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by

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Girth 5 graphs from relative difference sets.

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Abstract

We consider the problem of construction of graphs with given degree k and girth 5 and as few vertices as possible. We give a construction of a family of girth 5 graphs based on relative difference sets. This family contains the smallest known graph of degree 8 and girth 5 which was constructed by G. Royle, four of the known cages including the Hoffman-Singleton graph, some graphs constructed by G. Exoo and some new smallest known graphs.

Keywords: Cage, girth, Cayley graph, relative difference set.

A (k, g) graph is a k regular graph with girth g. Sachs [13] proved that for every $k \geq 3$ and $g \geq 5$ there exists a (k, g) graph. The number of vertices in the smallest (k, g) graph is denoted by $f(k, g)$. A (k, g) graph with $f(k, g)$ vertices is called a (k, g) cage. It is well-known that $f(k, g) \geq n(k, g)$ where $n(k, g)$ is the Moore bound

$$n(k,g) = \begin{cases} \frac{k(k-1)^{\frac{g-1}{2}}-2}{k-2} & \text{if } g \text{ is odd} \\ \frac{2(k-1)^{\frac{g}{2}}-2}{k-2} & \text{if } g \text{ is even.} \end{cases}$$

In this paper we consider the case $g = 5$. Then the Moore bound is $n(k, 5) = k^2 + 1$. For $k \leq 7$, the exact value of $f(k, 5)$ is known, but for $k \geq 8$ the difference between the upper and lower bound on $f(k, 5)$ is large. In particular, for $k = 8$ the Moore bound is $n(8, 5) = 65$ but the smallest known $(8, 5)$ graph is a Cayley graph of order 80 constructed by Royle [12].

For a table of smallest known (k, g) graphs we refer to Royle [12].

The unique cage of degree 7 is the graph constructed by Hoffman and Singleton [7]. It was observed by de Resmini and Jungnickel [6, Ex. 4.5] (see Example 7 below) that the Hoffman-Singleton graph can be constructed from a relative difference set in a group of order 25 acting semiregularly on the graph.

Exoo [5] gave a construction of some new smallest $(k, 5)$ graphs for $k = 8, 10, 11, 12, 13, 14$. This construction was also based on relative difference sets (or sets which are nearly relative difference sets) in a cyclic group acting semiregularly on the graph with two orbits of equal size.

Royle's Cayley graph on 80 vertices can be constructed in a similar way from a non-abelian group.

In this paper we give a general construction of graphs with girth 5 from relative difference sets and from subgraphs of Cayley graphs.

We will first give a short introduction to the concepts used in the construction.

Let G be any finite group and let $S \subset G$ be a subset not containing the group identity and with the property that $g \in S \Rightarrow g^{-1} \in S$. Then the Cayley graph of G with connection set S is the graph $\mathrm{Cay}(G, S)$ with vertex set G and edge set $\{\{x, y\} \mid x, y \in G, xy^{-1} \in S\}$, where $\{x, y\}$ denotes an edge joining the vertices x and y.

A (v, κ, λ) difference set in a group G of order v is a set $S \subseteq G$ with $|S| = \kappa$ such that for every non-identity element $g \in G$ there exists exactly λ pairs $(s, t) \in S \times S$ so that $g = st^{-1}$.

The following well known theorem of Singer [14] gives an important class of difference sets.

Theorem 1 *Let q be a prime power. Then there exists a $(\frac{q^{d+1}-1}{q-1}, \frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1})$ difference set in the cyclic group. In particular ($d = 2$), there exists a $(q^2 + q + 1, q + 1, 1)$ difference set in the cyclic group.*

It is also well known that for a prime power q and a $(q^2 + q + 1, q + 1, 1)$ difference set $S \subset \mathbb{Z}_{q^2+q+1}$, the graph with vertex set $\mathbb{Z}_{q^2+q+1} \times \{1, 2\}$ and edge set $\{\{(a, 1), (a + s, 2)\} \mid a \in \mathbb{Z}_{q^2+q+1}, s \in S\}$ is a $(q + 1, 6)$ cage.

