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Deadlocks and Dihomotopy in Mutual Exclusion Models

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1 Introduction: Mutual Exclusion Models

Already back in 1968, E.W. Dijkstra [Dij68] proposed to apply a geometric point of view in the consideration of coordination situations in *concurrency*. His *progress graphs* were at the basis of the *Higher Dimensional Automata* (HDA) introduced by V. Pratt[Pra91] and developed in the thesis of É. Goubault[Gou95] and in later research (cf. [FGR99]).

In this article, we stick to Dijkstra's simple continuous geometric model. A system of n concurrent processes will be represented as a subset of Euclidean space $\mathbf{R}^n$ with the usual partial order. Each coordinate axis corresponds to one of the processes performing a linear programme[1]; a state of the system is a point in $\mathbf{R}^n$ with its ith coordinate describing "local time" of the ith processor. A run of a concurrent program is modelled by a *continuous increasing* path – time increases for every participating processor – between two states.

Shared resources can often only be used by one or a limited number of processors at the same time. As a consequence, certain *hyperrectangles* – corresponding to conflict in the access to such a resource – have to be removed from the model; together, they form the *forbidden region*.

The resulting *mutual exclusion models* are more general than those modelling *semaphore* programs. They allow us to consider also k-semaphores, where a shared object may be accessed by k, but not by $k+1$ processors.

To get more formal, let $I=[0,1]$ denote the unit interval, and let $I^n \subset \mathbf{R}^n$ denote the unit hypercube. An (open) isothetic hyperrectangle is a subset

$$R = (a_1, b_1) \times \cdots \times (a_n, b_n) \subset I^n;$$

closed or half-open coordinate intervals are exceptionally allowed at the boundaries in the forms $[0,b)$, $(a,1]$, resp. $[0,1]$. The *forbidden region* $F = \bigcup_1^r R^i$ is then a finite union of n-hyperrectangles $R^i = (a_1^i, b_1^i) \times \cdots \times (a_n^i, b_n^i)$, and the *state space* is its complement $X = I^n \setminus F$. We assume that $\mathbf{0} = (0,\dots,0)$ and $\mathbf{1} = (1,\dots,1)$ are *not* contained in the forbidden region F; they represent the intitial, resp. the final state of the concurrent programme.

[1] The methods can be adapted to more general progams by replacing an axis by a graph and the state space by a product of graphs, cf. [FS00].

We address two questions in this article: How can one use the geometric/combinatorial description of the forbidden region to

1. detect *deadlocks* and associated *unsafe*, resp. *unreachable* regions? We give a survey of the results obtained in [FGR98] in Sect. 2 as a background for the following:
2. obtain information on the number of "essentially different" schedules between two states? These results are new and will be developed and explained in Sect. 4 and Sect. 5.

2 Deadlock Detection in Mutual Exclusion Models

The "Swiss flag" example from Fig. 1 below (the forbidden region is dashed) conveys the idea, that deadlocks – with no possible legal move – in such mutual exclusion models are associated to n-dimensional "lower corners" below the forbidden region.

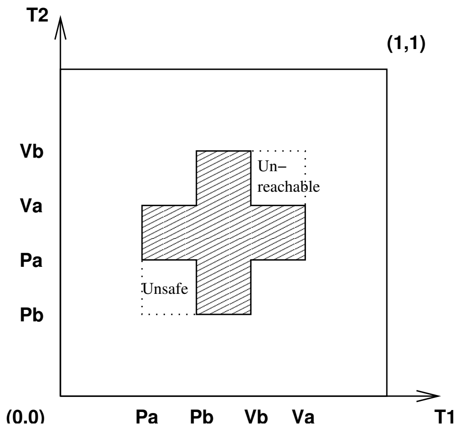

Fig. 1. "Swiss flag"

To make this formal, we call a continuous path[2] $\alpha : \boldsymbol{I} \to X \subset \boldsymbol{I}^n$ from $\mathbf{x} = \alpha(0)$ to $\mathbf{y} = \alpha(1)$ a *dipath* (directed path) if every composition $pr_i \circ \alpha$ is increasing. We introduce a new partial order $\preceq$ on X by

$$\mathbf{x} \preceq \mathbf{y} \Leftrightarrow \text{ there is a dipath } \alpha \text{ from } \mathbf{x} \text{ to } \mathbf{y} \text{ in } X.$$

As can be seen e.g. in Fig. 1, this partial order is in general finer than the one X inherits from the usual partial order on $\mathbf{R}^n$.

[2] We distinguish between the interval $\boldsymbol{I}$ with the *usual* order as (partial) order and the interval I neglecting order (or rather, with equality as the partial order relation.

An element $\mathbf{x} \in X$ is called *admissible* if $\mathbf{x} \preceq \mathbf{1}$ and *unsafe* else. An element $\mathbf{y} \in X$ is called *reachable* if $\mathbf{0} \preceq y$ and *unreachable* else. An element $\mathbf{1} \neq \mathbf{x} \in X$ is called a *deadlock* if $\mathbf{x} \preceq \mathbf{y} \Rightarrow \mathbf{y} = \mathbf{x}$; cf. Fig. 1.

