delta+V+\kappa^2)^{-1}$ near $\kappa=0$ have been written down in [10, 9, 17, 11]. For the first example we take results from [10]. For the second example we need to carry the computations further than was done in [11], in order to get explicit expressions for the coefficients. The last example has not been treated previously, so we give some details of the computations. We use both the weighted space technique, and the factorization technique.

4.1 Schrödinger operators in three dimensions

We first present the results in the case where the Schrödinger operator acts in three dimensions.

We will draw on the results in [10] to get the results on asymptotic expansion of the resolvent, and the explicit form of the expansion coefficients. The results in [10] are formulated using the weighted L^2-spaces. Here we use the factorization technique. It is quite straightforward to translate between the two formalisms, so we shall not elaborate on this point.

Let us briefly recall a few results from [10] in order to establish the notation. We have $H=-\Delta+V$ on $L^2(\mathbf{R}^3)$. The integral kernel for the free resolvent yields the following expansion, valid pointwise, and as an asymptotic expansion between weighted spaces, see [10, Lemma 2.3].

$$\begin{aligned}(-\Delta+\kappa^2)^{-1}(\mathbf{x},\mathbf{y}) &= \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}e^{-\kappa|\mathbf{x}-\mathbf{y}|} \\ &= G_0^0(\mathbf{x},\mathbf{y})+\kappa G_1^0(\mathbf{x},\mathbf{y}) \\ &\quad +\kappa^2 G_2^0(\mathbf{x},\mathbf{y})+\kappa^3 G_3^0(\mathbf{x},\mathbf{y})+\cdots.\end{aligned} \tag{4.19}$$

Note that the coefficients in [10, (2.2)] are here denoted by G_j^0. The expressions for the kernels are

$$G_j^0(\mathbf{x},\mathbf{y}) = \frac{(-1)^j}{4\pi j!}|\mathbf{x}-\mathbf{y}|^{j-1}, \quad j=0,1,2,\ldots. \tag{4.20}$$

Note the changes in signs due to our use of the expansion parameter κ. We use the convention that the expansion coefficients in (3.3) in the weighted space formalism, i.e. without the A-terms, are denoted by G_j (omitting the tilde). When we use the weighted spaces, $\langle\cdot,\cdot\rangle$ also is used for the duality between $L^{2,s}$ and $L^{2,-s}$.

The point zero is classified into four cases. It may be a regular point, in which case there is no singularity in the resolvent expansion. In the other three cases there exist at least one solution to $(-\Delta+V)\Psi=0$, in the space $L^{2,-s}(\mathbf{R}^3)$, $1/2<s\leq 3/2$. We have the result that $\Psi\in L^2(\mathbf{R}^3)$, if and only if $\langle V,\Psi\rangle=0$, see [10, Lemma 3.3]. In case there is a solution with $\langle V,\Psi\rangle\neq 0$, it is normalized by the condition $\langle V,\Psi\rangle=\sqrt{4\pi}$. It is called the *canonical zero resonance function.*

Our first result concerns the one channel case. Note that we take Ψ_0 to be realvalued.

Theorem 4.1 (One channel case). *Assume that V and W satisfy* (4.3) *and* (4.4) *with $\beta>7$ and $\gamma>5$, respectively. Assume that* (A1) *holds for $H=-\Delta+V$. Let*

$$X_j=\int_{\mathbf{R}^3}\Psi_0(\mathbf{x})V(\mathbf{x})x_j d\mathbf{x},\quad j=1,2,3. \tag{4.21}$$

(i) *Assume that $X_j\neq 0$ for at least one j. Then $\nu=-1$, and*

$$g_{-1}=\frac{b^2}{12\pi}(X_1^2+X_2^2+X_3^2). \tag{4.22}$$

(ii) *Assume that* 0 *is also a resonance for H, with canonical zero resonance function Φ_0. Assume either that $X_j\neq 0$ for at least one j, or that $\langle\Psi_0,W\Phi_0\rangle\neq 0$. Then $\nu=-1$, and we have*

$$g_{-1}=\frac{b^2}{12\pi}(X_1^2+X_2^2+X_3^2)+|\langle\Psi_0,W\Phi_0\rangle|^2. \tag{4.23}$$

Proof. Part (i): Under our assumptions on V we have an asymptotic expansion (3.3) with $N=-1$, see [10, Theorem 6.4], and furthermore

$$G_{-1}=P_0VG_3^0VP_0.$$

Insert this expression into (3.11) and use the explicit kernel (4.20) together with the result [10, Lemma 3.3]

$$\int_{\mathbf{R}^3}\Psi_0(\mathbf{x})V(\mathbf{x})d\mathbf{x}=0 \tag{4.24}$$

to get (4.22).