Definition 2 *Let G be a group of order nm and let $N \lhd G$ be a normal subgroup of order n. A subset $S \subseteq G$ is said to be a relative (m, n, κ, λ) difference set with forbidden subgroup N if $|S| = \kappa$ and for every non-identity element $g \in G$ the number of pairs $(t, s) \in S \times S$, where $g = ts^{-1}$ is exactly λ if $g \notin N$ and 0 if $g \in N$.*

We refer to Pott [10] for basic theory of relative difference sets.

We can now state our main theorem. We note that in the application of relative difference sets in the construction of $(k,5)$ graphs we could replace *exactly* λ by *at most* λ in the above definition.

Theorem 3 *Let G be a group of order nm and let $N \lhd G$ be a normal subgroup of order n. Let $Na_1, \ldots, Na_m$ be the cosets of N. Suppose that S is a relative $(m,n,\kappa,1)$ difference set in G with forbidden subgroup N. Let Δ be a Cayley graph of N and let H_1 and H_2 be ℓ-regular graphs with vertex set N and with girth at least 5, such that H_1 is a subgraph of Δ and H_2 is a subgraph of the complement of Δ.*

Let Γ denote the graph with vertex set $G \times \{1,2\}$ and edges of the following types

Type I $\{(g,1),(gs,2)\}$ *for $g \in G$ and $s \in S$,*

Type II.1 $\{ga_i,1),(ha_i,1)\}$ *for $\{g,h\} \in H_1$ and $i \in \{1,\ldots,m\}$,*

Type II.2 $\{ga_i,2),(ha_i,2)\}$ *for $\{g,h\} \in H_2$ and $i \in \{1,\ldots,m\}$.*

Then Γ has girth at least 5 and is regular of degree $\kappa + \ell$.

Proof Since each vertex is incident with κ edges of type I and ℓ edges of type II, Γ is $\kappa + \ell$ regular.

Suppose that C is a cycle in Γ of length at most 4.

Since the subgraphs spanned by $G \times \{1\}$ and $G \times \{2\}$ consist of disjoint copies of H_1 and H_2, respectively, and both H_1 and H_2 have girth at least 5, C contains at least two edges of type I.

Suppose that $\{(g,1),(x,2)\}$ and $\{(h,1),(x,2)\}$, $h \neq g$, are edges in Γ. Then g and h are in different cosets of N. This follows from the fact that there exists $s,t \in S$ so that $x = gs = ht$ and so $h^{-1}g = ts^{-1} \notin N$.

If $(y,2) \neq (x,2)$ was another vertex adjacent to both $(g,1)$ and $(h,1)$ then $y = gs_1 = ht_1$ for some $s_1, t_1 \in S$ and $h^{-1}g = ts^{-1} = t_1 s_1^{-1}$. Since this contradicts $\lambda = 1$ for the relative difference set S, C contains at least one edge of type II.

If $\{(g,1),(gs,2)\}$ and $\{(g,1),(gt,2)\}$, $s \neq t$, are edges in Γ, i.e. $s,t \in S$ then, since $ts^{-1} \notin N$ and N is normal, $(gt)(gs)^{-1} = gts^{-1}g^{-1} \notin N$ and so gt and gs are in different cosets of N.

It follows that if (g,i) and (h,i) have a common neighbour in $G \times \{3 - i\}$ then (g,i) and (h,i) are in different connected component of the graph spanned by $G \times \{i\}$.

Thus the only possible cycles of length at most 4 have vertices in the following cyclic order

$$(g_1, 1), (g_2, 1), (g_2 s, 2), (g_1 t, 2)$$

where $s, t \in S$. Since $(g_1, 1)$ and $(g_2, 1)$ are adjacent, g_1 and g_2 are in the same coset, say Na_i, and we can write $g_1 = h_1 a_i, g_2 = h_2 a_i$ for some $h_1, h_2 \in N$.

Since $(g_1 t, 2)$ and $(g_2 s, 2)$ are adjacent, $g_1 t = h_1 a_i t$ and $g_2 s = h_2 a_i s$ are in the same coset of N. Thus

$$(h_1 a_i t)(h_2 a_i s)^{-1} = h_1 a_i t s^{-1} a_i^{-1} h_2^{-1} \in N$$

and so $a_i t s^{-1} a_i^{-1} \in N$ and since $N \lhd G$, $ts^{-1} \in N$. Since N is the forbidden subgroup, it follows that $s = t$.

