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INEQUALITIES FOR INTEGRAL MEANS OVER SYMMETRIC SETS

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ABSTRACT. We prove that the integral of n functions over a symmetric set L in $\mathbb{R}^n$, with additional properties, increases when the functions are replaced by their symmetric decreasing rearrangements. The result is known when L is a centrally symmetric convex set, and our result extends it to nonconvex sets. We deduce as consequences, inequalities for the average of a function whose level sets are of the same type as L, over measurable sets in $\mathbb{R}^n$. The average of such a function on E is maximized by the average over the symmetric set E^*.

1. INTRODUCTION

If f is a real symmetric decreasing function ($f(x) = f(-x)$ and $f(x) \geq f(y)$ whenever $|x| \leq |y|$), then it is obvious that the integral of f over a set I in $\mathbb{R}$ is maximized over all sets $I \subset \mathbb{R}$ of the same measure when I is a symmetric interval, I^*, i.e.

$$\int_I f(x)\,dx \leq \int_{I^*} f(x)\,dx.$$

More generally, the above inequality is a special case of the Hardy-Littlewood inequality [14], $\int_{\mathbb{R}} f(x)g(x)\,dx \leq \int_{\mathbb{R}} f^*(x)g^*(x)\,dx$, where $f^*(x)$ and $g^*(x)$ are the symmetric decreasing rearrangements of $f(x)$ and $g(x)$, respectively.

Another special case of this inequality is the following:

$$\int_{I^*} f(x)\,dx \leq \int_{I^*} f^*(x)\,dx,$$

with I^* a symmetric interval.

The area of rearrangement inequalities is very rich and we mention here a few results related to our work. The Riesz rearrangement inequality involves three positive functions and states that:

$$\int_{\mathbb{R}^n} f(x)g(y)h(x-y)\,dxdy \leq \int_{\mathbb{R}^n} f^*(x)g^*(y)h^*(x-y)\,dxdy.$$

A generalization of this inequality to n functions is due to Brascamp-Lieb-Luttinger [6].

Recently, two types of rearrangement inequalities have been studied: the extended Hardy-Littlewood inequality:

$$\int \Psi(f_1(x), \dots, f_m(x))\,dx \leq \int \Psi(f_1^*(x), \dots, f_m^*(x))\,dx, \tag{1.1}$$

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and the extended Riesz inequality:

$$\int \dots \int \Psi(f_1(x_1), \dots f_m(x_m) \prod_{i<j} K_{ij}(d(x_i, x_j))\, dx_1 \dots dx_m \tag{1.2}$$

$$\leq \int \dots \int \Psi(f_1^*(x_1), \dots, f_m^*(x_m) \prod_{i<j} K_{ij}(d(x_i, x_j))\, dx_1 \dots dx_m$$

In (1.1) and (1.2) the integrals are taken over $\mathbb{R}^n$.

The main condition on Ψ required for (1.1) and (1.2) to hold was found by Lorentz [16], and states that:

$$\Psi(\mathbf{x} + te_i + se_j) + \Psi(\mathbf{x}) \geq \Psi(\mathbf{x} + te_i) + \Psi(\mathbf{x} + se_j),$$

for all $i \neq j$, $s, t > 0$ and all $\mathbf{x} \in \mathbb{R}^m_+$. This is a monotonicity condition of order 2. Indeed, if Ψ is C^2, then the above condition is equivalent to all mixed second order partials being nonnegative ($\partial_{ij}\Psi \geq 0$).

The case $m = 2$ for (1.1) is due to Crowe-Zweibel-Rosenbloom [10] for Ψ continuous which vanishes on the boundary of $\mathbb{R}^n_+$. They represented Ψ as the distribution function of a Borel measure μ,

$$\Psi(s_1, s_2) = \mu([0, s_1) \times [0, s_2)),$$

and they used this together with Fubini's theorem to write

$$\int \Psi(f_1(x), f_2(x))\, dx = \int_{\mathbb{R}^2_+} (\int \chi_{\{f_1(x)>s_1\}} \chi_{\{f_2(x)>s_2\}}\, dx)\, d\mu(s_1, s_2),$$

thus reducing (1.1) to the case when Ψ is the product of characteristic functions.