Part (ii): The proof ia analogous to the first part, using [10, Theorem 6.5]. Note the differences in signs due to our use of the expansion parameter κ here. □

Remark 4.2. One can understand the result (4.24) as follows. The eigenfunction satisfies

$$\Psi_0(\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{1}{|\mathbf{x}-\mathbf{y}|}V(\mathbf{y})\Psi_0(\mathbf{y})d\mathbf{y}.$$

By expanding the kernel for large $|\mathbf{x}|$ on finds the condition (4.24) in order to have $\Psi_0 \in L^2(\mathbf{R}^3)$.

Concerning the two channel case we have the following result.

Theorem 4.3 (Two channel case). *Assume that V and W satisfy (4.3) and (4.4) with $\beta > 7$ and $\gamma > 5$, respectively.*

(i) *Assume that $-\Delta+V$ has neither a threshold resonance nor a threshold eigenvalue. Then $\nu \geq 1$, and we have*

$$g_1 = \frac{-1}{4\pi}|\langle W_{12}, (I + G_0^0 V)^{-1}1\rangle|^2. \tag{4.25}$$

(ii) *Assume that $-\Delta+V$ has a threshold resonance, and no threshold eigenvalue. Let Φ_0 denote the canonical zero resonance function. Assume that $\langle W_{12}, \Phi_0\rangle \neq 0$. Then $\nu = -1$, and*

$$g_{-1} = |\langle W_{12}, \Phi_0\rangle|^2. \tag{4.26}$$

Proof. In the two channel case we have $\Psi_0 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. We start with part (i). We get the required resolvent expansion (3.3) from [10, Theorem 6.1]. In this case $\widetilde{G}_{-1} = 0$. Under the assumption on the potential V (3.3) holds for $N = 1$. We also have the expression

$$G_1 = \begin{bmatrix} \frac{-1}{4\pi}|(I + G_0^0 V)^{-1}1\rangle\langle(I + G_0^0 V)^{-1}1| & 0 \\ 0 & 0 \end{bmatrix},$$

see [10, (6.3)]. Now the proof consists in combining this expression with the definition (3.11) and the matrix W. This leads to the result stated in part (i). Concerning part (ii), then we use [10, Theorem 6.3] and perform the same computations as for part (i). □

Remark 4.4. Note that the function $\psi = (I + G_0^0 V)^{-1}1$ satisfies $(-\Delta + V)\psi = 0$ in the sense of distributions. Thus it is a generalized zero energy eigenfunction. Compare with the discussion in [3].

4.2 Schrödinger operators in one dimension

Since in the case dimension $m = 1$ and local short range potentials there is no bound state at the threshold, we can only consider the two channel case. For $m = 1$ the expansion of $\rho_\gamma(-\Delta + V + \kappa^2)^{-1}\rho_\gamma$ is much more complicated, due to the $1/\kappa$ singularity in the free resolvent. The result needed is obtained from [11]. Since it was not written down explicitly in [11], we reproduce some results needed to complete the computation.

The kernel of the free resolvent has the expansion

$$\begin{aligned}(-\Delta + \kappa^2)^{-1}(\mathbf{x}, \mathbf{y}) &= \frac{1}{2\kappa} e^{-\kappa|x-y|} \\ &= \frac{1}{2\kappa} - \frac{|x-y|}{2} + \kappa \frac{|x-y|^2}{4} + \mathcal{O}(\kappa^2) \\ &= \frac{1}{\kappa} G^0_{-1}(\mathbf{x}, \mathbf{y}) + G^0_0(\mathbf{x}, \mathbf{y}) + \mathcal{O}(\kappa),\end{aligned} \tag{4.27}$$

where we also introduced the notation used here. Note that the G^0_j here are different from those defined in (4.20). We also use the notation $v(x) = |V(x)|^{1/2}$, $U(x) = 1$, if $V(x) \geq 0$, $U(x) = -1$, if $V(x) < 0$, such that the factorization used is $V = vUv$. We write $w = vU$.