To formulate results, we need to introduce k-element intersections of the hyperrectangles R_i forming part of the forbidden region $F = \bigcup_1^r R^i$, cf. Sect. 1: For any k-element index set $J = \{i_1, \dots, i_k\} \subset \{1, \dots, r\}$, $k > 0$, let $R^J = R^{i_1} \cap \cdots \cap R^{i_k}$. Unless $R^J = \emptyset$, it is again a hyperrectangle $R^J = (a_1^J, b_1^J) \times \cdots \times (a_n^J, b_n^J)$ with $a_j^J = \max\{a_j^i \mid i \in J\}$ and $b_j^J = \min\{b_j^i \mid i \in J\}$. The minimal vertex of R^J is given by $\mathbf{a}^J = (a_1^J, \dots, a_n^J)$. Moreover, let $\tilde{a}_j^J$ denote the "second largest" of the j-th coordinates a_j^i; we need also consider the "unsafe corner" $Us^J =]\tilde{\mathbf{a}}^J, \mathbf{a}^J] =]\tilde{a}_1^J, a_1^J] \times \cdots]\tilde{a}_n^J, a_n^J] \subset X$.

Proposition 1. *1. An element $\mathbf{1} \neq \mathbf{x} \in X$ in the interior of $\boldsymbol{I}^n$ is a deadlock if and only if there is an n-element index set $J = \{i_1, \dots, i_n\}$ with $R^J \neq \emptyset$ and $\mathbf{x} = \mathbf{a}^J = \min R^J$.*

2. If $\mathbf{x} = \mathbf{a}^J = \min R^J$ is a deadlock, then all elements of the n-hyperrectangle Us^J are unsafe.

Remark 1. 1. In a similar way, one can find an "unreachable corner" Ur^J "above" the maximal element of an n-intersection R^J.

2. A simple trick allows to detect deadlock points that are contained in the boundary of I^n as well; cf. [FGR98] and also Sect. 5.3.

In [FGR98], we describe a fast incremental algorithm, that detects the *entire* unsafe region (consisting of *all* unsafe elements in X) in few steps – usually, many (discrete) states are detected in one single step. One has to take into account the (order) combinatorics of intersections of forbidden hyperrectangles and of those hyperrectangles that have found to be unsafe in previous steps. An implementation of this algorithm can be found on the URL `http://www.ens.fr/~goubault`.

3 The Dihomotopy Concept

An execution of a concurrent proces corresponds to a dipath (cf. Sect. 2) in the state space X. The most interesting dipaths are those starting at $\mathbf{0}$ and terminating at $\mathbf{1}$ (a complete run), but also dipaths starting and/or terminating at other elements need to be considered; both for practical purposes in state space analysis and as intermediate steps in theoretical calculations.

Many executions will "automatically" be equivalent; this means that all conceivable concurrent calculations along the corresponding schedules/paths yield the same result. In geometric language, this is the case when the dipaths corresponding to the executions are *dihomotopic*, cf. [FGR99] Dihomotopy is a modification of the notion *homotopy* – which is fundamental and well-studied in Algebraic Topology. With dihomotopy we take into account not only continuity but also partial order. There are several definitions for dihomotopy, all of which are equivalent in the case of our simple partially ordered state space; cf.Prop. 2, or in greater generality [Faj03]. We need to work with three of these definitions:

Definition 1. *A continuous 1-parameter deformation* (dihomotopy) $H : \boldsymbol{I} \times I \to X$ *with* $H(0,t) = \mathbf{x}, H(1,t) = \mathbf{y}$ *for all* $t \in I$ *is called*

1. *a* dihomotopy[FGR99] *if, for all* t*, the "interpolating" paths* $\alpha_s : t \mapsto H(s,t)$ *are* dipaths*.*
2. *an* elementary d-homotopy[Gra03] *if, for all* $s \in \boldsymbol{I}$ *and such that for all* s *and* t*, the "interpolating" paths* $\alpha_t : s \mapsto H(s,t)$ *and* $\alpha_s : t \mapsto H(s,t)$ *are* dipaths*.*

Two continuous dipaths $\alpha, \beta : \boldsymbol{I} \to X$ *from* $\mathbf{x} \in X$ *to* $\mathbf{y} \in X$ *are called*

1. dihomotopic[FGR99] *if there exists a dihomotopy* H *with* $H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$ *for all* $s \in \boldsymbol{I}$*.*
2. d-homotopic[Gra03] *if there exist dipaths* $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_{2s} = \beta$ *and elementary d-homotopies from* α_{2k} *to* α_{2k+1} *and from* α_{2k+2} *to* α_{2k+1} *– i.e., a "zig-zag homotopy" between* α *and* β*.*

Remark 2. Both notions are defined in far more general situations for maps between locally partially ordered spaces (dihomotopy), resp. for maps between d-spaces (d-homotopy). The latter notion is preferable for homotopy theoretic purposes.

Only the order requirement for the interpolating paths is characteristic for a di/d-homotopy compared to a homotopy (with fixed ends). Examples (cf. [FGR99] or Example 1 below) show, that dihomotopy in general is a finer relation than homotopy of dipaths. Moreover, it is important to notice that, unlike for homotopy, dihomotopy in general does *not* satify a cancellation property: $\alpha * \beta_1$ dihomotopic to $\alpha * \beta_2$ does not always imply that β_1 is dihomotopic to β_2. Examples for non-cancellation are given in [FGR99]; it also occurs in Example 1 below.

