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by

Leif K. Jørgensen

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Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 96 35 80 80 ▪ Telefax: +45 98 15 81 29
URL: http://www.math.aau.dk

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Rank of Adjacency Matrices of Directed (Strongly) Regular Graphs

L. K. Jørgensen
Dept. of Mathematical Sciences
Aalborg University
Fr. Bajers Vej 7, 9220 Aalborg, Denmark.
leif@math.aau.dk

Abstract

For a positive integer r we consider the set B_r of all values of $\frac{k}{n}$ for which there exists an $n \times n$ matrix with entries 0 and 1 such that each row and each column has exactly k 1's and the matrix has rank r. We prove that the set B_r is finite, for every r.

If there exists a k-regular directed graph on n vertices such that its adjacency matrix has rank r then $\frac{k}{n} \in B_r$. We use this to exclude existence of directed strongly regular graphs for infinitely many feasible parameter sets.

The investigation of $\{0,1\}$ matrices of low rank in this paper is motivated by the study of directed strongly regular graphs, which were introduced by A. Duval [2], see section 3. Fiedler, Klin and Muzychuk [4] excluded the existence of a 6-regular directed strongly regular graph on 16 vertices for which the adjacency matrix would have rank 4. In [8], we characterized the parameter sets for all directed strongly regular graphs with adjacency matrix of rank 3 or 4. In this paper we develop the techniques from [4] and [8] further and extend results to arbitrary regular directed graphs and to arbitrary rank.

In section 1, we consider the general problem about rank of square $\{0,1\}$ matrices with constant row and column sum. In sections 2 and 3 we apply the results to directed graphs (digraphs) and in particular to directed strongly regular graphs.

1 $\{0,1\}$ matrices with low rank

We consider matrices A such that

- every entry of A is either 0 or 1
- A is an $n \times n$ matrix, for some n
- A has exactly k 1's in each column and exactly k 1's in each row, for some k.

Previously, bounds on the rank of such matrices as a function of k and n have been studied. Houck and Paul [7] proved that if $0 < k < n$ and $(k, n) \neq (2, 4)$ then there is a matrix of rank n. Brualdi, Manber and Ross [1] proved that $\lceil \frac{k}{n} \rceil$ is a lower bound on the rank and that this bound can be attained if and only if k divides n.

In this paper we fix the rank and consider the ratio $\frac{k}{n}$. For a given number r, let B_r be the set of all rational numbers q for which there exists k and n with $\frac{k}{n} = q$ and a matrix with the above properties and with rank r.

The following example shows that it is natural to consider the ratio $\frac{k}{n}$.

Example 1 *Let A be an $n \times n$ matrix with entries 0 and 1 and with exactly k 1's in each row and in each column such that A has rank r. Let $B = J_m \otimes A$ be the Kronecker product of the $m \times m$ matrix with all entries equal to 1 and A. (B is a blockmatrix with $m \times m$ blocks all equal to A.) Then B also has rank r, it has size $mn \times mn$ and it has exactly mk 1's in each row and in each column. The ratio $\frac{mk}{mn} = \frac{k}{n}$ is the same as for A.*

Theorem 2 *Every set B_r is finite,*

$$|B_r| < 2^{r^2}.$$

Proof Let r be a fixed positive integer and let A be a square $\{0,1\}$ matrix with rank r for which there exists a number k so that $AJ = JA = kJ$ (i.e., each row and each column of A has exactly k 1's). Let n be the dimension of the matrix.

Let M be an $n \times r$ submatrix of A such that the columns of M is a set of r linearly independent columns of A. Since the sum of all columns of A is $(k, \ldots, k)^{\mathrm{t}}$, the vector $\mathbf{1}_n = (1, \ldots, 1)^{\mathrm{t}}$ is a linear combination of the columns of M. Thus there exists a vector $\alpha = (\alpha_1, \ldots, \alpha_r)^{\mathrm{t}}$ so that $M\alpha = \mathbf{1}_n$.

Since there are k 1's in each column of M, the sum of all equations in this system of linear equations is $k \sum_{i=1}^{r} \alpha_i = n$, i.e., $\frac{k}{n} = \frac{1}{\sum \alpha_i}$.

Let R be an $r \times r$ submatrix of M consisting of r linearly independent rows of M. Then $R\alpha = \mathbf{1}_r$.

From this equation, $\alpha = (\alpha_1, \ldots, \alpha_r)^{\mathrm{t}}$ and thus $\frac{k}{n}$ can be determined uniquely.

Since the number of $r \times r$ $\{0,1\}$ matrices with rank r is less that 2^{r^2}, we have $|B_r| < 2^{r^2}$. □

We will now find a lower bound on $|B_r|$. This is done by constructing matrices with rank r.