The expansion results are obtained by studying the operator

$$M(\kappa) = U + v(-\Delta + \kappa^2)^{-1} v,$$

and its inverse, see [11, (4.3)]. We have

$$M(\kappa) = \tfrac{1}{2}\alpha P \kappa^{-1} + M_0 + M_1\kappa + \kappa^2 r(\kappa), \tag{4.28}$$

where

$$P = \alpha^{-1}|v\rangle\langle v|, \quad \alpha = \|v\|^2, \tag{4.29}$$

and $M_0 - U$ and M_1 are the integral operators given by the kernels

$$(M_0 - U)(x, y) = -\tfrac{1}{2} v(x)|x - y| v(y), \tag{4.30}$$

$$M_1(x, y) = \tfrac{1}{4} v(x)|x - y|^2 v(y), \tag{4.31}$$

and, for $\beta > 7$, the remainder $r(\kappa)$ is uniformly bounded in norm. Let $Q = 1 - P$, and let $S\colon QL^2(\mathbf{R}) \to QL^2(\mathbf{R})$ be the orthogonal projection onto $\operatorname{Ker} QM_0Q$. Then (see [11, Theorem 5.2 and (5.18)]) $\operatorname{Rank} S \leq 1$, and the formula for $M(\kappa)^{-1}$ is as follows.

$$M(\kappa)^{-1} = \frac{2\kappa}{\alpha}(1 + \kappa\widetilde{M}(\kappa))^{-1}$$

$$
\begin{aligned}
&+\frac{2}{\alpha}(1+\kappa\widetilde{M}(\kappa))^{-1}Q(m_0+S+\kappa m_1(\kappa))^{-1}Q(1+\kappa\widetilde{M}(\kappa))^{-1}\\
&+\kappa^{-1}\frac{2}{\alpha}(1+\kappa\widetilde{M}(\kappa))^{-1}Q(m_0+S+\kappa m_1(\kappa))^{-1}Sq(\kappa)^{-1}S\\
&\quad\times(m_0+S+\kappa m_1(\kappa))^{-1}Q(1+\kappa\widetilde{M}(\kappa))^{-1},
\end{aligned}
\tag{4.32}
$$

where we use the notation

$$
\begin{aligned}
\widetilde{M}(\kappa) &= \frac{2}{\alpha}(M_0+\kappa M_1)+\mathcal{O}(\kappa^2),\\
m(\kappa) &= \frac{2}{\alpha}QM_0Q-\frac{2}{\alpha}\kappa Q(\frac{2}{\alpha}M_0^2-M_1)Q+\mathcal{O}(\kappa^2)\\
&\equiv m_0+\kappa(m_1+\kappa m_2(\kappa))\\
&\equiv m_0+\kappa m_1(\kappa),
\end{aligned}
\tag{4.33}
$$

and

$$
q(\kappa)=q_0+\mathcal{O}(\kappa) \tag{4.34}
$$

as an operator in $SL^2(\mathbf{R})$, with

$$
q(0)\equiv q_0=Sm_1S. \tag{4.35}
$$

In the formula (4.32), if QM_0Q is invertible as an operator in $QL^2(\mathbf{R})$, i.e. $S=0$, the last term vanishes. If $S\neq 0$, we have the following result (see [11, Theorem 5.2]).

