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R-2005-06

February 2005

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ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

ON A NEW BOUNDEDNESS RESULT FOR PSEUDODIFFERENTIAL OPERATORS WITH EXOTIC SYMBOLS

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Abstract. The α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$, $\alpha \in [0,1]$, form a family of spaces that contain the Besov and modulation spaces as special cases. In this paper we prove that a pseudodifferential operator $\sigma(x,D)$ with symbol in the exotic Hörmander class $S^b_{\rho,0}$ extends to a bounded operator $\sigma(x,D)\colon M^{s,\alpha}_{p,q}(\mathbb{R}^d) \to M^{s-b,\alpha}_{p,q}(\mathbb{R}^d)$ provided $\alpha \le \rho \le 1$, $0 < \alpha \le 1$, and $1 < p < \infty$. The result extends the well-known result that pseudodifferential operators with symbol in the class $S^b_{1,0}$ maps the Besov space $B^s_{p,q}(\mathbb{R}^d)$ into $B^{s-b}_{p,q}(\mathbb{R}^d)$.

1. Introduction

In this paper we extend certain well-known results about pseudodifferential operators on Besov spaces to the family of so-called α-modulation spaces. Let $\langle x\rangle := (1+|x|^2)^{1/2}$ for $x \in \mathbb{R}^d$, and recall that the Hörmander class $S^b_{\rho,\delta}(\mathbb{R}^d\times\mathbb{R}^d)$ is the family of functions $\sigma \in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ satisfying

$$|\sigma|^{(b)}_{N,M} := \max_{|\alpha|\le N,|\beta|\le M} \sup_{x,\xi\in\mathbb{R}^d} \langle\xi\rangle^{\rho|\alpha|-\delta|\beta|-b}|\partial^\alpha_\xi\partial^\beta_x\sigma(\xi,x)| < \infty,$$

for $M,N \in \mathbb{N}$. For $\sigma \in S^b_{\rho,\delta}(\mathbb{R}^d\times\mathbb{R}^d)$, we associate the pseudodifferential operator

$$\sigma(x,D)f(x) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\sigma(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

The family of operators associated with $S^b_{\rho,\delta}(\mathbb{R}^d\times\mathbb{R}^d)$ is denoted by $\mathrm{Op}S^b_{\rho,\delta}(\mathbb{R}^d\times\mathbb{R}^d)$. It is well-known that for symbols $\sigma \in S^b_{1,\delta}(\mathbb{R}^d\times\mathbb{R}^d)$, $0\le\delta<1$, $\sigma(x,D)$ extends to a bounded mapping on the Besov scale of spaces [19];

$$\sigma(x,D) : B^s_{p,q}(\mathbb{R}^d) \to B^{s-b}_{p,q}(\mathbb{R}^d), \qquad s\in\mathbb{R}, 1<p,q<\infty. \tag{1.1}$$

The main result of the present paper is to generalize (1.1) to the full scale of α-modulation spaces. It was pointed out by Feichtinger and Gröbner in the papers [6, 5] that Besov spaces are special cases of decomposition type Banach spaces $D(Q,B,Y)$. Gröbner [8] used the methods in [6] to define a family of spaces, called α-modulation spaces, depending on a parameter $\alpha\in[0,1]$ with the Besov scale of spaces being the $\alpha=1$ "fiber". The Besov spaces are based on coverings of frequency space $\mathbb{R}^d$ by balls $B(a_n,r_n)$ satisfying $|a_n| \asymp |B(a_n,r_n)|^{1/d}$, that is to say there exist constants $c,C\in(0,\infty)$ such that $c|a_n| \le |B(a_n,r_n)|^{1/d} \le C|a_n|$ for all the balls. Gröbners idea was to define spaces corresponding to coverings based on the rule $|a_n|^\alpha \asymp |B(a_n,r_n)|^{1/d}$, $0\le\alpha\le1$. The precise definition of an α-modulation space will be given in Section 2. The case $\alpha=0$ corresponds to the

2000 *Mathematics Subject Classification.* Primary 47G30, 46E35; Secondary 47B38.

Key words and phrases. Modulation spaces, α-modulation spaces, Besov spaces, pseudodifferential operator, exotic symbol class.

This work is in part supported by the Danish Technical Science Foundation, Grant no. 9701481.

classical Modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ introduced by Feichtinger [4], so in some abstract sense the α-modulation spaces "interpolate" between the Besov and Modulation spaces. The family of coverings yielding the α-modulation spaces was considered independently by Päivärinta and Somersalo in [14]. Päivärinta and Somersalo used the partitions to extend the Calderón-Vaillancourt boundedness result for pseudodifferential operators to the local Hardy spaces h_p.

The main result in the present paper is to prove that for $\alpha \in (0,1]$ and $\sigma \in S^b_{\rho,0}(\mathbb{R}^d \times \mathbb{R}^d)$ with $1 \geq \rho \geq \alpha$, $\sigma(x,D)$ extends to a bounded operator

$$\sigma(x,D) : M^{s,\alpha}_{p,q}(\mathbb{R}^d) \to M^{s-b,\alpha}_{p,q}(\mathbb{R}^d). \tag{1.2}$$

The proof of the result in the case $\alpha = 1$ [when $M^{s,\alpha}_{p,q}(\mathbb{R}^d) = B^s_{p,q}(\mathbb{R}^d)$] is relatively easy since it is possible to use wavelet bases to reduce the proof to a matrix estimate. However, we do not have this option in the general case since no nice bases are known for $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ when $d > 1$. Therefore we have to investigate a classical decomposition of the symbol, see e.g. [10, chapter 2] and references therein. Notice that for $\rho < 1$ we have a strict inclusion $S^b_{1,0}(\mathbb{R}^d \times \mathbb{R}^d) \subset S^b_{\rho,0}(\mathbb{R}^d \times \mathbb{R}^d)$ so the estimate (1.2) extends to symbols not included in the result for Besov spaces. An example of a symbol $\sigma \in S^b_{1/2,0}(\mathbb{R} \times \mathbb{R}) \backslash S^b_{1,0}(\mathbb{R} \times \mathbb{R})$ is the symbol associated with the convolution kernel $K(x) = e^{i/|x|}|x|^{-\gamma}$, $\gamma > 0$. It can be shown that the symbol $\hat{K}(\xi)$ behaves like $e^{ic|\xi|^{1/2}}|\xi|^{\gamma/2-3/4}$, see [15, Chap. VII]. Thus, $\hat{K}(\xi) \in S^{\gamma/2-3/4}_{1/2,0}(\mathbb{R}^2)$. Another example is the inverse of the heat operator, see Example 5.3 in Section 5.

Our result (1.2) does not apply to the limit case $\alpha = 0$. For $\alpha = 0$, $M^{s,0}_{p,q}(\mathbb{R}^d)$ is the family of modulation spaces. Pseudodifferential operators on modulation spaces have been studied in a number of paper by different authors, see e.g. [17, 1, 12, 9, 2, 16]. Pseudodifferential operators on α-modulation spaces has been studied by Nazaret and Holschneider in [13], and by on of the present authors in [3]. The results in [3] apply only to the one dimensional case, and they rely on the fact that nice orthonormal brushlet bases can be found for $M^{s,\alpha}_{p,q}(\mathbb{R})$.

The structure of the paper is as follows. In Section 2 we give the precise definition of the α-modulation spaces based on a so-called bounded admissible partition of unity (BAPU). The spaces are independent of the specific choice of BAPU, which we exploit to construct a partition with "nice" functions from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. The construction of the BAPU is done in Section 2.1. In Section 3 we warm up for the main result in Section 4 by proving boundedness results for multiplier operators on α-modulation spaces. The main result is proved in 4 using the multiplier result from Section 3. Since we do not have an atomic decomposition of $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$, $d > 1$, the idea of the proof is to expand the symbol $\sigma(x,\xi)$ in a Taylor series in x, and then estimate each contributing factor. Elliptic pseudodifferential operators on the α-modulation spaces are considered in Section 5. Finally, there is an appendix where we prove certain facts about α-coverings needed for the construction of the BAPU in Section 2.1.

2. Modulation spaces

In this section we define the α-modulation spaces. The α-modulation spaces, first introduced by Gröbner in [8], are a family of spaces that contain the classical modulation and Besov spaces as special "extremal" cases. The spaces are defined by a parameter α, belonging to the interval $[0,1]$. This parameter determines a segmentation of the frequency domain from which the spaces are built.