By the construction of type II edges, $\{h_1, h_2\}$ is an edge in H_1, and if we write $a_i s = h a_j$ where $h \in N$ then $g_1 t = h_1 a_i s = h_1 h a_j$ and $g_2 s = h_2 h a_j$ and so $\{h_1 h, h_2 h\}$ is an edge in H_2. Since $H_1 \subseteq \Delta$, $\{h_1, h_2\}$ is an edge in Δ and so $h_1 h_2^{-1}$ is in the connection set of Δ. Similarly, $\{h_1 h, h_2 h\}$ is not an edge in Δ and so the connection set of Δ does not contain $(h_1 h)(h_2 h)^{-1} = h_1 h h^{-1} h_2^{-1} = h_1 h_2^{-1}$.

This contradiction proves that Γ does not contain any cycle of length at most 4. □

The smallest value of ℓ for which the construction in this theorem is interesting is $\ell = 2$. In this case we need the following lemma. In the applications of the lemma, the group N is either cyclic or isomorphic to S_3.

Lemma 4 *Let N be a group of order $n \geq 5$. Then there exists graphs Δ, H_1, H_2 as in Theorem 3 with $\ell = 2$, except if N is the quaternion group of order 8.*

Proof We want to find Δ so that the complement of Δ has degree at least $\frac{n}{2}$. Then, by a theorem of Dirac [4], we can take H_2 to be a Hamiltonian cycle in the complement of Δ.

Suppose that N has an element g of order at least 5. Then we can take $H_1 = \Delta = \mathrm{Cay}(N, \{g, g^{-1}\})$. Thus we may assume that N does not have any element of order at least 5 and so, by Sylow's theorems, $n = 2^i 3^j$, for some i, j.

Suppose that $j \geq 2$. Then N has a subgroup H of order 9. Since N does not have any element of order at least 5, H is the non-cyclic group of order 9, $H \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$.

Since $S = \{(1,0),(2,0),(0,1),(0,2)\} \subset H$ has the property that $\mathrm{Cay}(H,S)$ is a self-complementary 4 regular Hamiltonian graph, we choose $\Delta = \mathrm{Cay}(N,S)$. So we assume that $j \in \{0,1\}$.

Suppose first that $i \leq 2$. Then $n = 6$ or $n = 12$. If $n = 6$ and every element has order at most 4 then $N = S_3$. In this case we take $H_1 = \Delta = \mathrm{Cay}(S_3, \{(1\ 2),(1\ 3)\})$. For $n = 12$ the lemma is true if N has a subgroup of order 6. If N does not have a subgroup of order 6 then $N = A_4$. In this case we choose $\Delta = \mathrm{Cay}(A_4, \{(1\ 2\ 3),(1\ 3\ 2),(1\ 2)(3\ 4)\})$ and H_1 is a Hamilton cycle in Δ.

Suppose now that $i \geq 3$. Then N has a (non-cyclic) subgroup H of order 8. If H is not the quaternion group then there exists $S \subset H$ so that $\mathrm{Cay}(H,S)$ is the cube graph and then we can take $\Delta = \mathrm{Cay}(N,S)$. Thus we may assume that every subgroup of order 8 is isomorphic to the quarternion group.

Since every group of order 16 has a subgroup of order 8 not isomorphic to the quaternion group, the lemma is true if 16 divides n.

Since every group of order 24 has a subgroup of order 6, the lemma is true for $n = 24$. □

We can now start constructing graphs with girth 5.

Example 5 $\{0\} \subset \mathbb{Z}_5$ *is trivially a relative* $(1,5,1,1)$ *difference set. The construction in Theorem 3 combined with Lemma 4 gives the Petersen graph.*

One general construction of relative difference sets was found by Dembowski and Ostrom [2].