Proposition 4.5. *Assume $S\neq 0$. Let $\Phi\in SL^2(\mathbf{R})$, $\|\Phi\|=1$. If Ψ is defined by*

$$
\Psi(x)=\frac{1}{\alpha}\langle v,M_0\Phi\rangle+\frac{1}{2}\int_{\mathbf{R}}|x-y|v(y)\Phi(y)dy, \tag{4.36}
$$

then

$$
w\Psi=\Phi, \tag{4.37}
$$

$\Psi\notin L^2(\mathbf{R})$, $\Psi\in L^\infty(\mathbf{R})$, and in the distribution sense

$$
H\Psi=0. \tag{4.38}
$$

Conversely, if there exists $\Psi\in L^\infty(\mathbf{R})$ satisfying (4.38) in the distribution sense, then

$$
\Phi=w\Psi\in SL^2(\mathbf{R}). \tag{4.39}
$$

In addition,

$$
q(0)=-\frac{2}{\alpha}\tilde{c}^2S, \tag{4.40}
$$

with

$$\tilde{c}^2 = \frac{2}{\alpha^2}|\langle v, M_0\Phi\rangle|^2 + \frac{1}{2}|\langle v, X\Phi\rangle|^2 > 0, \tag{4.41}$$

where X is the operator of multiplication with x.

We are prepared to state the main result of this subsection.

Theorem 4.6 (Two channel case). *Assume V satisfies* (4.3) *with $\beta > 7$, and W satisfies* (4.8) *with $\gamma > 5$. Then we have the following results.*

(i) *If in the open channel there is no threshold resonance* (*i.e.* $S = 0$), *then $\nu \geq 1$.*

(ii) *If there is a threshold resonance in the open channel* (*i.e.* $S \neq 0$), *and $\langle W_{12}, \Psi\rangle \neq 0$, where Ψ is the resonance function, then $\nu = -1$, and*

$$g_{-1} = \frac{|\langle W_{12}, \Psi\rangle|^2}{\tilde{c}^2}. \tag{4.42}$$

Proof. We have to insert the expansion (4.32) into

$$\begin{aligned}\rho_\gamma(-\Delta + V + \kappa^2)^{-1}\rho_\gamma &= \rho_\gamma(-\Delta + \kappa^2)^{-1}\rho_\gamma \\ &\quad - \rho_\gamma(-\Delta + \kappa^2)^{-1}vM(\kappa)^{-1}v(-\Delta + \kappa^2)^{-1}\rho_\gamma, \end{aligned} \tag{4.43}$$

and compute the $\frac{1}{\kappa}$ term. The main observation is that most of the singular terms vanish or cancel each other. Observe that

$$\begin{aligned} &Q(1 + \kappa\widetilde{M}(\kappa))^{-1}v(-\Delta + \kappa^2)^{-1}\rho_\gamma \\ &\quad = \frac{1}{\kappa}Q|v\rangle\langle\rho_\gamma| - \frac{2}{\alpha}QM_0vG^0_{-1}\rho_\gamma - QvG^0_0\rho_\gamma + \mathcal{O}(\kappa) \\ &\quad = -\frac{2}{\alpha}QM_0vG^0_{-1}\rho_\gamma - QvG^0_0\rho_\gamma + \mathcal{O}(\kappa), \end{aligned} \tag{4.44}$$

since by definition $Pv = v$ and $QP = 0$. Insertion of the expansion (4.32) into (4.43) gives four terms to be considered. From (4.44) follows that the third one is $\mathcal{O}(1)$. Computing the $\frac{1}{\kappa}$ contribution from the first two terms, one obtains (see (4.29))

$$\frac{1}{2}|\rho_\gamma\rangle\langle\rho_\gamma| - \frac{1}{2\alpha}\langle v^2, 1\rangle|\rho_\gamma\rangle\langle\rho_\gamma| = 0. \tag{4.45}$$

Since in the regular case (i.e. $S = 0$) the fourth term does not exist, the first part of the theorem follows from (4.44) and (4.45). Moreover, in the case $S \neq 0$, one has to consider only the fourth term. The computation of the $\frac{1}{\kappa}$

coefficient leads to (observe that $SQ = S$, and see also (4.27), (4.34), (4.40), (4.35), (4.36), and (4.41)),

$$\rho_\gamma(-\Delta + V + \kappa^2)^{-1}\rho_\gamma = \frac{1}{\tilde{c}^2\kappa}|\rho_\gamma\Psi\rangle\langle\rho_\gamma\Psi| + \mathcal{O}(1), \tag{4.46}$$

which gives (4.42), and the proof is finished. □

4.3 Schrödinger operators on the half line with $\ell \geq 1$

In this subsection we consider the operator

$$H_{0,\ell} = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}, \quad \ell = 1, 2, \dots, \tag{4.47}$$

on the space $\mathcal{H} = L^2(\mathbf{R}_+)$. It will provide us with examples of resolvent expansions, where we can verify Assumption (A5) with $\nu \geq 3$ odd and arbitrarily large. Note that the cases $\nu = -1$ and $\nu = 1$ were covered in the preceding sections.