Definition 2.1. A countable set $\mathcal{Q}$ of subsets $Q \subset \mathbb{R}^d$ is called an admissible covering if $\mathbb{R}^d = \cup_{Q\in\mathcal{Q}} Q$ and there exists $n_0 < \infty$ such that $\#\{Q' \in \mathcal{Q} : Q \cap Q' \neq \emptyset\} \leq n_0$ for all $Q \in \mathcal{Q}$. An admissible covering is called an α-covering, $0 \leq \alpha \leq 1$, of $\mathbb{R}^d$ if $|Q| \asymp \langle x\rangle^{\alpha d}$ (uniformly) for all $x \in Q$ and for all $Q \in \mathcal{Q}$.

We let $\mathcal{F}(f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx$, $f \in L_1(\mathbb{R}^d)$, denote the Fourier transform.

Definition 2.2. Let $\mathcal{Q}$ be an α-covering of $\mathbb{R}^d$. A corresponding bounded admissible partition of unity (BAPU) $\{\psi_Q\}_{Q\in\mathcal{Q}}$ is a family of functions satisfying

- $\operatorname{supp}(\psi_Q) \subset Q$
- $\sum_{Q\in\mathcal{Q}} \psi_Q(\xi) = 1$
- $\sup_Q \|\mathcal{F}^{-1}\psi_Q\|_{L_1} < \infty$.

Definition 2.3. Given $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha \leq 1$, let $\mathcal{Q}$ be an α-covering of $\mathbb{R}^d$ and let Ψ be a BAPU. Then we define the *α-modulation space*, $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ as the set of distributions $f \in S'(\mathbb{R}^d)$ satisfying

$$\|f\|_{M^{s,\alpha}_{p,q}} := \left(\sum_{Q\in\mathcal{Q}} \langle\xi_Q\rangle^{qs} \big\|\mathcal{F}^{-1}(\psi_I \mathcal{F}f)\big\|^q_{L_p}\right)^{1/q} < \infty, \tag{2.1}$$

with $\{\xi_Q\}_{Q\in\mathcal{Q}}$ a sequence satisfying $\xi_Q \in Q$. For $q = \infty$ we have the usual change of the sum to sup over $Q \in \mathcal{Q}$.

It is proved in [8] that the definition of $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ is independent of the α-cover and of the BAPU, see also [6, Theorem 2.3]. In Section 2.1 below we construct a BAPU $\{\psi_k\}_{k\in\mathbb{Z}^d\setminus\{0\}} \subset \mathcal{S}(\mathbb{R}^d)$, satisfying $|\partial^\beta\psi_k(\xi)| \leq C_\beta\langle\xi\rangle^{-\alpha|\beta|}$ for every multi-index $\beta \in \mathbb{N}_0^d$. We will use this particular BAPU to simplify the proof of our main result in Section 4.

2.1. Bounded admissible partitions of unity and their properties.

The α-modulation spaces are defined using a bounded admissible partition of unity, but the spaces are actually independent of the specific choice. The results in Sections 3 and 4 rely on the fact that it is possible to construct a smooth BAPU with certain "nice" properties. We have the following construction.

Proposition 2.4. *For $\alpha \in [0,1)$, there exists an α-covering of $\mathbb{R}^d$ with a corresponding BAPU $\{\psi_k\}_{k\in\mathbb{Z}^d\setminus\{0\}} \subset \mathcal{S}(\mathbb{R}^d)$ satisfying*

$$|\partial^\beta\psi_k(\xi)| \leq C_\beta\langle\xi\rangle^{-|\beta|\alpha},$$

for every multi-index β and $k \in \mathbb{Z}^d \setminus \{0\}$.

Proof. For $r > 0$, and $k \in \mathbb{Z}^d \setminus \{0\}$ we define the ball

$$B^r_k := \{\xi \in \mathbb{R}^d : \big|\,\xi - |k|^{\frac{\alpha}{1-\alpha}}k\,\big| < r|k|^{\frac{\alpha}{1-\alpha}}\}.$$

By Lemma A.1, there exists $r_1 > 0$ such that $\{B^{r_1}_k\}_{k\in\mathbb{Z}^d\setminus\{0\}}$ is an α-covering of $\mathbb{R}^d$. There also exists $0 < r_2 < r_1$, such that $\{B^{r_2}_k\}_{k\in\mathbb{Z}^d\setminus\{0\}}$ is pairwise disjoint.

Fix $r > r_1$. We take $\Phi \in C^\infty(\mathbb{R}^d)$ satisfying $\inf_{\xi\in B(0,r_1)} |\Phi(\xi)| := c > 0$ and $\operatorname{supp}(\Phi) \subset B(0,r)$. Let

$$g_k(\xi) := \Phi(|c_k|^{-\alpha}(\xi - c_k)), \qquad k \in \mathbb{Z}^d \setminus \{0\},$$

where $c_k := |k|^{\frac{\alpha}{1-\alpha}}k$. Clearly, $g_k \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(g_k) \subset B^r_k$. In fact, $\{\operatorname{supp}(g_k)\}_k$ is an α-covering of $\mathbb{R}^d$. The covering is admissible (see Lemma A.1) since $\{B^{r_2}_k\}_{k\in\mathbb{Z}^d\setminus\{0\}}$, with $B^{r_2}_k \subset \operatorname{supp}(g_k)$, is pairwise disjoint. It is easy to see that the partition has "finite height", i.e., $\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \chi_{\operatorname{supp}(g_k)}(\xi) \leq n_1$ for some uniform constant n_1.

Notice that

$$|\partial^\beta g_k(\xi)| = |c_k|^{-\alpha|\beta|}|(\partial^\beta \Phi)(|c_k|^{-\alpha}(\xi - c_k))| \le C_\beta |c_k|^{-\alpha|\beta|},$$

and since $|c_k| \ge 1$ for all $k \in \mathbb{Z}^d \setminus \{0\}$, we have

$$|\partial^\beta g_k(\xi)| \le C'_\beta \langle c_k\rangle^{-\alpha|\beta|} \asymp \langle \xi\rangle^{-\alpha|\beta|} \qquad \text{for all } \xi \in B_k^r.$$

Since we want a BAPU, we consider the sum $g(\xi) := \sum_{k\in\mathbb{Z}^d\setminus\{0\}} g_k(\xi)$. Now, $\{\mathrm{supp}(g_k)\}_k$ has finite height, so g is well defined, and the finite overlap ensures that $|\partial^\beta g(\xi)| \le C'_\beta \langle\xi\rangle^{-|\beta|\alpha}$. Recall that $g_k(\xi) \ge c$ for all $\xi \in B_k^{r_1}$, and since $\{B_k^{r_1}\}_{k\in\mathbb{Z}^d\setminus\{0\}}$ covers $\mathbb{R}^d$, we have $g(\xi) \ge c$. Thus, we can define

$$\psi_n(\xi) := \frac{g_n(\xi)}{\sum_{k=1}^\infty g_k(\xi)}.$$

By Lemma B.2 in Appendix B, $|\partial^\beta \psi_k(\xi)| \le C_\beta \langle\xi\rangle^{-|\beta|\alpha}$. In order to conclude, we need to verify that $\sup_k \|\mathcal{F}^{-1}\psi_k\|_{L_1} < \infty$. Let $\widetilde{\psi}_k(\xi) = \psi_k(|c_k|^\alpha \xi + c_k)$. By a simple substitution in each of the following integrals, we obtain

$$\begin{aligned}\|\mathcal{F}^{-1}\psi_k\|_{L_1} &= (2\pi)^{-d/2}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\psi_k(\xi)e^{ix\cdot\xi}\,d\xi\right|dx\\ &= (2\pi)^{-d/2}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\widetilde{\psi}_k(\xi)e^{ix\cdot\xi}\,d\xi\right|dx\\ &\le C_d\Big(\sum_{|\beta|\le d+1}\|\partial^\beta\widetilde{\psi}_k\|_{L_1}\Big)\int_{\mathbb{R}^d}\langle x\rangle^{-d-1}dx \le C'_d,\end{aligned}$$

where we have used Lemmas B.1 and B.3 for the last estimate. We conclude that $\{\psi_k\}_k$ is a BAPU corresponding to the α-covering $\{\mathrm{supp}(g_k)\}_k$. □

Let us briefly return to Definition 2.3. We rewrite (2.1) in terms of the BAPU from Proposition 2.4, using the multiplier operators $\psi_k(D)$,

$$\|f\|_{M^{s,\alpha}_{p,q}} = \left\|\langle k|k|^{\alpha/(1-\alpha)}\rangle^s \big\|\psi_k(D)f\big\|^q_{L_p}\right\|_{\ell_q(\mathbb{Z}^d)}.$$

We need the following result proved in [6, Theorem 2.3]. Define $\Psi_k := \sum_{k'}\psi_{k'}$, where the sum is taken over all $k' \in \mathbb{Z}^d \setminus \{0\}$ with $B^r_{k'} \cap B^r_k \neq \emptyset$. Then

$$\|f\|_{M^{s,\alpha}_{p,q}} \asymp \left\|\langle k|k|^{\alpha/(1-\alpha)}\rangle^s \big\|\Psi_k(D)f\big\|^q_{L_p}\right\|_{\ell_q}. \tag{2.2}$$

Recall that the definition of $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ does not depend on the particular choice of BAPU, see [8]. It is easy to see, using the BAPU above, that $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$, $1 < p, q < \infty$.