The corresponding result for (1.2) was proved by Almgren and Lieb [1]. For $m > 2$ the inequality (1.1) is due to Brock [7]. Inequality (1.2) was proved for special cases of Ψ in [9] and [17], and for Ψ continous in [12]. In a recent paper [8], Burchard and Hajaiej removed the continuity assumption on Ψ in (1.1) and (1.2).

In this paper we are interested in inequalities over a set L in $\mathbb{R}^n$ of n functions $f_1, \dots, f_n$ of the form:

$$\int_L f_1(x_1) \cdots f_n(x_n)\, dx \leq \int_L f_1^*(x_1) \cdots f_n^*(x_n)\, dx, \tag{1.3}$$

for L satisfying certain symmetry conditions.

It is an interesting unsolved problem to characterize the sets L for which inequality (1.3) holds for all nonnegative functions f_j.

Of course, if $L = \mathbb{R}^n$ then the above inequality trivially becomes an equality since each f_i is equimeasurable to f_i^*. So, the interesting cases are when L is not the whole $\mathbb{R}^n$. When L is a centrally symmetric convex set, the inequality (1.3) is essentially due to Pfiefer [18]. We will prove (1.3) for some sets L which are not necessarily convex, thus extending Pfiefer's result. Then, in section 3 we give an example of a nonconvex centrally symmetric set L, for which reverse inequality in (1.3) holds. The main ingredient in the proof of (1.3) is the Brunn-Minkowski inequality regarding the volume of the convex sum of two sets, which is mentioned in section 2.

Following the method of Crowe-Zweibel-Rosenbloom, once inequality (1.3) is established, one can replace the product of the n functions by a function Ψ, which can be thought of as the distribution function of a Borel measure μ on $\mathbb{R}^n_+$, $\Psi(y_1, \dots, y_n) = \mu([0, y_1) \times [0, y_2) \times \dots \times [0, y_n))$. If Ψ is n times continuously differentiable, then

$\frac{\partial^n \Psi}{\partial x_1 \ldots \partial x_n} \geq 0$. For another proof of the fact that Ψ can be approximated by products of characteristic functions see also [11]. Thus, the more general inequality holds:

$$\int_L \Psi(f_1(x_1), \ldots, f_n(x_n))\, dx \leq \int_L \Psi(f_1^*(x_1), \ldots, f_n^*(x_n))\, dx.$$

A consequence of (1.3) is that

$$\int_E h(x)\, dx \leq \int_{E^*} h(x)\, dx, \tag{1.4}$$

where h has level sets of type L, E is a product set (compact rectangle) in $\mathbb{R}^n$ and E^* is obtained from E using symmetrization, as defined in the next section.

Pfiefer used the reverse inequality in (1.4) for h quasiconvex, as an application to the expected value of the volume of a simplex. Let v_r denote the $r-$dimensional volume of the $r-$simplex formed by $r+1$ points chosen at random from a measurable set K. If $M(K)$ is the expected value of the random variable v_r and B is a ball with the same volume as K, Blaschke showed that $M(K) \geq M(B)$, for K any compact convex set, with equality if and only if K is an ellipsoid for $r = n$. Shöpf [19] showed that, if h is a strictly increasing function on $\mathbb{R}_+$, then the expected value of $h \circ v_r$, denoted by $M_h(K)$, satisfies $M_h(K) \geq M_h(B)$.

Pfiefer generalized the above results to arbitrary measurable sets, K, and showed that equality holds when K is a ball almost everywhere ($r < n$), or an ellipsoid ($r = n$).