The operator $H_{0,\ell}$ is essentially selfadjoint on $C_0^\infty((0,\infty))$. We now give the integral kernel of the resolvent $(H_{0,\ell} + \kappa^2)^{-1}$. To this end we need some results on special functions. We denote by $j_\ell(z)$ the spherical Bessel functions of the first kind, and by $h_\ell^{(1)}(z)$ the spherical Bessel functions of the third kind. We follow the notation and normalizations given in [1, Section 10.1]. We then define

$$u_\ell(z) = zj_\ell(z), \quad w_\ell(z) = izh_\ell^{(1)}(z).$$

We need the expansions of these two functions around zero. Using [1, (9.1.10),(10.1.1)], we get after some simplifications,

$$u_\ell(z) = z^{\ell+1}2^\ell \sum_{k=0}^\infty \frac{(-1)^k(k+\ell)!}{k!(2(k+\ell)+1)!}z^{2k} \tag{4.48}$$

For the function $w_\ell(z)$ we change the variable to get a simplified expression. Using [1, (10.1.16)], we get, computing as in [9],

$$w_\ell(i\zeta) = i^{-\ell}\zeta^{-\ell}\sum_{n=0}^\infty d_n\zeta^n, \tag{4.49}$$

$$d_n = (-1)^{n-1}\sum_{\substack{k=0\\ k\geq \ell-n}}^{\ell} \frac{(\ell+k)!(-2)^{-k}}{k!(\ell-k)!}\frac{1}{(n-\ell+k)!}. \tag{4.50}$$

We recall from [9] the following result on the expansion coefficients of $h_1^{(1)}$. Note that we have not made the ℓ-dependence in d_n explicit, in order to avoid a complicated notation.

Lemma 4.7. *The coefficients* (4.50) *have the following property*

$$d_n = 0 \quad \text{for } n = 1, 3, \ldots, 2\ell - 1. \tag{4.51}$$

We now recall (see any standard text, for example [2, 18]) that the kernel of the resolvent is given as

$$(H_{0,\ell} + \kappa^2)^{-1}(r, r') = -\frac{i}{\kappa} u_\ell(i\kappa r_<) w_\ell(i\kappa r_>). \tag{4.52}$$

Here we have introduced the standard notation

$$r_> = \max\{r, r'\}, \quad r_< = \min\{r, r'\}. \tag{4.53}$$

The expansion results for u_ℓ and w_ℓ then lead to asymptotic expansions for the resolvents. We keep the same notation as in the previous subsection, so we introduce the weight function $\rho_\gamma(r) = \langle r \rangle^{-\gamma/2}$, now for $r \in \mathbf{R}_+$.

Proposition 4.8. *Assume that* $\gamma > 2p + 3$. *We then have an expansion*

$$\rho_\gamma (H_{0,\ell} + \kappa^2)^{-1} \rho_\gamma = \sum_{j=0}^{p-1} \kappa^j \widetilde{G}_j + \kappa^p r_p(\kappa). \tag{4.54}$$

Here the expansion coefficients are bounded operators on $\mathcal{H}$, *and the error term* $r_p(\kappa)$ *is uniformly bounded for* κ *small. We have*

$$\widetilde{G}_j = 0, \quad j = 1, 3, \ldots, 2\ell - 1. \tag{4.55}$$

We have the following integral kernel expressions (*assuming* $\gamma > 2\ell + 5$)

$$\widetilde{G}_0(r, r') = \rho_\gamma(r) \frac{(r_<)^{\ell+1}(r_>)^{-\ell}}{2\ell + 1} \rho_\gamma(r'), \tag{4.56}$$

$$\widetilde{G}_2(r, r') = \rho_\gamma(r) \frac{(r_<)^{\ell+1}(r_>)^{-\ell}}{2\ell + 1} \left[-\frac{1}{2}(r_<)^2 + \frac{\ell}{2\ell - 1}(r_>)^2 \right] \rho_\gamma(r'), \tag{4.57}$$

$$\widetilde{G}_{2\ell+1}(r, r') = 2^\ell \frac{\ell! d_{2\ell+1}}{(2\ell + 1)!} \rho_\gamma(r)(-r \cdot r')^{\ell+1} \rho_\gamma(r'). \tag{4.58}$$