Theorem 6 *Let q be an odd prime power and let G be the additive group of $GF(q)$. Then $\{(x,x^2) \mid x \in GF(q)\} \subseteq G \times G$ is a relative $(q,q,q,1)$ difference set with forbidden subgroup $\{0\} \times G$.*

Example 7 *For $q = 5$, we find that $\{(0,0),(1,1),(2,4),(3,4),(4,1)\} \subset \mathbb{Z}_5 \times \mathbb{Z}_5$ is a relative difference set. The construction in Theorem 3 combined with Lemma 4 gives a 7 regular graph with girth 5 and 50 vertices, i.e. the Hoffman Singleton graph.*

For other values of q we get smaller graphs from the following construction of relative difference sets. This construction was found by Bose [1] and Elliot and Butson [3].

Theorem 8 *For every prime power q and every positive integer d there exists a relative*

$$(\frac{q^d - 1}{q - 1}, q - 1, q^{d-1}, q^{d-2})$$

difference set in the cyclic group of order $q^d - 1$. In particular, (for $d = 2$) there exists a cyclic relative $(q + 1, q - 1, q, 1)$ difference set.

Combining Theorem 3, Theorem 8 and Lemma 4 we get the following result which is essentially one of two constructions in Exoo [5]

Corollary 9 *For every prime power $q \geq 7$, there exists a $q+2$ regular graph of girth 5 with $2(q^2 - 1)$ vertices.*

In order to get other values of the degree, we may consider subgraphs of the graph constructed in Theorem 3.

Theorem 10 *Let $q \geq 7$ be a prime power and let $k \leq q + 2$. Then there exists a k regular graph with girth 5 and with $2(k - 1)(q - 1)$ vertices.*

Proof Let G be the cyclic group of order $(q + 1)(q - 1)$ and let N be the subgroup of order $q - 1$. Let $S \subset G$ be a relative $(q + 1, q - 1, q, 1)$ difference set with forbidden subgroup N. Let Γ be the graph constructed in Theorem 3 with $\ell = 2$.

Since elements in N do not occur as the difference of two elements in S, S contains at most one element from each coset of N.

Since the parameters of the relative difference set satisfy $m - \kappa = 1$ there is a unique coset of N containing no elements of S. Thus, for each coset Na_i there is a unique coset $Na_{i'}$ so that Γ has no edges from $Na_i \times \{1\}$ to $Na_{i'} \times \{2\}$.

Then the subgraph of Γ spanned by

$$\cup_{i=1}^{k-1} Na_i \times \{1\} \quad \cup \quad \cup_{i=1}^{k-1} Na_{i'} \times \{2\}$$

has the required properties. □

Similarly, we obtain the following result from Theorem 6.

Theorem 11 *Let $q \geq 5$ be a prime power and let $k \leq q + 2$. Then there exists a k regular graph with girth 5 and with $2q(k - 2)$ vertices.* □

With $k = 6$ and $q = 5$ we get a graph with 40 vertices. O'Keefe and Wong [9] and Wong [16] proved that this is the unique $(6, 5)$-cage. With $k = q = 5$ we get a graph with 30 vertices. This is one of four $(5, 5)$-cages, see Wegner [15], Yang and Zhang [17] and Meringer [8]. The Petersen graph can also be obtained from Theorem 11 with $k = 3$ and $q = 5$. The unique $(4, 5)$ cage has 19 vertices and was constructed by Robertson [11].

The smallest number of vertices in a k regular graph of girth 5 is not known for any $k \geq 8$. For $8 \leq k \leq 16$, the following table lists the smallest number n of vertices in a k regular graph with girth 5 constructed in this paper. For $k = 10$ and $k = 13$ these graphs are exactly the graphs constructed by Exoo [5] and for $k = 8$ the graph was constructed by Royle [12].

k	n	Construction	First constructed by
8	80	Ex. 13	Royle
9	96	Cor. 9	
10	126	Cor. 9	Exoo
11	156	Ex. 12	
12	216	Ex. 14	
13	240	Cor. 9	Exoo
14	288	Thm. 17, $q = 13$	
15	312	Thm. 17, $q = 13$	
16	336	Thm. 17, $q = 13$	

Example 12 *In the group $\mathbb{Z}_{13} \times S_3$ of order 78 the set*

$$\{(1, I), (10, I), (11, I), (0, (1\,2)), (5, (1\,2)), (2, (2\,3)), (8, (2\,3)), (7, (1\,3)), (9, (1\,3))\}$$

where I is the identity permutation, is a $(13, 6, 9, 1)$ relative difference set with forbidden subgroup $\{0\} \times S_3$, see Pott [10]. The construction in Theorem 3 gives an 11 regular graph with girth 5 and 156 vertices.