3. Differential operators on α-modulation spaces

In this Section we consider a special class of pseudodifferential operators, namely Fourier multipliers, and show that this class is well behaved on $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$. One important example of such an operator is the Bessel potential $J^b := (I-\Delta)^{b/2}$ defined by $\widehat{J^b f}(\xi) = \langle\xi\rangle^b \hat{f}(\xi)$. It is well-known that for the Besov spaces, $J^b B^s_{p,q}(\mathbb{R}^d) = B^{s-b}_{p,q}(\mathbb{R}^d)$, and it is perhaps surprising that J^b has exactly the same lifting property when considered on $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$, $\alpha > 0$.

Proposition 3.1. *Let $L = \lceil d/2 \rceil$. Suppose the function $\sigma \in C^L(\mathbb{R}^d)$ satisfies $|\partial^\beta \sigma(\xi)| \leq C\langle\xi\rangle^{b-|\beta|\rho}$ for $|\beta| \leq L$, $b \in \mathbb{R}$ and $0 \leq \rho \leq 1$. Let T be the Fourier multiplier given by $\widehat{Tf} := \sigma\hat{f}$. Then T extends to a bounded operator, $T : M^{s,\alpha}_{p,q}(\mathbb{R}^d) \to M^{s-b,\alpha}_{p,q}(\mathbb{R}^d)$, for $0 \leq \alpha \leq \rho$, $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \leq q \leq \infty$, i.e.,*

$$\|Tf\|_{M^{s-b,\alpha}_{p,q}} \leq C\|f\|_{M^{s,\alpha}_{p,q}} \qquad \text{for all } f \in M^{s,\alpha}_{p,q}(\mathbb{R}^d).$$

Proof. For $\alpha = 1$ (i.e., in the Besov space case) the result is well known, see e.g. [18, Chap. 2]. Suppose $\alpha < 1$. Let $\{\psi_k\}_{k\in\mathbb{Z}^d\setminus\{0\}}$ be the BAPU from Proposition 2.4, and let $c_k = k|k|^{(1-\alpha)/\alpha}$ be the center of the ball B^r_k, see Section 2.1. Define $\Psi_k := \sum_{k'} \psi_{k'}$, where the sum is taken over all $k' \in \mathbb{Z}^d \setminus \{0\}$ with $B^r_{k'} \cap B^r_k \neq \emptyset$. By Proposition 2.4,

$$|\partial^\beta \Psi_k(\xi)| \leq C\langle\xi\rangle^{-\alpha|\beta|},$$

with C independent of $k \in \mathbb{Z}^d \setminus \{0\}$. Define

$$\sigma_k(\xi) := \langle c_k\rangle^{-b}\sigma(\xi)\Psi_k(\xi).$$

Since $\alpha \leq \rho$, we have

$$\begin{aligned}|\partial^\beta \sigma_k(\xi)| &\leq \langle c_k\rangle^{-b} \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} |\partial^\gamma \sigma(\xi)| \cdot |\partial^{\beta-\gamma}\Psi_k(\xi)| \\ &\leq C\langle c_k\rangle^{-b} \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \langle\xi\rangle^{b-|\gamma|\rho}\langle\xi\rangle^{-\alpha(|\beta|-|\gamma|)} \leq C_\beta\langle c_k\rangle^{-b}\langle\xi\rangle^{b-\alpha|\beta|}.\end{aligned}$$

Moreover, for $\xi \in \mathrm{supp}(\Psi_k)$; $\langle c_k\rangle \asymp \langle\xi\rangle$, and $|\xi - c_k|^d \leq C|B^{r_1}_k| \asymp \langle\xi\rangle^{\alpha d} \Rightarrow \langle\xi\rangle^{-\alpha|\beta|} \leq C|\xi - c_k|^{-|\beta|}$. Therefore,

$$|\partial^\beta \sigma_k(\xi)| \leq C'\langle\xi\rangle^{-|\beta|\alpha} \leq C''|\xi - c_k|^{-|\beta|}.$$

Now, by the Hörmander-Mihlin multiplier theorem (applied to the multiplier $\tilde{\sigma}_k(\xi) := \sigma_k(\xi + c_k)$) we deduce that σ_k extends to a bounded multiplier on L_p, $1 < p < \infty$, with bound independent of $k \in \mathbb{Z}^d \setminus \{0\}$. Since $\Psi_k(\xi) = 1$ for $\xi \in \mathrm{supp}(\psi_k)$, this implies

$$\|\mathcal{F}^{-1}(\psi_k\sigma\hat{f})\|_p \leq C\langle c_k\rangle^b\|\mathcal{F}^{-1}(\psi_k\hat{f})\|_p,$$

with C independent of k. The result now follows from Definition 2.3,

$$\begin{aligned}\|Tf\|^q_{M^{s,\alpha}_{p,q}} &\asymp \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \langle c_k\rangle^{qs}\big\|\mathcal{F}^{-1}(\psi_k\sigma\hat{f})\big\|^q_{L_p} \\ &\leq C \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \langle c_k\rangle^{q(s+b)}\big\|\mathcal{F}^{-1}(\psi_k\hat{f})\big\|^q_{L_p} \asymp \|f\|^q_{M^{s+b,\alpha}_{p,q}}.\end{aligned}$$

□

As a corollary, we deduce the following about $J^b = (1-\Delta)^{b/2}$, which will be used to simplify the proof of our main result, Theorem 4.1.

Corollary 3.2. *Given $b \in \mathbb{R}$, let $J^b = (1-\Delta)^{b/2}$. Then for $0 \leq \alpha \leq 1$, $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \leq q \leq \infty$ we have $J^b M^{s,\alpha}_{p,q}(\mathbb{R}^d) = M^{s-b,\alpha}_{p,q}(\mathbb{R}^d)$, in the sense that*

$$\|f\|_{M^{s,\alpha}_{p,q}} \asymp \|J^b f\|_{M^{s-b,\alpha}_{p,q}} \qquad \text{for all } f \in M^{s,\alpha}_{p,q}(\mathbb{R}^d).$$

Proof. The result follows by Proposition 3.1 using the identity $(J^b)^{-1} = J^{-b}$. □

4. Pseudodifferential operators on α-modulation spaces

This section contains the main result of the present paper. We prove that pseudodifferential operators with symbols in the class $S^b_{\rho,0}(\mathbb{R}^d\times\mathbb{R}^d)$ maps $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ into $M^{s-b,\alpha}_{p,q}(\mathbb{R}^d)$ for $\rho\geq\alpha$, $\alpha\in(0,1]$. The result is.

Theorem 4.1. *Suppose $b\in\mathbb{R}$, $\alpha\in(0,1]$, $\sigma\in S^b_{\rho,0}(\mathbb{R}^d\times\mathbb{R}^d)$, $\rho\geq\alpha$, $s\in\mathbb{R}$, $p\in(1,\infty)$, and $q\in[1,\infty]$. Then*

$$\sigma(x,D):M^{s,\alpha}_{p,q}(\mathbb{R}^d)\to M^{s-b,\alpha}_{p,q}(\mathbb{R}^d).$$

The proof of Theorem 4.1 in the case $\alpha=1$ [where $M^{s,\alpha}_{p,q}(\mathbb{R}^d)=B^s_{p,q}(\mathbb{R}^d)$] is relatively easy since it is possible to use wavelet bases to reduce the proof to a matrix estimate. However, we do not have this option in the general case since no nice bases are known for $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ when $d>1$. In the case $d=1$, so-called brushlet bases for $M^{s,\alpha}_{p,q}(\mathbb{R})$ are available and it is indeed possible to use discrete methods as demonstrated by one of the authors in [3]. Therefore, our proof of Theorem 4.1 is more in the spirit of the analytic methods used to prove the Besov space case before wavelets and other atomic decompositions became available.