In the case of the state space of a mutual exclusion model (more generally, for a cubical complex), one may restrict attention to dipaths on the *1-skeleton* of $X \subseteq \boldsymbol{I}^n$ and to *combinatorial* dihomotopies [FGR99]. To explain these notions in our simple case, one considers the projections of *all* hyperrectangles within the forbidden region to the coordinate axes. This gives rise to a subdivision of the axes $[0,1]$ into subintervals – at requests for shared resources or terminations of such. The 1-skeleton corresponding to that subdivision consists of the *line sections* parallel to one of the axes and constant at one of the subdivision points for all other directions. A (locally serial) dipath along this 1-skeleton proceeds at every time along one of these line sections. An *elementary* dipath proceeding with "unit speed and one step" parallel to the x_i-axis will be denoted σ_i – the i-th process proceeds one step forward while the others wait. (This notation is not unambiguous, but good enough for our purposes). Two such elementary dipaths σ_i, σ_j can be concatenated to yield the dipath $\sigma_i * \sigma_j$ if the target of the first agrees with the source of the second.

Definition 2. *1. Two dipaths* $\alpha = \sigma_i * \sigma_j$ *and* $\beta = \sigma_j * \sigma_i$ *in* X *with the same source* $\mathbf{x}$ *and target* $\mathbf{y}$ *are called* elementarily dihomotopic *if the 2-dimensional rectangle with lower vertex in* $\mathbf{x}$ *and upper vertex in* $\mathbf{y}$ *is contained in* X*.*

2. *The (combinatorial) dihomotopy relation is obtained from elementary dihomotopy as the closure under concatenation, reflexivity and transitivity. We write* $\alpha \overset{\sim}{\to} \beta$ *to denote that* α *(combinatorially) dihomotopic to* β.

More general definitions for combinatorial dihomotopy are given in [FGR99] and [Faj03].

An elementary dihomotopy (given by such a rectangle in the state space) reflects the fact that the result of the compound execution of σ_i and σ_j is independent of the order in which these are performed (even after possible subdivisions into smaller pieces).

Within the state space I^n – no mutual exclusion – any such skeletal dipath can be obtained from any other (in I^n) by permutations and thus by a succession of transpositions and hence elementary dihomotopies. In particular, any two dipaths with the same source and target are combinatorially dihomotopic in I^n. In a (smaller) state space $X \subset I^n$ however – with mutual exclusion, e.g., $X = I^n \setminus F$ – a chain of elementary dihomotpies within I^n might contain a particular elementary dihomotopy along a 2-dimensional rectangle that is *not* contained in the state space X although its boundary is. If this is the case *for all* such chains between two given dipaths, these two dipaths are *not* combinatorially dihomotopic in X.

In the “Swiss flag” example from Fig. 1 in Sect. 2, there are two dihomotopy classes of dipaths connecting $\mathbf{0}$ and $\mathbf{1}$. A complete classification algorithm for dipaths up to dihomotopy in *2*-dimensional models had previously been given in [Rau00]. It is the aim of this article to pave the way for a generalisation of those results to the general n-dimensional case.

In this article, we will allow ourselves to use whatever notion of dihomotopy, d-homotopy or combinatorial dihomotopy is most suitable for our purposes. We may do so because of the following result, which applies in particular to the geometric cubical complex $X = \boldsymbol{I}^n \setminus F$:

Proposition 2. ([Faj03], Thm. 5.1 and Thm. 5.6) *All three notions are equivalent in geometric cubical complexes.*

4 Dihomotopy and Deadlocks in Mutual Exclusion Models

The purpose of this section is to make a link between the detection of deadlocks and unsafe regions in mutual exclusion models and the occurence of non-dihomotopic dipaths in such models. It had been conjectured for a long time, that, just as n intersecting n-rectangles give rise to deadlocks, unsafe and unreachable regions, so should likewise $(n-1)$ intersecting n-rectangles give rise to non-trivial non-local dihomotopy.[3] We discuss here when and why this in fact is the case.

[3] Even *a single* n-rectangle in the forbidden region creates dihomotopy, but only between points that are “sufficiently close” to that n-rectangle, cf. the discussion in dimension 3 in [FGHR04].

Forgetting about the last coordinate (processor) amounts to projecting the forbidden hyperrectangles and the forbidden region under $\pi : \mathbf{R}^n \to \mathbf{R}^{n-1}$, $\mathbf{x} = (x_1, \ldots, x_n) \mapsto \pi\mathbf{x} = (x_1, \ldots, x_{n-1})$, arriving at a forbidden region $\bar{F} = \pi(F)$ and a state space $\bar{X} = \boldsymbol{I}^{n-1} \setminus \bar{F}$ (different from $\pi(X)$, in general!).