Example 13 *In the group $G = \langle x, y \mid x^8 = y^5 = 1, yx = xy^2 \rangle$ of order 40 with normal subgroup $N = \langle y \rangle$ the set $S = \{1, x, x^3, x^5y^4, x^6y, x^7y^3\}$ has the property that no non-identity element in N can be written as st^{-1} where $s, t \in S$ and all other elements in G can be written as st^{-1} for at most one pair $s, t \in S$. Using the construction in Theorem 3 we get an 8 regular graph with 80 vertices and girth 5. This graph was first constructed by Royle [12].The graph is vertex transitive with automorphism group of order 160. It is a Cayley graph of two groups of order 80.*

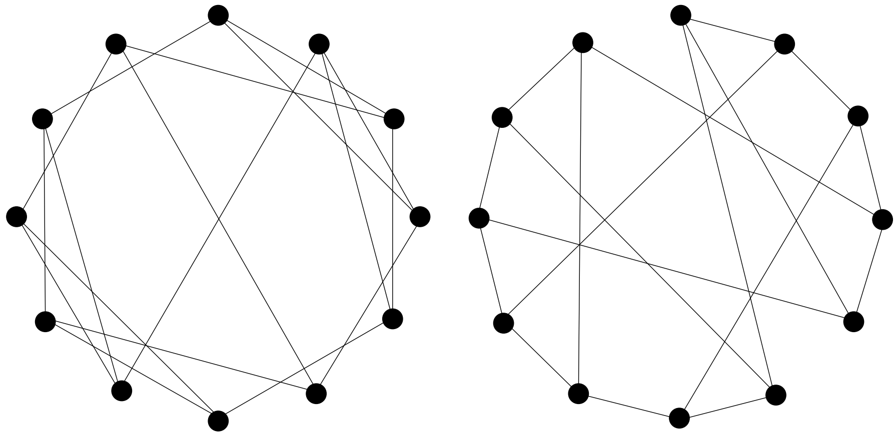

Figure 1: Two cubic graphs with girth 5 and order 12.

Example 14 *In the group* $G = \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ *of order 108 with normal subgroup* $N = \langle(2,1,0,0)\rangle$ *the set* $S = \{(0,0,0,0), (0,0,0,2), (0,0,1,0), (0,1,1,1), (1,0,1,2), (1,1,0,2), (1,1,2,1), (1,2,2,0), (2,1,2,2), (3,1,2,2)\}$ *has the property that no non-identity element in* N *can be written as* $s-t$ *where* $s,t \in S$ *and all other elements in* G *can be written as* $s-t$ *for at most one pair* $s,t \in S$*. Using the construction in Theorem 3 we get a 12 regular graph with 216 vertices and girth 5.*

We next consider the case $\ell = 3$ in Theorem 3. In this case n must be even and $n \geq f(3,5) = 10$. It can be shown that $n = 10$ is not possible. Thus $n = 12$ is the first case where it is possible to have $\ell = 3$ in Theorem 3. In the next example we show that it is possible to have $\ell = 3$ if $n = 12$, except maybe if $N = A_4$.