Before we give the proof of Theorem 4.1, let us state and prove a technical lemma. We let $\check{f}$ denote the inverse Fourier transform of f.

Lemma 4.2. *Suppose $\sigma\in S^0_{\rho,0}$, $\alpha\leq\rho\leq 1$. Then for $|\gamma|\leq K$ and $m\geq 0$, we have*

$$\int_{\mathbb{R}^d}\big|\sup_{z\in\mathbb{R}^d}(\partial^\gamma_x\sigma(z,\xi)\partial^\nu_\xi\psi_k(\xi))^\vee(x)\big|\langle x\rangle^m\,dx\leq C|\,\sigma|^{(0)}_{L,K},$$

where $L\in\mathbb{N}$ satisfies $L>m+d$, and C does not depend on $k\in\mathbb{Z}^d\setminus\{0\}$.

Proof. Let $\sigma^\gamma_\eta(x,\xi):=\partial^\gamma_x\partial^\eta_\xi\sigma(x,\xi)$. We have the estimate

$$\begin{aligned}\int_{\mathbb{R}^d}&\big|\sup_{z\in\mathbb{R}^d}(\partial^\gamma_x\sigma(z,\xi)\partial^\nu_\xi\psi_k(\xi))^\vee(x)\big|\langle x\rangle^m\,dx\\&=\int_{\mathbb{R}^d}\sup_{z\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}e^{ix\cdot\xi}\sigma^\gamma(z,\xi)\partial^\nu_\xi\psi_k(\xi)\,d\xi\right|\langle x\rangle^m\,dx.\end{aligned}$$

Let $\widetilde{\psi}_k(\xi):=\psi_k(|c_k|^\alpha\xi+c_k)$. Then a substitution in each integral gives

$$=|c_k|^{-\alpha|\nu|}\int_{\mathbb{R}^d}\sup_{z\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}e^{ix\cdot\xi}\sigma^\gamma(z,|c_k|^\alpha\xi+c_k)\partial^\nu_\xi\widetilde{\psi}_k(\xi)\,d\xi\right|\langle x\rangle^m\,dx.$$

We now use Lemma B.1 in the inner integral.

$$\leq C\int_{\mathbb{R}^d}\sup_{z\in\mathbb{R}^d}\sum_{|\beta|\leq L}\int_{\mathbb{R}^d}\left|\partial^\beta_\xi\Big[\sigma^\gamma(z,|c_k|^\alpha\xi+c_k)\partial^\nu_\xi\widetilde{\psi}_k(\xi)\Big]\right|d\xi\langle x\rangle^{-L+m}\,dx$$

and by Leibniz's rule, we obtain

$$\begin{aligned}
&\leq C \int_{\mathbb{R}^d} \sup_{z\in\mathbb{R}^d} \sum_{|\beta|\leq L} \int_{\mathbb{R}^d} \sum_{0\leq\eta\leq\beta} \binom{\beta}{\eta} |c_k|^{\alpha|\eta|} \big|\sigma_\eta^\gamma(z,|c_k|^\alpha\xi + c_k)\partial_\xi^{\nu+\beta-\eta}\widetilde{\psi}_k(\xi)\big|\, d\xi \langle x\rangle^{-L+m}\, dx. \\
&\leq C' \sum_{\substack{|\beta|\leq L\\ 0\leq\eta\leq\beta}} \sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d} |c_k|^{\alpha|\eta|} |\sigma_\eta^\gamma(z,|c_k|^\alpha\xi + c_k)||\partial_\xi^{\nu+\beta-\eta}\widetilde{\psi}_k(\xi)|\, d\xi \int_{\mathbb{R}^d} \langle x\rangle^{-L+m}\, dx \\
&\leq C'' \sum_{\substack{|\beta|\leq L\\ 0\leq\eta\leq\beta}} |\,\sigma|^{(0)}_{|\eta|,K} \int_{\mathbb{R}^d} |\partial_\xi^{\nu+\beta-\eta}\widetilde{\psi}_k(\xi)|\, d\xi \leq C'''|\,\sigma|^{(0)}_{L,K},
\end{aligned}$$

where we used Lemma B.3, $\alpha \leq \delta$, and the fact that for $\xi \in \operatorname{supp}(\widetilde{\psi}_k)$,

$$|\sigma_\eta^\gamma(z,|c_k|^\alpha\xi + c_k)| \leq |\,\sigma|^{(0)}_{|\eta|,K}\langle |c_k|^\alpha\xi + c_k\rangle^{-\rho|\eta|} \leq C|\,\sigma|^{(0)}_{|\eta|,K}\langle c_k\rangle^{-\rho|\eta|}.$$

□

Proof of Theorem 4.1. Let us begin by showing that by standard arguments the proof can be reduced to the case where $s > 2d$ and $b = 0$. Suppose the result holds for $b = 0$ and $s > 2d$. Given $s \leq 2d$ and $b = 0$, choose $\tilde{s}$ such that $\tilde{s} + s > 2d$. Recall that $J^a \in \mathrm{Op}S^a_{1,0} \subset \mathrm{Op}S^a_{\rho,0}$, so we can write

$$\sigma(x,D) = J^{\tilde{s}}[J^{-\tilde{s}}\sigma(x,D)J^{\tilde{s}}]J^{-\tilde{s}},$$

with $J^{-\tilde{s}}\sigma(x,D)J^{\tilde{s}} \in \mathrm{Op}S^0_{\rho,0}$, see e.g. [11, Chapter 18]. By Corollary 3.2, $J^{-\tilde{s}}M^{s,\alpha}_{p,q} = M^{s+\tilde{s},\alpha}_{p,q}$, and by assumption $J^{-\tilde{s}}\sigma(x,D)J^{\tilde{s}}$ is bounded on $M^{s+\tilde{s},\alpha}_{p,q}$. The result for $s \leq 2d$ now follows by the fact that $J^{\tilde{s}}M^{s+\tilde{s},\alpha}_{p,q} = M^{s,\alpha}_{p,q}$. Now, we can extend the result to arbitrary $s \in \mathbb{R}$ and $b \in \mathbb{R}$ by using the fact that $J^b M^{s,\alpha}_{p,q} = M^{s-b,\alpha}_{p,q}$, and then write

$$\sigma(x,D) = [\sigma(x,D)J^{-b}]J^b,$$

with $\sigma(x,D)J^{-b} \in \mathrm{Op}S^0_{\rho,0}$, since $J^{-b} \in \mathrm{Op}S^{-b}_{1,0} \subset \mathrm{Op}S^{-b}_{\rho,0}$.

We consider the case $b = 0$ and $s > 2d$. It suffices to prove that $\|\sigma(x,D)f\|_{M^{s,\alpha}_{p,q}} \leq C\|f\|_{M^{s,\alpha}_{p,q}}$ for $f \in \mathcal{S}(\mathbb{R}^d)$ since $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$. Fix $f \in \mathcal{S}(\mathbb{R}^d)$. We need to estimate the L_p-norm of the terms $\psi_k(D)\sigma(x,D)f$. Notice that for $g \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
[\psi_k(D)g](x) &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{ix\cdot y}\psi_k(y)\hat{g}(y)\, dy \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} \check{\psi}_k(y)g(x-y)\, dy \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} \hat{\psi}_k(y)g(x+y)\, dy. \qquad (4.1)
\end{aligned}$$