Let us compare the forbidden regions in X and in $\bar{X}$: Consider an $(n-1)$-element index set J with non-empty intersection hyperrectangle $R^J \subset F$. If the participating hyperrectangles intersect *generically* – in particular, if $R^J \neq R^K$ for every smaller index set $K \subset J$ – then every of the $(n-1)$ hyperrectangles R_i will "contribute" at least one coordinate to the minimum $\mathbf{a}^J = [a_1^J, \ldots, a_n^J]$ of R^J – and similarly to its maximum $\mathbf{b}^J$. We may then suppose without restriction, that

$$a_1^J = a_1^1, \ldots, a_{n-2}^J = a_{n-2}^{n-2}, a_{n-1}^J = a_{n-1}^{n-1}, a_n^J = a_n^{n-1}.$$

The $(n-1)$ hyperrectangles $\pi(R^i)$ in $\boldsymbol{I}^{n-1}$ intersect in $\pi(R^J) = \pi(R)^J$ – for short πR^J – a hyperrectangle with minimal vertex $\pi\mathbf{a}^J = (a_1^J, \ldots, a_{n-1}^J)$, which is a *deadlock* for the model space $\bar{X}$. The intersection $\pi R^J = [\pi\mathbf{a}^J, \pi\mathbf{b}^J]$ gives furthermore rise to an unsafe region $Us(\pi R^J) =]\pi\tilde{\mathbf{a}}^J, \pi\mathbf{a}^J] \subset \bar{X}$. As in Sect. 2, the point $\tilde{\mathbf{a}}^J$ has the "second" largest coordinates among the a_j^i as its coordinates.

In a similar way, we can consider the projection $\pi' : \mathbf{R}^n \to \mathbf{R}^{n-1}, (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-2}, x_n)$, giving rise to the deadlock $\pi'\mathbf{a}^J = (a_1^J, \ldots, a_{n-2}^J, a_n^J)$ and the unsafe region $Us(\pi' R^J) \subset \boldsymbol{I}^{n-1} \setminus \pi' F$.

Lemma 1. 1. *Let $\mathbf{x}, \mathbf{y} \in X$ satisfy*

$$\pi\mathbf{x} \in Us(\pi R^J) \text{ or } \pi'\mathbf{x} \in Us(\pi' R^J), \mathbf{x} \preceq \mathbf{a}^J, \mathbf{b}^J \preceq \mathbf{y}.$$

A dipath $\alpha = (\alpha_1, \ldots, \alpha_n)$ from $\mathbf{x}$ to $\mathbf{y}$ satisfies either
(P1) $\alpha_n(t) \leq b_n \Rightarrow \pi\alpha(t) \in Us(\pi R^J)$ *or*
(P2) $\alpha_n(t) > a_n \Rightarrow \alpha^n(t) \notin Us(\pi R^J)$.
Two dihomotopic dipaths satisfy either both **(P1)** *or* both **(P2)**.

2. *Let $\mathbf{u}, \mathbf{v} \in X$ chosen such that*

$$\pi\mathbf{v} \in Ur(\pi R^J) \text{ or } \pi'\mathbf{v} \in Ur(\pi R^J), \mathbf{u} \preceq \mathbf{a}^J, \mathbf{b} \preceq \mathbf{v}^J.$$

A dipath $\beta = (\beta_1, \ldots, \beta_n)$ from $\mathbf{u}$ to $\mathbf{v}$ satisfies either
(P3) $\beta_n(t) \geq a_n \Rightarrow \pi\beta(t) \in Ur(\pi R^J)$ *or*
(P4) $\beta_n(t) < b_n \Rightarrow \pi\beta(t) \notin Ur(\pi R^J)$.
Two dihomotopic dipaths satisfy either both **(P3)** *or* both **(P4)**.

Corollary 1. *Two dipaths α, β from $\mathbf{x}$ to $\mathbf{y}$ with α satisfying (P1) and β satisfying (P2),* cannot *be dihomotopic.* □

An instructive example is given by two dipaths α, β from $\mathbf{a}^J$ to $\mathbf{b}^J$: While all the other coordinates remain fixed, we let α_{n-1} grow from a_{n-1}^J to b_{n-1}^J before α_n grows from a_n to b_n; for β, the nth coordinate grows before the $(n-1)$st. Remark for later use that there are no *upward* restrictions for the *end* point $\mathbf{y}$.

Proof. of Lemma 1

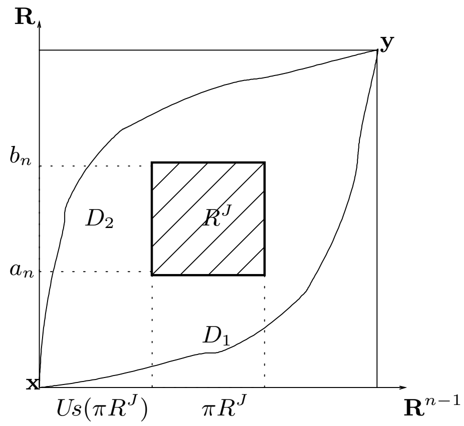

Fig. 2. Non-dihomotopic dipaths

1. The crucial property is:

$$\pi R^J \times]a_n, b_n[) \subset F.$$

   A dipath in X from $\mathbf{x}$ to $\mathbf{y}$ has thus to pass through

   $D = D_1 \cup D_2 = (\pi R^J \times [x_n, a_n]) \cup (Us(\pi R^J) \times]a_n + \varepsilon, b_n - \varepsilon[)$ for a small $\varepsilon > 0$,

   since adding D to F disconnects $\mathbf{x}$ from $\mathbf{y}$. Since D_1 and D_2 are *disconnected* from each other, any dipath from $\mathbf{x}$ to $\mathbf{y}$ has to pass through one and only one of those sets. There cannot be a dihomotopy between a dipath intersecting the first and a dipath intersecting the second, since this would yield a division of the connected parameter interval I into two open non-empty sets, cf. [Rau00].
2. is proved by a symmetric argument.