Example 15 *Let* $\Delta = Cay(\mathbb{Z}_{12}, \{\pm 2, \pm 3, 6\})$*. There are two cubic graphs with girth 5 and 12 vertices. In Figure 1, one these is shown as a subgraph of* Δ *and the other is shown as a subgraph of the complement of* Δ*. Thus we can take the graphs in Figure 1 to be* H_1 *and* H_2 *in Theorem 3.*

Δ *is a Cayley of every group of order 12, except* A_4*.*

Theorem 16 *Let* N *be a cyclic or dihedral group of order* $n \geq 12$*,* n *even. Then there exists graphs* Δ, H_1, H_2 *as in Theorem 3 with* $\ell = 3$*.*

Proof The case $n = 12$ was considered in Example 15. Thus we may assume that $n \geq 14$. Let $m = \frac{n}{2} \geq 7$. Then all differences of distinct elements in $\{0, 1, 3\}$ are different in $\mathbb{Z}_m$. Thus the graph H_1 with vertex set $\mathbb{Z}_m \times \{1, 2\}$ and edges $\{(i, 1), (i + s, 2)\}$ where $i \in \mathbb{Z}_m$ and $s \in \{0, 1, 3\}$ has girth 6. The similar graph H_2 with $s \in \{2, 4, 5\}$ also has girth 6.

H_1 and H_2 are edge-disjoint Cayley graphs of the dihedral group.

Now denote the vertex (i, j) by x_{2i-j+1}. Then H_1 is a subgraph of $\Delta = \mathrm{Cay}(\mathbb{Z}_n, \{\pm 1, \pm 5\})$ and H_2 is a subgraph of $\mathrm{Cay}(\mathbb{Z}_n, \{\pm 3, \pm 7, \pm 9\})$. If $n \geq 16$ these graphs are disjoint.

If $n = 14$ then let $p = (1, 3, 4, 2)(5, 12, 11, 13, 8, 10, 9, 6)$ and redefine H_2 to be the graph with vertex set $\{x_i \mid i \in \mathbb{Z}_{14}\}$ and edge set $\{\{x_{p(i)}, x_{p(j)}\} \mid \{x_i, x_j\} \in H_1\}$. □

As in Theorem 10 we get the following.

Theorem 17 *Let $q \geq 13$ be an odd prime power and let $k \leq q + 3$. Then there exists a k regular graph with girth 5 and with $2(k - 2)(q - 1)$ vertices.*

For large values of k we can get better results with $\ell > 3$.

Theorem 18 *Let $\ell \geq 4$ and let $n \geq 16\ell^2$ be even. Let N be a cyclic group of order n. Then there exists graphs Δ, H_1, H_2 as in Theorem 3.*

Proof By Chebyshev's Theorem, there exists a prime p, so that $\ell - 1 \leq p < 2(\ell - 1)$. By Singer's theorem there exists numbers $t_1, \ldots, t_{p+1}$ that form a difference set with $\lambda = 1$ modulo $p^2 + p + 1$. We may assume $-2\ell^2 < t_1 < \ldots < t_\ell < 2\ell^2$. Let $r = \frac{n}{2}$. Then the differences $t_i - t_j,\ 1 \leq i, j \leq \ell, i \neq j$ are all different modulo r. Thus the graph H_1 with vertex set $\mathbb{Z}_r \times \{1, 2\}$ and edges $\{(a, 1), (a + t_i, 2)\}$, for $a \in \mathbb{Z}_r, 1 \leq i \leq \ell$ has girth at least 6.

Now denote the vertex (i, j) in H_1 by x_{2i-j+1}. Then x_{2a} is adjacent to $x_{2(a+t_i)-1}$, for $a \in \mathbb{Z}_n, 1 \leq i \leq \ell$. Thus H_1 is a subgraph of $\Delta = \mathrm{Cay}(\mathbb{Z}_n, \{\pm(2t_i - 1) \mid 1 \leq i \leq \ell\}) \subseteq \mathrm{Cay}(\mathbb{Z}_n, \{i \mid -4\ell^2 < i \leq 4\ell^2\})$.

Similarly, the graph H_2 with vertex set $\mathbb{Z}_r \times \{1, 2\}$ and edges $\{(a, 1), (a + t_i + 4\ell^2, 2)\}$, for $a \in \mathbb{Z}_r, 1 \leq i \leq \ell$ has girth at least 6 and is a subgraph of the complement of Δ. □

Combining the Theorems 3, 8 and 18, we get the following.

Corollary 19 *Let q be an odd prime power. Then there exists a $q + \lfloor \frac{\sqrt{q-1}}{4} \rfloor$ regular graph of girth 5 and with $2(q^2 - 1)$ vertices.*

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