Letting $\sigma_\eta^\gamma(x,\xi) := \partial_x^\gamma \partial_\xi^\eta \sigma(x,\xi)$, we obtain

$$
\begin{aligned}
\sigma(x+y,D)f(x+y) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x+y)\cdot\xi} \sigma(x+y,\xi)\hat{f}(\xi)\,d\xi \\
&= (2\pi)^{-d/2} \sum_{|\gamma|\le K-1} \frac{y^\gamma}{\gamma!} \int_{\mathbb{R}^d} e^{i(x+y)\cdot\xi} \sigma^\gamma(x,\xi)\hat{f}(\xi)\,d\xi \\
&+ (2\pi)^{-d/2} \sum_{|\gamma|=K} K\frac{y^\gamma}{\gamma!} \int_{\mathbb{R}^d} e^{i(x+y)\cdot\xi} \int_0^1 (1-\tau)^{K-1} \sigma^\gamma(x+\tau y,\xi)\hat{f}(\xi)\,d\tau\,d\xi \\
&:= T(x,y) + R(x,y), \qquad (4.2)
\end{aligned}
$$

where we have expanded $\sigma(x+y,\xi)$ in a Taylor series around x. The order K is chosen such that $K\alpha > \min\{0, s+(1-\alpha)(1+3d)\}$. Using (4.2) in (4.1), we obtain

$$
\psi_k(D)\sigma(x,D)f = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{\psi}_k(y) T(x,y)\,dy + (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{\psi}_k(y) R(x,y)\,dy. \qquad (4.3)
$$

We estimate each of the two terms separately.

$$
\begin{aligned}
\int_{\mathbb{R}^d} \hat{\psi}_k(y) T(x,y)\,dy &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{\psi}_k(y) \sum_{|\gamma|\le K-1} \frac{y^\gamma}{\gamma!} \int_{\mathbb{R}^d} e^{i(x+y)\cdot\xi} \sigma^\gamma(x,\xi)\hat{f}(\xi)\,d\xi\,dy \\
&= (2\pi)^{-d/2} \sum_{|\gamma|\le K-1} \frac{1}{\gamma!} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma^\gamma(x,\xi)\hat{f}(\xi) \int_{\mathbb{R}^d} e^{iy\cdot\xi} \hat{\psi}_k(y) y^\gamma\,dy\,d\xi \\
&= (2\pi)^{-d/2} \sum_{|\gamma|\le K-1} \frac{1}{\gamma!} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma^\gamma(x,\xi) \partial_\xi^\gamma \psi_k(\xi) \hat{f}(\xi)\,d\xi. \qquad (4.4)
\end{aligned}
$$

We apply the L_p norm to (4.4). Define $\Psi_k := \sum_{k'} \psi_{k'}$, where the sum is taken over all $k' \in \mathbb{Z}^d \setminus \{0\}$ with $B_{k'}^r \cap B_k^r \neq \emptyset$. Using Minkowski's inequality and the fact that $\Psi_k(\xi) = 1$ on $\mathrm{supp}(\psi_k)$, we get

$$
\begin{aligned}
&(2\pi)^{d/2} \Big\{ \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \hat{\psi}_k(y) T(x,y)\,dy \Big|^p dx \Big\}^{1/p} \\
&\le \sum_{|\gamma|\le K-1} \frac{1}{\gamma!} \Big\{ \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma^\gamma(x,\xi) \partial_\xi^\gamma \psi_k(\xi) \hat{f}(\xi)\,d\xi \Big|^p dx \Big\}^{1/p} \\
&\le \sum_{|\gamma|\le K-1} \frac{1}{\gamma!} \Big\{ \int_{\mathbb{R}^d} \sup_{z\in\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma^\gamma(z,\xi) \partial_\xi^\gamma \psi_k(\xi) \hat{f}(\xi)\,d\xi \Big|^p dx \Big\}^{1/p} \\
&= \sum_{|\gamma|\le K-1} \frac{1}{\gamma!} \Big\{ \int_{\mathbb{R}^d} \sup_{z\in\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma^\gamma(z,\xi) \partial_\xi^\gamma \psi_k(\xi) \Psi_k(\xi) \hat{f}(\xi)\,d\xi \Big|^p dx \Big\}^{1/p}.
\end{aligned}
$$

We now use the relation $(\hat{f}\hat{g})^\vee = f * g$

$$
\begin{aligned}
&\le \sum_{|\gamma|\le K-1} \frac{1}{\gamma!} \Big\{ \int_{\mathbb{R}^d} \sup_{z\in\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \big|(\sigma^\gamma(z,\xi) \partial_\xi^\gamma \psi_k(\xi))^\vee(y)\big| \big|\Psi_k(D) f(x-y)\big|\,dy \Big|^p dx \Big\}^{1/p} \\
&\le \sum_{|\gamma|\le K-1} \frac{1}{\gamma!} \Big\{ \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \sup_{z\in\mathbb{R}^d} \big|(\sigma^\gamma(z,\xi) \partial_\xi^\gamma \psi_k(\xi))^\vee(y)\big| \big|\Psi_k(D) f(x-y)\big|\,dy \Big|^p dx \Big\}^{1/p}
\end{aligned}
$$

and by standard norm estimates for convolutions

$$\leq \sum_{|\gamma|\leq K-1} \frac{1}{\gamma!}\Big\| \sup_{z\in\mathbb{R}^d} (\sigma^\gamma(z,\xi)\partial_\xi^\gamma \psi_k(\xi))^\vee \Big\|_{L_1} \|\Psi_k(D)f\|_{L_p}.$$

Hence, by Lemma 4.2 we may conclude that

$$\left\{ \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \hat{\psi}_k(y) T(x,y)\, dy \right|^p dx \right\}^{1/p} \leq C|\,\sigma|^{(0)}_{d+1,K} \|\Psi_k(D)f\|_{L_p}. \tag{4.5}$$

Now we turn to the second term in (4.3). We let $\mu_k(\xi) = \psi_k(a_k\xi)$, where $a_k := \langle k|k|^{\alpha/(1-\alpha)}\rangle$. Notice that

$$\int_{\mathbb{R}^d} \hat{\psi}_k(y) R(x,y)\, dy = \int_{\mathbb{R}^d} \hat{\mu}_k(y) R(x, a_k^{-1}y)\, dy.$$

We have,

$$\Big| \sum_{|\gamma|=K} \frac{a_k^{-K}}{\gamma!} \int_{\mathbb{R}^d} y^\gamma \hat{\mu}_k(y) \int_{\mathbb{R}^d} e^{i(x+a_k^{-1}y)\cdot\xi} \int_0^1 (1-\tau)^{K-1} \sigma^\gamma(x+a_k^{-1}\tau y,\xi) \hat{f}(\xi)\, d\tau\, d\xi\, dy \Big|$$

$$\leq C a_k^{-K} \sum_{|\gamma|=K} \int_{\mathbb{R}^d} \langle y\rangle^K \big|\hat{\mu}_k(y)\big| \Big| \int_0^1 (1-\tau)^{K-1} \int_{\mathbb{R}^d} e^{i(x+a_k^{-1}y)\cdot\xi} \sigma^\gamma(x+a_k^{-1}\tau y,\xi)\hat{f}(\xi)\, d\xi\, d\tau \Big|\, dy.$$

Using Lemma B.4 with $m = K+d+1+\theta d$ for a fixed $1<\theta<2$ we get,

$$\begin{aligned}
&\leq C' a_k^{-\tilde{K}} \sum_{|\gamma|=K} \int_{\mathbb{R}^d} \frac{\langle y\rangle^{-d-1}}{\langle y\rangle^{\theta d}} \sup_{z\in\mathbb{R}^d} \big|[\sigma^\gamma(z,D)f](x+a_k^{-1}y)\big|\, dy \\
&= C' a_k^{-\tilde{K}} \sum_{|\gamma|=K} \int_{\mathbb{R}^d} \langle y\rangle^{-d-1} \sup_{z\in\mathbb{R}^d} \frac{\big|[\sigma^\gamma(z,D)f](x+a_k^{-1}y)\big|}{\langle y\rangle^{\theta d}}\, dy \\
&\leq C' a_k^{-\tilde{K}} \sum_{|\gamma|=K} \int_{\mathbb{R}^d} \langle y\rangle^{-d-1} \sup_{z,\eta\in\mathbb{R}^d} \frac{\big|[\sigma^\gamma(z,D)f](x+\eta)\big|}{\langle a_k\eta\rangle^{\theta d}}\, dy \\
&\leq C' a_k^{-\tilde{K}} \sum_{|\gamma|=K} \int_{\mathbb{R}^d} \langle y\rangle^{-d-1} \sup_{z,\eta\in\mathbb{R}^d} \frac{\big|[\sigma^\gamma(z,D)f](x+\eta)\big|}{\langle \eta\rangle^{\theta d}}\, dy, \qquad \text{since } \alpha_k \geq 1, \\
&\leq C' a_k^{-\tilde{K}} \sum_{|\gamma|=K} \sup_{z,\eta\in\mathbb{R}^d} \frac{\big|[\sigma^\gamma(z,D)f](x+\eta)\big|}{\langle \eta\rangle^{\theta d}},
\end{aligned}$$

where $\tilde{\tilde{K}} = K\alpha - (1+\theta d)(1-\alpha) \geq K\alpha - (1+2d)(1-\alpha) > s + d(1-\alpha)$. Now,