A *single* arrangement of $(n-1)$ intersecting hyperrectangles will in general *not* lead to non-dihomotopic dipaths from $\mathbf{0}$ to $\mathbf{1}$. This can be seen e.g. for the state space with a *single* wedge (cf. Example 1 below) as the forbidden region. We have to consider (at least) *two disjoint* arrangements J, K consisting of $(n-1)$ intersecting n-rectangles each within the forbidden region F; as usual, $X = \boldsymbol{I}^n \setminus F$. The two intersection n-rectangles and their projections will be called

$$R^J = [\mathbf{a}^J, \mathbf{b}^J], R^K = [\mathbf{a}^K, \mathbf{b}^K], \pi R^J = [\pi\mathbf{a}^J, \pi\mathbf{b}^J], \pi R^K = [\pi\mathbf{a}^K, \pi\mathbf{b}^K]$$

with unsafe, resp. unreachable regions

$$Us(\pi R^K) =]\pi\tilde{\mathbf{a}}^K, \pi\mathbf{a}^K], Ur(\pi R^J) = [\pi\mathbf{b}^J, \pi\tilde{\mathbf{b}}^J[.$$

We suppose that $a_n^J < b_n^K$.

A dipath $\alpha = (\alpha_1, \ldots, \alpha_n) : \boldsymbol{I} \to X$ from $\mathbf{0}$ to $\mathbf{1}$ is called *inter-JK* if it satisfies

$$a_n^J < \alpha_n(t) < b_n^K \Rightarrow \pi\mathbf{b}^J < \pi\alpha(t) < \pi\mathbf{a}^K. \tag{1}$$

– where the <-relation on the right-hand side is understood for all $n-1$ coordinates.

Proposition 3. *Let $F = \bigcup_{i=1}^r R_i \subset \boldsymbol{I}^n$ denote he forbidden region. Let $J, K \subset \{1, \ldots k\}$ denote two disjoint subsets indexing $(n-1)$ intersecting hyperrectangles $R_i$each and such that $\pi\tilde{\mathbf{a}}^K < \pi\mathbf{b}^J < \pi\mathbf{a}^K < \pi\tilde{\mathbf{b}}^J$. Any dipath $\beta : \boldsymbol{I} \to X$ from $\mathbf{0}$ to $\mathbf{1}$ that is dihomotopic to an inter-JK-dipath α is then an inter-JK-dipath itself.*

Corollary 2. *Under the assumptions of Prop. 3, a dipath in X from $\mathbf{0}$ to $\mathbf{1}$ that is not inter-JK, e.g., a dipath on the boundary of $\boldsymbol{I}^n$, is* not *dihomotopic to an inter-JK-dipath. In particular, if there exist both an inter-JK-dipath from $\mathbf{0}$ to $\mathbf{1}$ and another one that is not inter-JK, then these two are not dihomotopic to each other.* □

Remark 3. 1. From an application point ov view, Cor. 2 implies the existence of different terminating schedules that can possibly yield different results of distributed calculations according to different schedules.
2. Cor. 2 applies in particular to the example of a 3-dimensional PV-programme given in [CR87].

Example 1. The situation from Prop. 3 arises in three dimensions, when the forbidden region is a *cylinder* (with a "thick" rectangle as cross-section). More strikingly, there are state spaces with *trivial fundamental group*, that allow non-dihomotopic dipaths: It suffices to consider a forbidden region consisting of two "wedges", one behind the other and not connected to each other; one of them yields a deadlock after projection (to the "front") and the other unreachable points; cf. Fig. 3 below. A dipath (from lower left to upper right) through the area between the wedges is homotopic (relative to the end points) but not dihomotopic to a dipath avoiding it.

Proof. of Prop. 3. By Prop. 2, it is enough to show that α and β are not d-homotopic. To this end, we apply Marco Grandis' van Kampen theorem [Gra03], Thm. 3.6., to the decomposition $X = X_1 \cup X_2$ with $X_1 = X \cap (\boldsymbol{I}^{n-1} \times [0, b_n^K[)$, $X_2 = X \cap (\boldsymbol{I}^{n-1} \times]a_n^J, 1])$. Given a decomposition of the dipath $\alpha = \alpha^1 * \alpha^2$ with α_i in X_i and division point $\mathbf{u} = \alpha^1(1) = \alpha^2(0) \in [\pi\mathbf{b}^J, \pi\mathbf{a}^K] \times]a_n^J, b_n^K[$ by assumption. It is sufficient to prove, for β_i dihomotopic to α_i *in* X_i (with fixed end points $\mathbf{0}$ and $\mathbf{u}$, resp. $\mathbf{u}$ and $\mathbf{1}$), that $\beta = \beta^1 * \beta^2$ is an inter-JK dipath, as well.