The method used by Pfiefer extends partially to the case when $h \circ v_r$ is replaced by $h \circ v_r \circ \tau^{-1}$, with τ a coordinate transformation which is concave in each coordinate. If we denote by $M_h^\tau(K)$ the expected value of the composition $h \circ v_r$ at $\tau^{-1}(p_i)$, $i = 0, \ldots, r$, then we can show that $M_h^\tau(K) \geq M_h^\tau(\tilde{K})$, where $\tilde{K}$ is obtained by Steiner symmetrizing K along the coordinate axes, and h is increasing, or $M_h^\tau(K) \leq M_h^\tau(\tilde{K})$, if h is decreasing. The idea is to prove $M_h^\tau(K) \geq M_h^\tau(\tilde{K})$, with $\tilde{K}$ replaced by K^*, where K^* is the Steiner symmetrization of K with respect to any coordinate hyperplane.

2. Preliminary results

For a finite measurable subset A_k of $\mathbb{R}$ we define A_k^* to be the open interval $(-|A_k|/2, |A_k|/2)$, where $|A_k|$ denotes the $1-$dimensional Lebesgue measure of A_k. The process of obtaining A_k^* from A_k is called the Steiner symmetrization of A_k with respect to the origin.

For $E \subset \mathbb{R}^n$ a *compact rectangle*, i.e., $E = A_1 \times \ldots \times A_n$ where each A_i is a compact set in $\mathbb{R}$, we define $E^* = A_1^* \times \ldots \times A_n^*$. Therefore E^* is obtained by symmetrizing each coordinate set.

A subset K of $\mathbb{R}^n$ is called convex if $\lambda x + (1-\lambda)y \in K$ whenever $x, y \in K$ and $0 < \lambda < 1$. If, in addition $K = -K$, then we call K a centrally symmetric convex set.

A real-valued function f defined on a convex subset K of $\mathbb{R}^n$ is called quasiconcave if each level set $K_t = \{x \in K \mid f(x) > t\}$ is convex. If, in addition, each K_t is centrally symmetric, then we call f a centrally symmetric quasiconcave function. A function f is called quasiconvex if $-f$ is quasiconcave.

Recall that f is concave on K if and only if $x, y \in K$ and $0 < \lambda < 1$ imply $f(\lambda x+(1-\lambda)y) \geq \lambda f(x)+(1-\lambda)f(y)$. Correspondingly, f is quasiconcave on K if $f(\lambda x+(1-\lambda)y) \geq \min(f(x), f(y))$. In particular, a concave function is quasiconcave. Positive quasiconcave functions are also called unimodal functions.

Anderson [2] showed that if E is a centrally symmetric convex set in $\mathbb{R}^n$ and f a positive centrally symmetric quasiconcave function then

$$\int_E f(x+ky)dx \geq \int_E f(x+y)dx, \text{ for } 0 \leq k \leq 1,$$

or equivalently $\int_{E+ky} f(x)dx \geq \int_{E+y} f(x)dx$, with equality for $k < 1$ if $(E+y)\cap K_t = E \cap K_t + y$, for every t.

The proof is reduced to f being the characteristic function of a level set and uses the Brunn-Minkowski inequality [13]

$$|(1-\lambda)E_1 + \lambda E_2|^{1/n} \geq (1-\lambda)|E_1|^{1/n} + \lambda |E_2|^{1/n} \tag{2.1}$$

with E_1, E_2 nonempty measurable subsets of $\mathbb{R}^n$ and $0 < \lambda < 1$. Here $|\cdot|$ indicates the $n-$ dimensional volume of the set and $(1-\lambda)E_1 + \lambda E_2$ is the set obtained by taking all linear combinations $(1-\lambda)x + \lambda y$, with $x \in E_1$ and $y \in E_2$.