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} a_k^{sq} \Big\| \int_{\mathbb{R}^d} \hat{\psi}_k(y) R(x,y)\, dy \Big\|_{L_p}^q \Big)^{1/q} \\
&\qquad \leq \Big\{ C \sum_{k\in\mathbb{Z}^d\setminus\{0\}} a_k^{(s-\tilde{K})q} \Big(\sum_{|\gamma|=K} \Big\| \sup_{\eta,z\in\mathbb{R}^d} \frac{\big|[\sigma^\gamma(z,D)f](x+\eta)\big|}{\langle\eta\rangle^{\theta d}} \Big\|_{L_p} \Big)^q \Big\}^{1/q}.
\end{aligned}$$

Since $L^q := C\sum_{k\in\mathbb{Z}^d\setminus\{0\}} a_k^{(s-\tilde{K})q} \le C\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{-d-1}$ is finite, we obtain

$$\begin{aligned}
&= L\sum_{|\gamma|=K}\left\| \sup_{\eta,z\in\mathbb{R}^d} \frac{\big|[\sigma^\gamma(z,D)f](x+\eta)\big|}{\langle\eta\rangle^{\theta d}}\right\|_{L_p(dx)}\\
&= L\sum_{|\gamma|=K}\left\| \sup_{\eta,z\in\mathbb{R}^d} \frac{\big|[\sigma^\gamma(z,D)\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\psi_k(D)f](x+\eta)\big|}{\langle\eta\rangle^{\theta d}}\right\|_{L_p(dx)}\\
&\le L\sum_{|\gamma|=K}\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\left\| \sup_{\eta,z\in\mathbb{R}^d} \frac{\big|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)\big|}{\langle\eta\rangle^{\theta d}}\right\|_{L_p(dx)}.
\end{aligned}$$

We estimate the term $A_k := \big|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)\big|$. Let $f_k(x) := [\Psi_k(D)f](x)$. We have

$$\begin{aligned}
A_k &= \left|\int_{\mathbb{R}^d} (\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(x+\eta-y) f_k(y)\,dy\right|\\
&\le \int_{\mathbb{R}^d} |(\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(x+\eta-y)||f_k(y)|\,dy\\
&\le \sup_{u\in\mathbb{R}^d}\frac{|f_k(u)|}{\langle x-u\rangle^{\theta d}}\int_{\mathbb{R}^d} |(\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(x+\eta-y)|\langle x-y\rangle^{\theta d}\,dy.
\end{aligned}$$

Now, by Petree's inequality, $\langle x-y\rangle^{\theta d} \le 2^{d/(2r)}\langle x-y+\eta\rangle^{\theta d}\langle\eta\rangle^{\theta d}$, so

$$\begin{aligned}
\sup_{z,\eta\in\mathbb{R}^d}\frac{A_k}{\langle\eta\rangle^{\theta d}} &\le C\sup_{\eta\in\mathbb{R}^d}\frac{|f_k(x-\eta)|}{\langle\eta\rangle^{\theta d}}\sup_{z\in\mathbb{R}^d}\int_{\mathbb{R}^d}|(\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(u)|\langle u\rangle^{\theta d}\,du\\
&\le C'\sup_{\eta\in\mathbb{R}^d}\frac{|f_k(x-\eta)|}{\langle\eta\rangle^{\theta d}}|\,\sigma|^{(0)}_{3d+1,K},
\end{aligned}$$

where we used Lemma 4.2 and the fact that $\theta d \le 2d$. Hence,

$$\begin{aligned}
&\sum_{|\gamma|=K}\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\left\| \sup_{\eta,z\in\mathbb{R}^d} \frac{\big|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)\big|}{\langle\eta\rangle^{\theta d}}\right\|_{L_p(dx)}\\
&\qquad= \sum_{|\gamma|=K}\sum_{k\in\mathbb{Z}^d\setminus\{0\}} a_k^{\theta d}\left\| \sup_{\eta,z\in\mathbb{R}^d} \frac{\big|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)\big|}{a_k^{\theta d}\langle\eta\rangle^{\theta d}}\right\|_{L_p(dx)}\\
&\qquad\le C'L|\,\sigma|^{(0)}_{3d+1,K}\sum_{k\in\mathbb{Z}^d\setminus\{0\}} a_k^{\theta d}\left\|\sup_{\eta\in\mathbb{R}^d}\frac{|f_k(x-\eta)|}{\langle a_k\eta\rangle^{\theta d}}\right\|_{L_p}.
\end{aligned}$$

Let

$$\hat{g}_k(\xi) := a_k^d\hat{f}_k(a_k\xi) = a_k^d\Psi_k(a_k\xi)\hat{f}(a_k\xi),$$

and notice that $\mathrm{supp}(\hat{g}_k) \subset B(0,c)$ for some $c>0$ independent of k. The following maximal inequality is proved in Triebel [18, p. 16]

$$\sup_{z\in\mathbb{R}^d}\frac{|g_k(x-z)|}{\langle z\rangle^{\theta d}} \le CM[|g_k|^{1/\theta}(x)]^\theta.$$

Expressing this in terms of f_k, we get

$$\sup_{z\in\mathbb{R}^d}\frac{|f_k(x-z)|}{\langle a_k z\rangle^{\theta d}} \le C\big[(M|f_k|^{1/\theta})(x)\big]^\theta,$$

where C does not depend on k. We apply L_p-norms and use the maximal inequality to obtain

$$\left\|\frac{|f_k(x-z)|}{\langle a_k z\rangle^{\theta d}}\right\|_{L_p(dx)} \le C\left\|\left[(M|f_k|^{1/\theta})(x)\right]^{\theta}\right\|_{L_p(dx)} = C\left\|\left[(M|f_k|^{1/\theta})(x)\right]\right\|^{\theta}_{L_{p\theta}(dx)}$$
$$\le C'\left\||f_k|^{1/\theta}\right\|^{\theta}_{L_{p\theta}(dx)} = C'\|f_k\|_{L_p}.$$

Putting these estimates together yields,

(4.6)
$$\sum_{|\gamma|=K}\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\left\|\sup_{\eta,z\in\mathbb{R}^d}\frac{\left|[\sigma^{\gamma}(z,D)\psi_k(D)f](x+\eta)\right|}{\langle\eta\rangle^{\theta d}}\right\|_{L_p(dx)} \le C''|\,\sigma|^{(0)}_{3d+1,K}\sum_{k\in\mathbb{Z}^d\setminus\{0\}} a_k^{\theta d}\|\Psi_k(D)f\|_{L_p},$$

and consequently

$$\left(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} a_k^{sq}\left\|\int_{\mathbb{R}^d}\hat{\psi}_k(y)R(x,y)\,dy\right\|^q_{L_p}\right)^{1/q} \le C''|\,\sigma|^{(0)}_{3d+1,K}\sum_{k\in\mathbb{Z}^d\setminus\{0\}} a_k^{\theta d}\|\Psi_k(D)f\|_{L_p}. \tag{4.7}$$

Finally, we can put the estimates together to close the case $b=0$ and $s>2d$. We have

$$\begin{aligned}
&\|\sigma(x,D)f\|_{M^{s,\alpha}_{p,q}}\\
&\asymp \left\|a_k^s\|\psi_k(D)\sigma(x,D)f\|_{L_p}\right\|_{\ell^q(\mathbb{Z}^d\setminus\{0\})}\\
&\le C\left\{\left\|a_k^s\left\|\int_{\mathbb{R}^d}\hat{\psi}_k(y)T(x,y)\,dy\right\|_{L_p(dx)}\right\|_{\ell_q}+\left\|a_k^s\left\|\int_{\mathbb{R}^d}\hat{\psi}_k(y)R(x,y)\,dy\right\|_{L_p(dx)}\right\|_{\ell_q}\right\}\\
&\le C'\left(|\,\sigma|^{(0)}_{d+1,K}\left\|\langle a\rangle_k^s\left\|\Psi_k(D)f\right\|_{L_p}\right\|_{\ell_q}+|\,\sigma|^{(0)}_{3d+1,K}\left\|a_k^{\theta d}\left\|\Psi_k(D)f\right\|_{L_p}\right\|_{\ell_q}\right)\\
&\le C'|\,\sigma|^{(0)}_{3d+1,K}\left[\|f\|_{M^{s,\alpha}_{p,q}}+\|f\|_{M^{\theta d,\alpha}_{p,q}}\right],\qquad\text{and since }\theta d<2d<s,\\
&\le C''|\,\sigma|^{(0)}_{3d+1,K}\|f\|_{M^{s,\alpha}_{p,q}}.
\end{aligned}$$

This concludes the proof of the theorem. ☐

Remark 4.3. A closer examination of the arguments used in the proof reveals that there exist $M,N>0$ (depending on s,q, and ρ) such that the norm of the operator

$$\sigma(x,D)\colon M^{s,\alpha}_{p,q}(\mathbb{R}^d)\to M^{s-b,\alpha}_{p,q}(\mathbb{R}^d)$$

is bounded by $C|\,\sigma|^{(b)}_{M,N}$, with C a constant.