The assumption of Prop. 3 has the following consequence:

$$Ur(\pi(R)^J) \cap Us(\pi(R)^K) = [\pi\mathbf{b}^J, \pi\mathbf{a}^K] = Ur(\pi(R)^J) \cap \downarrow \pi\mathbf{a}^K = Us(\pi(R)^K) \cap \uparrow \pi\mathbf{b}^J. \tag{2}$$

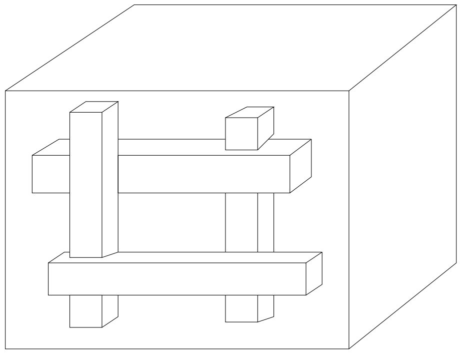

Fig. 3. Two wedges

Hence, $\alpha_n^1(t) > a_n^J$ implies $\pi\alpha(t) \in Ur(\pi(R)^J) \cap Us(\pi(R)^K)$. From Lemma 1, we may conclude, that $\beta_n^1(t) > a_n^J$ implies $\pi\beta^1(t) \in Ur(\pi(R)^J)$, as well. Moreover, $\pi\beta^1(t) \leq \pi\mathbf{u} \leq \pi\mathbf{a}^K$ for all t, and we can conclude from (2): $\pi\beta^1(t) \in [\pi\mathbf{b}^J, \pi\mathbf{a}^K]$. In the same way, it can be shown that $\beta_n^2(t) < b_n^K$ implies $\pi\beta^2(t) \in [\pi\mathbf{b}^J, \pi\mathbf{a}^K]$.

5 Trivial dihomotopy for models with less complicated constraints

In contrast, for a model space with a less complicated forbidden region, we can show by a simple essentially combinatorial argument and using the characterisation of dihomotopy from Def. 2:

Proposition 4. *For a model space X with the property that $R^J = \emptyset$ for all index sets J of cardinality $n-1$,* every *two dipaths from* $\mathbf{0}$ *to* $\mathbf{1}$ *are dihomotopic to each other.*

Remark 4. 1. A similar result holds also in the classical non-directed case: Using duality and Čech-type cohomology, it is easy to see that the complement of a forbidden region with $R^J = \emptyset$ for all index sets J of cardinality $n-1$ has a trivial first *homology* group.
2. From an application point of view, the criterion from Prop. 4 is easy to check and ensures that *all* runs in such a distributed calculation will yield the *same* result. This should also be interesting for data base scheduling; compare [Gun94] and [FGR99], Sect. 8. On the other hand, it cannot be applied for traditional semaphore scheduling, since those will always generate lots of intersections. One would have to restrict to k-semaphores with $k > 1$.

5.1 Local futures

Given a point $\mathbf{x} \in X = \boldsymbol{I}^n \setminus F$ and $F = \bigcup_{i=1}^n R_i$ intersecting in general position. We describe the local future $\uparrow_l \mathbf{x}$ of such a point, i.e., the intersection of a small cube with lower vertex $\mathbf{x}$ in I^n within the state space X (Reversing inequalities yields similar results for the local past $\downarrow_l \mathbf{x}$ of $\mathbf{x}$):

For a hyperrectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$, let

$$\partial_-^j R = \{\mathbf{x} = (x_1, \dots, x_n) \mid x_j \leq b_j \wedge \exists 1 \leq i_1 < \cdots < i_j \leq n : x_{i_k} = a_{i_k}\},$$

i.e., the intersection of j of its lower faces. In particular, the lower boundary of a standard s-cube $\mathbf{I}^S$ is $\partial_- \mathbf{I}^s = \partial_-^1 \mathbf{I}^s = \{(a_1, \dots, a_s) \mid \exists i : a_i = 0\}$.

Lemma 2. *Assume* $\mathbf{x} \in X$ *is contained in* $\partial_-^{j_1} R_1 \cap \cdots \cap \partial_-^{j_k} R_k$ *– with* $k \geq 0$ *and* $j_s \geq 1$ *maximal – and* $j := j_1 + \cdots + j_k \leq n$. *Then* $\uparrow_l \mathbf{x}$ *is dihomeomorphic to* $\partial_- \mathbf{I}^{j_1} \times \cdots \times \partial_- \mathbf{I}^{j_k} \times \mathbf{I}^{n-j}$.

(In the case that $\mathbf{x}$ is *not* contained in *lower boundary* $\partial_- R$ of any forbidden hyperrectangle R, then $j = 0$ and $\uparrow_l \mathbf{x}$ is dihomeomorphic to $\mathbf{I}^n$).

Proof. An element $\mathbf{y} \in \mathbf{I}^n$ with $\mathbf{x} \leq \mathbf{y}$ close to $\mathbf{x}$ is contained in the state space X if and only if $\mathbf{y} \notin (R_1 \cup \cdots \cup R_k)$, i.e., if and only if at least one of the j_i "critical" coordinates in $\partial_-^{j_i} R_i$ of $\mathbf{y}$ conincides with the respective coordinate of $\mathbf{x}$ – which gives rise to a factor homeomorphic to $\partial_- \mathbf{I}^{j_i}$.