Using Anderson's result and the method of coalescing rectangles, Pfiefer [18] proved, for K a centrally symmetric convex set in $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$ a compact rectangle, the following inequality:

$$|E^* \cap K| \geq |E \cap K|. \tag{2.2}$$

Let $f : \mathbb{R} \to \mathbb{R}_+$ be a non-negative measurable function, which vanishes at ∞ in the following sense:

$$\mu(\{f > t\}) < \infty, \qquad \forall\, t > 0 \tag{2.3}$$

where we write

$$\{f > t\} := \{x \in \mathbb{R} |\, f(x) > t\}.$$

Its *distribution function* λ_f is defined to be

$$\lambda_f(t) = |\{f > t\}|, \qquad t \in [0, \infty).$$

Two functions f and g are said to be *equimeasurable* if they have the same distribution function and we write $f \sim g$. Functions which are equimeasurable are also said to be *rearrangements* of each other. If f is in L^p, i.e. $\int f^p < \infty$, $p > 0$, then f satisfies (2.3).

Given $f : \mathbb{R} \to \mathbb{R}_+$ satisfying (2.3) we define its *symmetric decreasing rearrangement* $f^* : \mathbb{R} \to \mathbb{R}_+$ as follows :

$$f^*(x) = \inf\{t |\, \lambda_f(t) \leq 2|x|\},$$

It is immediate that $f(x) = f(-x)$ and f decreases as $|x|$ increases. Moreover, f^* is lower semicontinous and $f^* \sim f$. See, e.g., [4] for a more general exposition. In particular, if f is the characteristic function of a compact set A in $\mathbb{R}$, then f^* is the characteristic function of A^* a.e., as defined previously.

For a nonnegative function f we use the *layer-cake representation* to write f in terms of the characteristic functions of its level sets [15]:

$$f(x) = \int_0^\infty \chi_{\{f>t\}}(x)\, dt \tag{2.4}$$

The advantage of this representation is that $\chi_{\{f^*>t\}} = \chi_{\{f>t\}^*}$ and thus, $f^*(x) = \int_0^\infty \chi_{\{f^*>t\}}(x)\,dt = \int_0^\infty \chi_{\{f>t\}^*}(x)\,dt$.

3. Rearrangement inequalities

The following lemma can be deduced from Pfiefer's inequality (2.2).

Lemma 3.1. *Let K be a centrally symmetric convex set in $\mathbb{R}^n$ and $g_1,\dots,g_n : \mathbb{R} \to \mathbb{R}_+$ be n nonnegative functions satisfying (2.3). Then, the following inequality holds:*

$$\int_K g_1(y_1)\cdots g_n(y_n)\,dy \le \int_K g_1^*(y_1)\cdots g_n^*(y_n)\,dy.$$

Here, and in what follows, $dy = dy_1 \dots dy_n$.

The next inequality is a consequence of the Brunn-Minkowski inequality (2.1), and it will be used in the proof of our main result.

Lemma 3.2. *Let K be as in Lemma 3.1, and $g_2^*,\dots,g_n^*$ be positive symmetric decreasing functions on $\mathbb{R}$. For every $x \in \mathbb{R}$, define the $x-$slice of K by:*

$$K(x) = \{(x_2,\dots,x_n) |\, (x, x_2,\dots,x_n) \in K\},$$

with $K(x)$ possibly empty. If, for every $x \in \mathbb{R}$ we define

$$f(x) = \int_{K(x)} g_2^*(x_2)\cdots g_n^*(x_n)\,dx_2\dots dx_m,$$

then f is symmetric decreasing on $\mathbb{R}$.

Proof. It is easy to see that $f(x) = f(-x)$, using the fact that $K(-x) = -K(x)$ and each g_i^* is symmetric decreasing.

To prove that f is non-increasing on $\mathbb{R}_+$, we will use (2.1). Let $0 \le x < y$ and we will show that $f(y) \le f(x)$. Using the layer-cake representation (2.4), it is enough to assume that each g_i^* is the characteristic function of a symmetric interval I_i. Then $f(x) = |K(x) \cap E|$, the $(n-1)-$dimensional volume of $K(x) \cap E$, with $E = I_2 \times \dots \times I_n$, and $I_2 = I_2^*, \dots, I_n = I_n^*$. Obviously, E is a centrally symmetric convex set.