Proof. The results (4.48), (4.49), and (4.52) yield, after some computations, the existence of an asymptotic expansion of the form given in (4.54). The result (4.55) is a consequence of Lemma 4.7, since the expansion of $z^{-\ell-1} u_\ell(z)$ only contains even powers of z. The kernel expressions follow after some tedious computations, which we omit. In the expression for (4.58) we used the relation $r_< \cdot r_> = r \cdot r'$. □

We can now describe our results. We consider the two channel set-up, where we now take the Hilbert space $\mathcal{H} = L^2(\mathbf{R}_+) \oplus \mathbf{C}$, and replace (4.5) by

$$H = \begin{bmatrix} H_{0,\ell} & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.59}$$

Theorem 4.9 (Two channel case). *Consider the two channel case with H given by* (4.59)*. Assume that W given by* (4.6) *satisfies* (4.8) *with $\gamma > 2\ell+5$. Assume that $\langle W_{12}, r^{\ell+1}\rangle \neq 0$. Then we have $\nu = 2\ell + 1$ and*

$$g_\nu = (-1)^{\ell+1} \left[\frac{\sqrt{\pi}}{2^{\ell+1}\mathbf{\Gamma}(\ell + \frac{3}{2})} \right]^2 |\langle W_{12}, r^{\ell+1}\rangle|^2, \tag{4.60}$$

where $\mathbf{\Gamma}$ denotes the usual Gamma function.

Proof. We insert the expansion coefficients into (3.11), and after some simple computations, the result follows. A computer algebra computation using (4.50) yields the closed form of the coefficient, given in the theorem. □

Remark 4.10. The above result should be compared with the results in [3]. Here the same Hamiltonian is investigated using analytic continuation of the resolvent. The expression in (4.60) agrees with the one in [3].

Remark 4.11. One can also consider the operator

$$H = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + V(r), \tag{4.61}$$

where V decays sufficiently rapidly at infinity. The analysis of the threshold can be carried out along the same lines as above. In this case one can get a simple eigenvalue at the threshold for suitable V. The detailed analysis shows that also in the one channel case one can get examples, where (A5) is satisfied with ν arbitrarily large.

Remark 4.12. The results obtained here are similar to those in [9] for $-\Delta + V$ on $L^2(\mathbf{R}^m)$, $m \geq 5$ and odd. The free Schrödinger operator has an asymptotic expansion with coefficients having the same properties as above. The link between the two cases is given by $m = 2\ell + 3$. One could use the results in [9] to get one channel examples similar to those mentioned in the previous remark.

5 Further results

In this short section we list a few possible straightforward generalizations of the results obtained above.

(i) More examples, e.g. the one channel case in one dimension with nonlocal interactions, and both a bound state and a resonance at the threshold, higher dimensions etc. The only problem is that the computations are more tedious.

(ii) Even dimensions. Although there are no basic difficulties in extending the theory developed in Section 3, one has to cope with the more complex asymptotic expansions for the resolvents [9, 11].

(iii) Degenerate case, i.e the case when 0 is a m-fold degenerate eigenvalue, $m < \infty$. If all the eigenvalues $b_1 < b_2 < \cdots < b_m$ of P_0WP_0 on $P_0\mathcal{H}$ are strictly positive and nondegenerate, one can apply the method in Section 3 to each of them by replacing I_ε (see (3.16)) with $I_{j,\varepsilon} = (\varepsilon(b_j - a), \varepsilon(b_j + a))$, where

$$a = \frac{1}{2}\min_{j\neq k}\{b_1, |b_j - b_k|\}.$$

(iv) The case where the eigenvalue at zero is embedded in the continuum, while still being a threshold eigenvalue. This is most easily realized in a two channel model with both channels open. Here the asymptotics of the resonance may be different from the one found in Theorem 3.7.

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