Remark 5. Another way to phrase Lemma 2 is, that the local building blocks of a mutual exclusion state space are of type $\partial_- \mathbf{I}^{j_1} \times \cdots \times \partial_- \mathbf{I}^{j_k} \times \mathbf{I}^{n-j}$. These simple ingredients can thus be seen as the building blocks of the state space for any mutual exclusion model. This should be of independent interest!

Lemma 3. *Let* σ_j, σ_k *denote two elementary dipaths (cf. Sect. 3 starting at* $\mathbf{x}$ *– as above. There is a elementary dipath* σ_l *commuting with both* σ_k *and* σ_l *up to dihomotopy if at least one of the following conditions is satisfied:*

1. $j := j_1 + \cdots + j_k < n$*;*
2. $j_s \geq 3$ *for at least one index* s*;*
3. $j_{s_1}, j_{s_2} \geq 2$ *for two different indices* s_1, s_2.

Example 2. In the case $n = 2, j_1 = 2$, the local future of such a point x is of the form $\partial_-^2 \mathbf{I}^2$, which is a 1-dimensional wedge (like a letter L). In this case the two dipaths σ_1, σ_2 along the legs of the wedge do *not* commute locally up to dihomotopy. In the case $n = 3, j_1 = 2$, the local future of a point is the product of a wedge and an interval. In this case, the dipaths σ_1, σ_2 commute both with the dipath σ_3 in the third direction.

Proof. By Lemma 2, the local future $\uparrow_l \mathbf{x}$ is of the from $\partial_- \mathbf{I}^{j_1} \times \cdots \times \partial_- \mathbf{I}^{j_k} \times \mathbf{I}^{n-j}$. Remark that a factor $\partial_- \mathbf{I}^1$ consists of a single element and thus has no effect on the product.

1. Let σ_l denote a elementary dipath within $\mathbf{I}^{n-j}$. Then, for $j \neq l$, the rectangle "spanned" by σ_j and σ_l is contained in the state space X. In particular, σ_l commutes with σ_j.

2. For given indices j, k choose an index $j \neq l \neq k$ referring to one of the axes in the cube $\mathbf{I}^3$ the lower boundary $\partial_-\mathbf{I}^3$ of which is a factor in $\uparrow_l x$. Same argument as in 1. above.
3. The only case not yet covered is a product including a factor $\partial_-\mathbf{I}^2 \times \partial_-\mathbf{I}^2$. Two dipaths in *different* factors of this product span a rectangle in X and thus do commute with each other.

Lemma 4. *The conditions of Prop. 4 ensure that Lemma 3 is applicable.*

Proof. By assumption $j := j_1 + \cdots + j_k \leq n$ with $k \leq n-2$. Hence $j < n$ or $j_i \geq 3$ for at least one i or $j_{i_1} = j_{i_2} = 2$ for at least two indices i_1 and i_2.

5.2 Proof of Prop.4

As explained in Sect. 3, within the state space $X \subset \boldsymbol{I}^n$ we need only consider dipaths of the form $\sigma = \sigma_{i_1} * \cdots * \sigma_{i_N}, 1 \leq i_j \leq n$ from $\mathbf{0}$ to $\mathbf{1}$. By Prop. 2, we are done if we can show that all those are *combinatorially* dihomotopic.

Proof. The proof is by induction on the length l of dipaths ending at $\mathbf{1}$ – and thus starting at an (arbitrary) element $\mathbf{x}$ at "taxi cab distance" l from $\mathbf{1}$. For $l = 0$ and $l = 1$, there is nothing to prove. Assume inductively that, for all elements $\mathbf{x}$ at distance k from $\mathbf{1}$, all dipaths starting at $\mathbf{x}$ and ending at $\mathbf{1}$ are combinatorially dihomotopic to each other.

Let $\mathbf{y}$ denote a vertex of X at distance $k+1$ from $\mathbf{1}$ and let $\sigma = \sigma_{i_{n-k}} * \sigma_{i_{n-k-1}} * \cdots * \sigma_n =: \sigma_{i_{n-k}} * \bar{\sigma}$ and $\sigma' = \sigma'_{i_{n-k}} * \sigma'_{i_{n-k-1}} * \cdots * \sigma'_n =: \sigma'_{i_{n-k}} * \bar{\sigma'}$ denote two elementary dipaths from $\mathbf{y}$ to $\mathbf{1}$. By Lemma 3 and 4, there exists an elementary dipath σ_l with source $\mathbf{y}$ that commutes with both $\sigma_{i_{n-k}}$ and $\sigma'_{i_{n-k}}$.

We denote by $\mathbf{z}$ the target of $\sigma_{n-k} * \sigma_l \overset{\sim}{\to} \sigma_l * \sigma_{n-k}$. The condition of Prop. 4 assures also that the future $\uparrow \mathbf{z}$ of $\mathbf{z}$ is *deadlock-free* [FGR98]. In particular, there exists a dipath $\hat{\sigma}$ from $\mathbf{z}$ to $\mathbf{1}$. By induction, $\sigma_l * \hat{\sigma} \overset{\sim}{\to} \bar{\sigma}$. Likewise, $\bar{\sigma'}$ is dihomotopic to a dipath $\sigma_l * \hat{\sigma}'$. But then

$$\sigma = \sigma_{i_{n-k}} * \bar{\sigma} \overset{\sim}{\to} \sigma_{i_{n-k}} * \sigma_l * \hat{\sigma} \overset{\sim}{\to} \sigma_l * \sigma_{i_{n-k}} * \hat{\sigma} \overset{\sim}{\to} \sigma_l * \sigma'_{i_{n-k}} * \hat{\sigma}' \overset{\sim}{\to} \sigma'_{i_{n-k}} * \sigma_l * \hat{\sigma}' \overset{\sim}{\to} \sigma'_{i_{n-k}} * \bar{\sigma}';$$

the combinatorial dihomotopy in the middle exists by induction.