Since K is convex, we have for every $0 < \lambda < 1$ that

$$\lambda K(y) + (1-\lambda)K(-y) \subset K(\lambda y + (1-\lambda)(-y)). \tag{3.1}$$

Next, choose λ such that $x = \lambda y + (1-\lambda)(-y)$. Using (2.1) and the fact that $f(y) = f(-y)$, we obtain:

$$|\lambda[K(y) \cap E] + (1-\lambda)[K(-y) \cap E]| \ge |K(y) \cap E| \tag{3.2}$$

Using (3.1) we conclude that $f(x) \ge f(y)$ and the proof is now complete. □

Next, we introduce the Hardy-Littlewood-Pólya preorder relation $\prec$. For g and h positive functions satisfying (2.3) we say that $g \prec h$ if and only if $\int_0^t g^*(s)\,ds \le \int_0^t h^*(s)\,ds$, for all $t > 0$. The last inequality is an inequality between maximal functions. See, e.g., [5].

The following assertions are equivalent:

(a) $g \prec h$
(b) $\int_{\mathbb{R}} fg^* \le \int_{\mathbb{R}} fh^*$, for every f positive symmetric decreasing on $\mathbb{R}$
(c) $\int_{\mathbb{R}} fg \le \int_{\mathbb{R}} f^*h^*$, for every positive f satisfying (2.3)

(d) $\int_{\mathbb{R}} \Phi(g(x))\,dx \le \int_{\mathbb{R}} \Phi(h(x))\,dx$, for every increasing convex function Φ [3].

We are now ready to state and prove our main result.

Theorem 3.3. *Let K be a centrally symmetric convex set in $\mathbb{R}^n$, $\tau_1, \dots, \tau_n : \mathbb{R} \to \mathbb{R}$ be n odd increasing functions, concave on the interval $[0, \infty)$, and $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}_+$ be n functions vanishing at ∞ in the sense of (2.3). Define $L = \tau(K)$, the image of K under the coordinate transformation $\tau(x) = (\tau_1(x_1), \dots, \tau_n(x_n))$, where $x = (x_1, \dots, x_n)$, and*

$$I[f_1, \dots, f_n] = \int_L f_1(x_1) \cdots f_n(x_n)\,dx, \tag{3.3}$$

with $dx = dx_1 \dots dx_n$. Under these conditions the following inequality holds:

$$I[f_1, \dots, f_n] \le I[f_1^*, \dots, f_n^*]. \tag{3.4}$$

First let us notice that Theorem 3.3 is a generalization of Lemma 3.1 to nonconvex sets. Let us consider a simple example. Take K to be the rectangle in Figure 1 obtained by rotating the rectangle $\{(x, y) | -3\sqrt{2}/2 \le x \le 3\sqrt{2}/2, -1/\sqrt{2} \le y \le 1/\sqrt{2}\}$ by a positive angle of $\pi/4$. Define $\tau(x, y) = (x, y^{1/3})$ and $L = \tau(K)$. Then L is a concave set as shown in Figure 1.

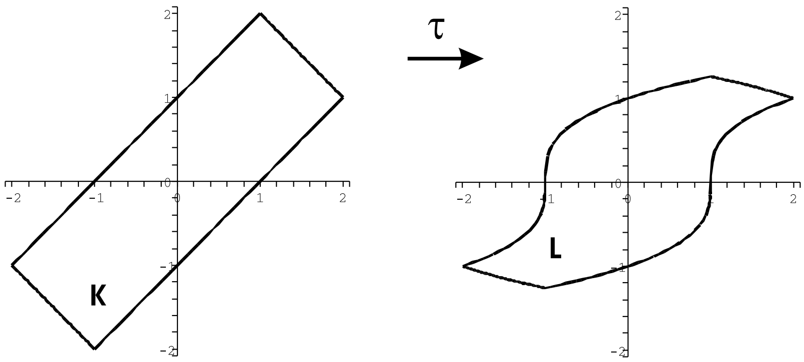

Figure 1

These being said, we now proceed to prove Theorem 3.3.