5. Elliptic pseudodifferential operators

In this final section we consider an application of the result in the previous section to elliptic pseudodifferential operators. Let us introduce some notation. Let

$$S^{\infty}_{\rho,\delta} := \bigcup_{m\in\mathbb{R}} S^b_{\rho,\delta},\quad\text{and}\quad S^{-\infty}_{\rho,\delta} := \bigcap_{m\in\mathbb{R}} S^b_{\rho,\delta}$$

We define the twisted product $\sigma\#\tau$ of two symbols $\sigma,\tau\in S^{\infty}_{\rho,\delta}$ by

$$\sigma\#\tau(x,D)=\sigma(x,D)\tau(x,D).$$

We say that a pseudodifferential operator with symbol $\sigma\in S^b_{\rho,\delta}$ is *elliptic* if there exist two constants $a,c>0$ such that

$$|\sigma(x,\xi)|\ge a\langle\xi\rangle\qquad\text{for }\langle\xi\rangle\ge c.$$

We have the following nice characterization, see [7, Theorem 2.64].

Theorem 5.1. *Suppose $\sigma \in S^b_{\rho,\delta}$, with $\rho > \delta$. Then σ is elliptic if and only if there exists $\tau \in S^{-b}_{\rho,\delta}$ such that $1 - \sigma\#\tau$ and $1 - \tau\#\sigma$ are both in $S^{-\infty}_{\rho,\delta}$.*

Let $M^{-\infty,\alpha}_{p,q}(\mathbb{R}^d) = \cup_{s\in\mathbb{R}} M^{s,\alpha}_{p,q}(\mathbb{R}^d)$. Using Theorem 5.1 and the result from the previous section we have

Theorem 5.2. *Suppose $\sigma \in S^b_{\rho,0}$ is elliptic, and $f \in M^{-\infty,\alpha}_{p,q}(\mathbb{R}^d)$. If $\sigma(\cdot, D)f \in M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ for some $s \in \mathbb{R}$, then $f \in M^{s+b,\alpha}_{p,q}(\mathbb{R}^d)$.*

Proof. Let $S = \sigma(\cdot, D)$, and let $T = \tau(\cdot, D)$ be as in Theorem 5.1. Notice that $f = T(Sf)+(I-TS)f$. By Theorem 4.1, T maps $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ to $M^{s+b,\alpha}_{p,q}(\mathbb{R}^d)$ and $(I-TS)$ maps $M^{-\infty,\alpha}_{p,q}(\mathbb{R}^d)$ to $M^{s+b,\alpha}_{p,q}(\mathbb{R}^d)$. □

The following example will conclude the paper.

Example 5.3. Consider the heat operator L given by

$$L(u) := \frac{\partial u}{\partial t} - \sum_{j=1}^{d} \frac{\partial 2u}{\partial x_j 2}.$$

The symbol of L is given by

$$l(\tau, \xi) = (i\tau + |\xi|^2), \qquad (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d,$$

so one can easily verify that L is elliptic. We consider an approximate inverse P to L with symbol

$$a(\tau, \xi) = (i\tau + |\xi|^2)^{-1}\eta(\tau, \xi), \qquad (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d,$$

where η is a smooth cut-off function that vanishes near the origin and equals 1 for large (τ, ξ). One can verify that $a(\tau, \xi) \in S^{-1}_{1/2,0}(\mathbb{R}^{d+1})$, see [15, Chap. VII]. Hence, if $u \in M^{-\infty,1/2}_{p,q}(\mathbb{R}^{d+1})$, $1 < p, q < \infty$, and $P(u) \in M^{s,1/2}_{p,q}(\mathbb{R}^{d+1})$, then $u \in M^{s-1,1/2}_{p,q}(\mathbb{R}^{d+1})$.

Appendix A. Admissible coverings

In this section we discuss a general construction of an α-covering of $\mathbb{R}^d$. This type of covering was considered in [8] and in [14]. A Proof of Lemma A.1 below can be found in [8], but since Gröbner's work has never been published, we have included a proof for the sake of completeness. Construction of α-coverings are also considered (from another perspective) in [14].

Notice that the set of balls $\{B(z, \sqrt{d})\}_{z\in\mathbb{Z}^d\setminus\{0\}}$ is an admissible 0-covering of $\mathbb{R}^d$. Define for some $\beta \in (-1, \infty)$, the bijection δ_β on $\mathbb{R}^d$ by $\delta_\beta(\xi) := \xi\,|\xi|^\beta$ (with inverse $\sigma_{\beta'}$, $\beta' = -\beta/(1 + \beta)$). Since the set $\{B(z, R)\}_{z\in\mathbb{Z}^d\setminus\{0\}}$ is admissible for $R \geq \sqrt{d}$, so is $\{\delta_\beta(B(z, R))\}_{z\in\mathbb{Z}^d\setminus\{0\}}$. Moreover, we have the following result.

Lemma A.1. *Suppose $\beta \geq 0$. Given $R > 0$, there exists an $r > 0$, such that*

$$\delta_\beta(B(z, R)) \subseteq B(\delta_\beta(z), r|z|^\beta), \qquad \text{for all } z \in \mathbb{R}^d, \text{ with } |z| \geq 1. \tag{A.1}$$

Likewise, given $r > 0$ there exists an $R > 0$, such that

$$B(\delta_\beta(z), r|z|^\beta) \subseteq \delta_\beta(B(z, R)), \qquad \text{for all } z \in \mathbb{R}^d. \tag{A.2}$$

Proof. The proof is based on the following observation. For two points $x, z \in \mathbb{R}^d$ and $\beta \in (-1, \infty)$, we have

$$\begin{aligned} |\delta_\beta(x) - \delta_\beta(z)| &= \Big| x|x|^\beta - z|z|^\beta \Big| \\ &\le \Big| x|x|^\beta - x|z|^\beta \Big| + \Big| x|z|^\beta - z|z|^\beta \Big| \\ &= |x| \Big| |x|^\beta - |z|^\beta \Big| + |z|^\beta |x - z| \\ &= \Big(|\beta||x||\tilde{x}|^{\beta-1} + |z|^\beta \Big) |x - z| \end{aligned} \tag{A.3}$$

for some $\tilde{x} \in L(x, z)$, by the mean value theorem.

Given $R > 0$, suppose $x \in B(z, R)$. Then (A.3) yields $|\delta_\beta(x) - \delta_\beta(z)| \le r|z|^\beta$ for some $r > 0$ depending only on β and R. Now, take any $y \in \delta_\beta(B(z, R))$, i.e., $y = \delta_\beta(x)$ for some $x \in B(z, R)$. Then $|y - \delta_\beta(z)| \le r|z|^\beta$, which proves (A.1).

We turn to (A.2). Suppose first that $|z| \le K$ for some $K > r^{1+\beta}$. Then it is easy to verify that there exists a radius $P > 0$ such that $B(\delta_\beta(z), r|z|^\beta) \subset B(0, P)$ for all z. Likewise, there exists a radius R such that $B(0, P) \subset \delta_\beta(B(z, R))$ for all z. This proves (A.2) for $|z| \le K$.