5.3 Dipaths up to dihomotopy between arbitrary points

As mentioned in the introduction to Sect. 4, Footnote 3, dihomotopy between intermediate states may be non-trivial although dihomotopy between the initial and the terminal state is trivial. A simple example for this phenomenon occurs for $X = \boldsymbol{I}^3 \setminus \mathbf{J}^3$ with $\mathbf{J} \subset \boldsymbol{I}$ an open subinterval: All dipaths in X from $\mathbf{0}$ to $\mathbf{1}$ are dihomotopic to each other, but there are two dihomotopy classes of dipaths between $\mathbf{x}$ and $\mathbf{y}$ in $(\boldsymbol{I} \setminus \mathbf{J}) \times (\boldsymbol{I} \setminus \mathbf{J}) \times \mathbf{J}$ whenever $x_1, x_2 \leq a \leq y_1, y_2$ for all $a \in \mathbf{J}$. This example is studied in detail in [FGHR04] which determines the *components of the fundamental category* of that state space X.

To study dipaths up to dihomotopy between $\mathbf{x}$ and $\mathbf{y}$ in $X = \boldsymbol{I}^n \setminus F$, we have to work with the state space $X_{\mathbf{xy}} = \{\mathbf{z} \in X \mid x_i \leq z_i \leq y_i,\ 1 \leq i \leq n\}$. Similar to the techniques in [FGR98] and [Rau00], it can be regarded as
$X_{\mathbf{xy}} = \boldsymbol{I}^n \setminus (F \cup F_{\mathbf{xy}})$ with

$$F_{\mathbf{xy}} = \bigcup_{1 \leq i \leq n} (\boldsymbol{I} \times \cdots \times \boldsymbol{I} \times [0, x_i[\times \boldsymbol{I} \times \cdots \times \boldsymbol{I}) \cup (\boldsymbol{I} \times \cdots \times \boldsymbol{I} \times]y_i, 1[\times \boldsymbol{I} \times \cdots \times \boldsymbol{I}).$$

This means that $2n$ additional (outer) hyperrectangles are added to the forbidden region.

The techniques from Sect. 4 and Sect. 5 apply. In particular, if $(n-1)$ of the rectangles in $F \cup F_{\mathbf{xy}}$ have a non-empty intersection (apart from the trivial intersections among the hyperrectangles in $F_{\mathbf{xy}}$), then Prop. 3 and Cor. 2 will in many cases show that there exist non-dihomotopic dipaths from $\mathbf{x}$ to $\mathbf{y}$. This applies e.g. to the example $X = \boldsymbol{I}^3 \setminus \boldsymbol{J}^3$ discussed above. If, on the other hand, all relevant intersections of $(n-1)$ hyperrectanles in $F \cup F_{\mathbf{xy}}$ are empty, then Prop. 4 shows that all dipaths from $\mathbf{x}$ to $\mathbf{y}$ are dihomotopic to each other.

6 Concluding Remarks. Future Work

The main results of [FGR98] as described in Prop. 1 and the end of Sect. 2 show that the "ordered combinatorics" of intersections of hyperrectangles in the forbidden region associated to a mutual exclusion model can be applied to yield a very efficient algorithm determining deadlocks, unsafe and unreachable regions for such a model space. We have modified these techniques to attack a more difficult problem, i.e., to determine the (number of) essentially different computation paths in such a model. The results indicate that the ordered combinatorics of intersections of hyperrectangles in the forbidden region (at one level lower) again will play a key role.

The ultimate goal for the work initiated in this paper is the construction of an algorithm determining the set of dihomotopy classes between two given states, building on the deadlock algorithm from [FGR98] and generalising the algorithm given in [Rau00] in the two-dimensional case. To this end, one has to investigate the "directed combinatorics" between situations as they arise in Prop. 3 more closely. Several unreachable and unsafe regions (associated to a projection of the forbidden region) can have an interplay that is not that easy to analyse, as you can see in Fig. 4 below. Moreover, one has to get to grips with situations where projections along various different axes have to be combined.

It should also be interesting to see how the components of the fundamental category of X from [FGHR04] relate to this approach.

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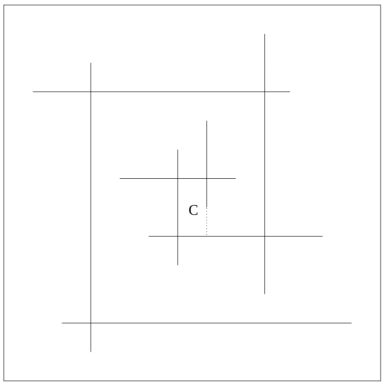

Fig. 4. A labyrinth state space

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