Proof. Since τ_i are concave functions, they are locally absolutely continuous, and using change of coordinates we can rewrite (3.3) as

$$I[f_1, \dots, f_n] = \int_K \prod_{i=1}^{n} (f_i \circ \tau_i)(y_i) {\tau'}_i(y_i)\,dy.$$

Each function ${\tau'}_i$, defined a.e., is symmetric decreasing, since τ_i is an odd function, concave on the interval $[0, \infty)$.

We claim: For every measurable set $I \subset \mathbb{R}$, we have

$$\int_I (f_i \circ \tau_i)(x) {\tau'}_i(x)\,dx \le \int_{I^*} (f_i^* \circ \tau_i)(x) {\tau'}_i(x)\,dx. \tag{3.5}$$

The claim is equivalent to,

$$\int_{\tau_i(I)} f_i(y)dy \leq \int_{\tau_i(I^*)} f_i^*(y)dy.$$

Because $\int_I \tau'_i(y)dy \leq \int_{I^*} \tau'_i(y)dy$ (here we use the fact that τ'_i is symmetric decreasing), it follows that $|\tau_i(I^*)| \geq |\tau_i(I)|$. Now since $\tau_i(I^*)$ is a symmetric interval, inequality (3.5) follows from the first inequality in the Introduction.

In view of the preceding presentation (part (c), with $f = \chi_I$), we know that (3.5) is equivalent to:

$$(f_i \circ \tau_i) \cdot \tau'_i \prec (f_i^* \circ \tau_i) \cdot \tau'_i, \qquad \text{for all } i = 1, \dots, n. \tag{3.6}$$

Next, we use Lemma 3.1 with $g_i = (f_i \circ \tau_i) \cdot \tau'_i$, $i = 1, \dots, n$, to obtain the following inequality:

$$\int_K \prod_{i=1}^n g_i(y_i)\,dy \leq \int_K \prod_{i=1}^n g_i^*(y_i)\,dy. \tag{3.7}$$

Letting $h_i = (f_i^* \circ \tau_i) \cdot \tau'_i$, $i = 1, \dots, n$, implies $h_i = h_i^*$, and from (3.6) we have $g_i \prec h_i$. Now inequality (3.4) follows if we prove that for $g_i \prec h_i$, $i = 1, \dots, n$, the following inequality holds:

$$\int_K \prod_{i=1}^n g_i^*(y_i)\,dy \leq \int_K \prod_{i=1}^n h_i(y_i)\,dy. \tag{3.8}$$

If f is symmetric decreasing and $g_1 \prec h_1$, we have:

$$\int_{\mathbb{R}} f(x) g_1^*(x)\,dx \leq \int_{\mathbb{R}} f(x) h_1(x)\,dx.$$

Taking f to be the function in Lemma 3.2, we obtain

$$\int_K \prod_{i=1}^n g_i^*(y_i)\,dy \leq \int_K h_1(y_1) \prod_{i=2}^n g_i^*(y_i)\,dy. \tag{3.9}$$

Applying this last inequality (3.9) $n-1$ times we obtain (3.8) which is equivalent to (3.4). The proof of Theorem 3.3 is now complete. □

Next, we give an example to illustrate that the set L in Theorem 3.3 cannot be replaced by an arbitrary centrally symmetric set. Let L be the set in $\mathbb{R}^2$ given by $\{(x,y) |\ -2 \leq x \leq 2, -|x|/2 \leq y \leq |x|/2\}$, and $f(x) = \chi_{(-2,0)}(x)$ and $g(y) = \chi_{(-1,1)}(y)$. Then $f^*(x) = \chi_{(-1,1)}(x)$ and $g^*(y) = g(y)$. In this case the reverse inequality (3.4) holds:

$$\begin{aligned}\int_L f(x)g(y)\,dxdy &= |(-2,0) \times (-1,1) \cap L| = 2 \\ &> 1 = |(-1,1) \times (-1,1) \cap L| = \int_L f^*(x) g^*(y)\,dxdy\end{aligned}$$