Suppose now that $|z| > r^{1+\beta}$. Recall that $\delta_\beta^{-1} = \sigma_{\beta'}$, where $\beta' := -\beta/(\beta + 1)$. Thus, to show the inclusion (A.2) is equivalent to show that

$$\sigma_{\beta'}(B(z, r|z|^{-\beta'})) \subseteq B(\sigma_{\beta'}(z), R). \tag{A.4}$$

Suppose $x \in B(z, r|z|^{-\beta'})$ for some $\beta' > -1$, then $(1 - r|z|^{-(1+\beta')})|z| \le |x| \le (1 + r|z|^{-(1+\beta')})|z|$. Since $1 + \beta = (1 + \beta')^{-1}$, (A.3) yields

$$|\delta_{\beta'}(x) - \delta_{\beta'}(z)| \le R|z|^{\beta'}|z|^{-\beta'} = R$$

for some $R > 0$ depending only on r and β'. Now, take any $y \in \delta_{\beta'}(B(z, r|z|^{-\beta'}))$, i.e., $y = \delta_{\beta'}(x)$ for some $x \in B(z, r|z|^{-\beta'})$. Then, $|y - \delta_{\beta'}(z)| \le R$, which proves (A.4). □

Remark A.2. By (A.1) there exists a radius r_1 such that $\mathbb{R}^d \subset \cup_{z \in \mathbb{Z}^d \setminus \{0\}} B(\delta_\beta(z), r|z|^\beta)$ for all $r \ge r_1$. Fix such an r and let $R := R(r)$ be given such that (A.2) holds. Then, since $\{\delta_\beta(B(z, R(r)))\}_{z \in \mathbb{Z}^d \setminus \{0\}}$ is an admissible covering of $\mathbb{R}^d$, so is $\{B(\delta_\beta(z), r|z|^\beta)\}_{k \in \mathbb{Z}^d \setminus \{0\}}$.

Suppose $\beta \ge 0$, and let $\alpha = \beta/(\beta + 1)$. Then it is easy to see that $|B(\delta_\beta(z), r|z|^\beta)| \asymp \langle y \rangle^{d\alpha}$ for all $y \in B(\delta_\beta(z), r|z|^\beta)$ independent of $z \in \mathbb{Z}^d \setminus \{0\}$. Therefore, by Remark A.2,

$$\{B(\delta_\beta(z), r|z|^\beta)\}_{k \in \mathbb{Z}^d \setminus \{0\}} \tag{A.5}$$

is an α-covering for any $r > r_1$.

Appendix B. Some Technical Lemmas

In this brief section, we have incuded the proofs of some of the technical lemmas used in Section 2.1.

The following two lemmas are well-known, but we have included the proofs for the sake of completeness. We let $W^{K,1}(\mathbb{R}^d)$ denote the Sobolev space of functions with derivaties of order up to K in $L_1(\mathbb{R}^d)$.

Lemma B.1. *Let $K \in \mathbb{N}$ and suppose $h \in W^{K,1}(\mathbb{R}^d)$. Then there exists a constant $C_K < \infty$ such that*

$$\langle \xi \rangle^K |\hat{h}(\xi)| \le C \sum_{|\beta| \le K} \|\partial^\beta h\|_{L_1} \le C \|h\|_{W^{K,1}}.$$

Proof. It is straightforward to verify that $\langle\xi\rangle^K \leq C_K \sum_{|\beta|\leq K} |(-i\xi)^\beta|$ for some finite constant C_K. We have

$$\langle\xi\rangle^K|\hat{h}(\xi)| \leq C_K \sum_{|\beta|\leq K} |(-i\xi)^\beta \hat{h}(\xi)| \leq C_K \sum_{|\beta|\leq K} \|(\partial^\beta h)^\wedge\|_{L_\infty} \leq C_K \sum_{|\beta|\leq K} \|\partial^\beta h\|_{L_1}.$$

□

Lemma B.2. *Let $f, g \in C^\infty(\mathbb{R}^d)$ and $\gamma \in \mathbb{R}$. Suppose for each multi-index β there exists a constant $C_\beta < \infty$ such that $|\partial^\beta f(x)|, |\partial^\beta g(x)| \leq C_\beta \langle x\rangle^{\gamma|\beta|}$. If $0 < c \leq |g(x)| \leq C < \infty$, then there exists constants C'_β such that*

$$\left|\partial^\beta\left(\frac{f}{g}\right)(x)\right| \leq C'_\beta\langle x\rangle^{\gamma|\beta|}.$$

Proof. Clearly, $f/g \in C^\infty(\mathbb{R}^d)$ so the mixed partial derivatives can be shuffled as we please. Let $\mathcal{K}_K$ be the family of pairs of functions (f_1, g_1) with $f_1, g_1 \in C^\infty(\mathbb{R}^d)$ satisfying

$$\sup_x \langle x\rangle^{-\gamma(|\beta|+K)}|\partial^\beta f_1(x)|,\ \sup_x \langle x\rangle^{-\gamma|\beta|}|\partial^\beta g_1(x)| < \infty,$$

and $0 < \inf_x |g_1(x)| \leq \sup_x |g_1(x)| < \infty$. Let $(f_1, g_1) \in \mathcal{K}_K$, and notice that

$$\partial_{x_j}\left(\frac{f_1}{g_1}\right) = \frac{[\partial_{x_j} f_1]g_1 - f_1\partial_{x_j} g_1}{g_1^2(x)} := \frac{f_2}{g_2}.$$

By Leibniz' rule,

$$\partial^\beta([\partial_{x_j} f_1]g_1) = \sum_\eta \binom{\beta}{\eta}(\partial^\eta[\partial_{x_j} f_1])\partial^{\beta-\eta} g_1,$$

and similarly for $\partial^\beta(f_1\partial_{x_j} g_1)$, so $\sup_x\langle x\rangle^{-\gamma(|\beta|+K+1)}|\partial^\beta f_2(x)| < \infty$, and

$$\sup_x \langle x\rangle^{-\gamma(K+1)}\left|\partial_{x_j}\left(\frac{f_1}{g_1}\right)\right| < \infty.$$

Moreover, one easily checks that $(f_2, g_2) \in \mathcal{K}_{K+1}$. An iteration of this argument starting from $f_1/g_1 = f/g$, $(f, g) \in \mathcal{K}_0$, proves the claim. □

The final two lemmas give estimates on the BAPU after each function has been dilated to have support near the origin. The estimates are used in the proof of Theorem 4.1.

Lemma B.3. *Define $\widetilde{\psi}_k(\xi) = \psi_k(|c_k|^\alpha\xi + c_k)$. Then for every $\beta \in \mathbb{N}^d$ there exists a constant C_β independent of $k \in \mathbb{Z}^d \setminus \{0\}$ such that*

$$|\partial_\xi^\beta \widetilde{\psi}_k(\xi)| \leq C_\beta \chi_{B(0,r)}(\xi).$$

Proof. Notice that

$$\widetilde{\psi}_k(\xi) = \frac{\Phi(\xi)}{\sum_{k'} \Phi(|c_k|^\alpha|c_{k'}|^{-\alpha}(\xi - c_{k'}) + c_k)}.$$

Thus, the result follows by Lemma B.2 □

Lemma B.4. *Let $c_k := k|k|^{\alpha/(1-\alpha)}$, $k \in \mathbb{Z}^d\backslash\{0\}$, and define $\mu_k(\xi) = \psi_k(a_k\xi)$, where $a_k := \langle c_k\rangle$. Then for every $m \in \mathbb{N}$ there exists a constant C_m independent of k such that*

$$|\hat{\mu}_k(y)| \leq C_m a_k^{(m-d)(1-\alpha)}\langle y\rangle^{-m}.$$

Proof. By Proposition 2.4 we have for any $\beta \in \mathbb{N}^d$

$$|\partial^\beta \mu_k(\xi)| = a_k^{|\beta|} |(\partial^\beta \psi_k)(a_k \xi)| \leq C a_k^{(1-\alpha)|\beta|} \chi_{B(1,a_k^{-(1-\alpha)})}(\xi),$$

since $\chi_{B_k^r}(a_k\xi) = \chi_{B(1,a_k^{-(1-\alpha)})}(\xi)$. By Lemma B.1 we get

$$\langle y \rangle^m |\hat{\mu}_k(y)| \leq C_m \sum_{|\beta| \leq m} \|\partial^\beta \mu_k\|_{L_1} \leq C'_m a_k^{-(1-\alpha)d} \sum_{|\beta| \leq m} a_k^{(1-\alpha)|\beta|} \leq C''_m a_k^{-(1-\alpha)d} a_k^{(1-\alpha)m}.$$

□

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