Corollary 3.4. *For L as in Theorem 3.3 and E a compact rectangle in $\mathbb{R}^n$ the following inequality between the measure of sets holds:*

$$|E^* \cap L| \geq |E \cap L|.$$

If equality holds and $L \supset E^*$, then $L \supset E'$, where E' is the smallest convex rectangle which contains E a.e. This follows from the fact that $L = \tau(K)$ is convex in each direction parallel to the coordinate axes. Indeed, if we fix $(x_2, \ldots, x_n)$ such that $J_1 = \{x_1 | (x_1, x_2, \ldots, x_n) \in K\}$ is a nonempty interval (recall that K is convex), then $\tau_1(J_1) = \{y_1 | (y_1, \tau_2(x_2), \ldots, \tau_n(x_n)) \in L\}$ is an interval since τ_1 is continuous. This implies that L is convex in the first coordinate. Similarly, for the other coordinates.

Thus, one is tempted to replace the set L in Theorem 3.3 by a centrally symmetric set which is convex in each coordinate direction. Unfortunately, there are counterexamples in this direction. Consider the following function:

$$f(x) = \begin{cases} \sqrt{1-x^2}, & -1 \le x \le 0 \\ (1-x)^2, & 0 < x \le 1. \end{cases}$$

We define L to be the area contained between the graph of f and its reflection in the origin, i.e.

$$L = \{(x,y) | -1 \le x \le 1, -f(-x) \le y \le f(x)\}.$$

Since $f(-\sqrt{3}/2) = f(1 - 1/\sqrt{2}) = 1/2$, it follows that $E := (-\sqrt{3}/2, 1 - 1/\sqrt{2}) \times (0, 1/2)$ is contained in L. On the other hand, $E^* = (-(2+\sqrt{3}-\sqrt{2})/4, (2+\sqrt{3}-\sqrt{2})/4) \times (-1/4, 1/4)$ is not entirely contained in L since $f(2+\sqrt{3}-\sqrt{2})/4) < 1/4$. This shows that with this choice of L and E, the reverse inequality holds, i.e. $|E^* \cap L| < |E \cap L|$.

Theorem 3.5. *With L and $f_1, \ldots, f_n$ as in Theorem 3.3, and Ψ the distribution function of a Borel measure μ on $\mathbb{R}^n_+$ defined in the Introduction, we have:*

$$\int_L \Psi(f_1(x_1), \ldots, f_n(x_n))\, dx \le \int_L \Psi(f_1^*(x_1), \ldots, f_n^*(x_n))\, dx.$$

Following the method of Crowe-Zweibel-Rosenbloom, Ψ can be approximated by a product of characteristic functions.

$$\int \Psi(f_1(x_1), \ldots, f_n(x_n))\, dx = \int_{\mathbb{R}^n_+} (\int \chi_{\{f_1(x_1)>s_1\}} \chi_{\{f_2(x_2)>s_2\}} \cdots \chi_{\{f_n(x_n)>s_n\}}\, dx)\, d\mu(s),$$

with $s = (s_1, \ldots, s_n)$. Using Theorem 3.3 and the fact that $\chi_{\{s_i < f_i^*\}} = \chi_{\{s_i < f_i\}^*}$, the conclusion of Theorem 3.5 follows.

Pfiefer established inequalities for the integral of a positive centrally symmetric quasiconcave function over a compact rectangle E in $\mathbb{R}^n$, and showed that it increases over E^*. More specifically, $\int_E f(x)\, dx \le \int_{E^*} f(x)\, dx$. His proof is based on the fact that the integral of a positive integrable function over a measurable set E in $\mathbb{R}^n$ is equal to the integral of the measure of each level set K_t, i.e., $\int_0^\infty |K_t \cap E|\, dt = \int_E f\, dx$.

The next propositions can be viewed as generalizations of Pfiefer's result to functions whose level sets are not necessarily convex. In what follows, *τ is the coordinate transformation defined in Theorem 3.3*, with each τ_i strictly increasing.

Given a function f, we define a new function $f \circ \tau^{-1}$. The preimage of a set A under $f \circ \tau^{-1}$ is $(f \circ \tau^{-1})^{-1}(A) = \tau(f^{-1}(A))$, so that the level sets of $f \circ \tau^{-1}$ are just $\tau(K_t)$.

The following propositions follow directly from Theorem 3.3 via Corollary 3.4.

Proposition 3.6. *Let C be a centrally symmetric convex set in $\mathbb{R}^n$, $C \subset Ran\,\tau$ which contains a compact rectangle E. Let f be a nonnegative, centrally symmetric quasiconcave function, such that $f \circ \tau^{-1}$ is integrable on C. Then:*

$$\int_E (f \circ \tau^{-1})(x)\,dx \leq \int_{E^*} (f \circ \tau^{-1})(x)\,dx.$$

Proof. The proof is very similar to Pfiefer's proof. Since C is a centrally symmetric convex set, $C \supset E^*$ and

$$\int_E (f \circ \tau^{-1})(x)\,dx = \int_0^\infty |\tau(K_t) \cap E|\,dt$$
$$\leq \int_0^\infty |\tau(K_t) \cap E^*|\,dt = \int_{E^*} (f \circ \tau^{-1})(x)\,dx.$$

□

Proposition 3.7. *Let f be a centrally symmetric quasiconvex function, bounded above, such that $f \circ \tau^{-1}$ is integrable on a centrally symmetric convex set $C \subset Ran\,\tau$ containing the compact rectangle E in $\mathbb{R}^n$. Then we have*

$$\int_E (f \circ \tau^{-1})(x)\,dx \geq \int_{E^*} (f \circ \tau^{-1})(x)\,dx.$$

Proof. Let M be such that $f \leq M$ and define $g(x) = M - f(x)$. Then $g \geq 0$ and by Theorem 3.6 we have

$$M|E| - \int_E (f \circ \tau^{-1})(x)\,dx = \int_E (M - f \circ \tau^{-1})(x)\,dx$$
$$\leq \int_{E^*} (M - f \circ \tau^{-1})(x)\,dx = M|E^*| - \int_{E^*} (f \circ \tau^{-1})\,dx.$$

Since $|E| = |E^*|$, it follows that $\int_E (f \circ \tau^{-1})(x)\,dx \geq \int_{E^*} (f \circ \tau^{-1})(x)\,dx$. □

Proposition 3.8. *Let f be a centrally symmetric quasiconcave function, bounded below, such that $f \circ \tau^{-1}$ is integrable on a centrally symmetric convex set $C \subset Ran\,\tau$ containing the compact rectangle E in $\mathbb{R}^n$. Then*

$$\int_E (f \circ \tau^{-1})(x)\,dx \leq \int_{E^*} (f \circ \tau^{-1})(x)\,dx.$$

In Proposition 3.6, if f is concave, then equality holds if and only if $|\tau(K_t) \cap E| = |\tau(K_t) \cap E^*|$ for all t because each of these measures is a continuous function of t. If h is strictly increasing and g is centrally symmetric concave function with $f = h \circ g$ integrable on C, then $|\tau(K_t) \cap E|$ and $|\tau(K_t) \cap E^*|$ are again continuous functions of t, and f is centrally symmetric quasiconcave. Indeed, $\{f > t\} = g^{-1}(\{h > t\})$ is concave and centrally symmetric. Thus, equality holds in Proposition 3.6 if and only if $|\tau(K_t) \cap E| = |\tau(K_t) \cap E^*|$, for all t.

Similarly, equality holds in Proposition 3.7 under the same conditions, if $f = h \circ g$, with g convex and h strictly increasing.

Corollary 3.9. *Let $f = h \circ g$ as in the cases mentioned above. If equality holds in Theorems 3.6-3.8 and $\tau(K_t) \supset E^*$, then $\tau(K_t) \supset E'$, a.e. where E' is the smallest convex rectangle which contains E, a.e..*

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