Theorem 3 *The sets B_r satisfy the following properties for every $r \geq 2$.*

1. $B_r \subset B_{r+1}$.
2. $\frac{1}{2} \in B_r$.
3. *If $q \in B_r$ then $1 - q \in B_r$.*
4. *If $q \in B_r$ then $\frac{q}{q+1} \in B_{r+1}$.*

Proof 1. Suppose that $q \in B_r$. Let A be an $n \times n$ $\{0,1\}$ matrix with k 1's in each row and in each column, with rank r and $\frac{k}{n} = q$. Then there exists at least $r-1$ columns in A which are not equal to the first column. Let $B = J_3 \otimes A$ (or $B = J_2 \otimes A$, if $r \geq 3$). Then B has rank r and there exists $r+1$ columns with numbers $j_1, \ldots, j_{r+1}$ in B which are not equal to the first column. Let $\mathbf{v_i}$, $i = 1, \ldots, r+1$, be the row vector with 1 in the first entry, -1 in entry j_i and 0 in all other entries. Since $\mathbf{v_1}, \ldots, \mathbf{v_{r+1}}$ are linearly independent, there exists i, so that $\mathbf{v_i}$ is not contained in the row space of B. There exists $r, s > n$ so that the entries $(r,1), (r,j_i), (s,1), (s,j_i)$ in B are $1,0,0,1$, respectively. Let B' be the matrix obtained from B by interchanging 0's and 1's on these four entries. This operation does not change row and column sums. The row space of B' is spanned by the first n rows of B and the vector $\mathbf{v_i}$. Thus B' has rank $r+1$ and $q \in B_{r+1}$.

2. The 2×2 identity matrix proves that $\frac{1}{2} \in B_2$. The result follows from 1.

3. Let A be an $n \times n$ $\{0,1\}$ matrix so that $AJ = JA = kJ$, rank $A = r$ and $\frac{k}{n} = q$. Since $\mathbf{1}$ is a linear combination of the columns of A, $\mathbf{1} - \mathbf{v}$ is a linear combination of the columns of A for every column vector $\mathbf{v}$. Thus

rank $(J-A) \leq$ rank A. By symmetry, rank $(J-A) =$ rank A. The matrix $J-A$ proves that $\frac{n-k}{n} = 1-q \in B_r$.

4. Suppose that $q \in B_r$. Then there exists k, n and an $n \times n$ $\{0,1\}$ matrix A so that $AJ = JA = kJ$ and rank $A = r$ and $\frac{k}{n} = q$. Then the matrix

$$\begin{bmatrix} A & 0 \\ 0 & J_k \end{bmatrix}$$

is an $(n+k) \times (n+k)$ matrix with exactly k 1's in each row and each column and it has rank $r+1$. Thus $\frac{k}{n+k} = \frac{q}{q+1} \in B_{r+1}$. □

Corollary 4 *For $r \geq 2$ we have*

1. $|B_{r+1}| \geq 2|B_r| + 1$.
2. $|B_r| \geq 2^{r-1} - 1$.

Proof 1. For each $\frac{a}{b} \in B_r$, we have $\frac{a}{a+b}, 1 - \frac{a}{a+b} = \frac{b}{a+b} \in B_{r+1}$ and also $\frac{1}{2} \in B_{r+1}$ and $\frac{a}{a+b} < \frac{1}{2} < \frac{b}{a+b}$.

2. follows by induction. □

The proof of Theorem 2 shows that the following algorithm constructs a finite set containing B_r.

Algorithm 5 *This algorithm constructs a set S_r of cardinality less than 2^{r^2} containing B_r.*
Let $S_r = \emptyset$.
For *each non-singular $r \times r$ $\{0,1\}$ matrix R* **do**
let $\alpha = R^{-1}\mathbf{1}_r$ and let a be the sum of all entries of α.
if *$a \neq 0$* **then** *add $\frac{1}{a}$ to the set S_r.*

Remark. If R_1 and R_2 are non-singular $r \times r$ $\{0,1\}$ matrices then we say that R_1 and R_2 are equivalent if R_2 can be obtained from R_1 by permuting the rows and permuting the columns. Clearly, equivalent matrices gives the same number in S_r. Thus we need only consider one element from each equivalence class. We choose the unique matrix in maximal form, i.e., the matrix with the property that the binary number obtained by reading the entries in row 1, followed the entries in row 2, etc, is maximal among all matrices in the equivalence class. This method is known as orderly search, see Read [9] or Faradžev [3].

For each number in S_r we want to determine if it belongs to B_r. The following theorem excludes some numbers.

Theorem 6 *If $q \in B_r$, $r \geq 2$ then $\frac{1}{r} \leq q \leq \frac{r-1}{r}$.*

Proof Let A be an $n \times n$ $\{0,1\}$ matrix so that $AJ = JA = kJ$ and rank $A = r$, and let M be an $n \times r$ submatrix of A with rank r. Then there is at least one 1 in each row of M. Thus $kr \geq n$, i.e., $\frac{k}{n} \geq \frac{1}{r}$. From Theorem 3 we then get $\frac{k}{n} \leq \frac{r-1}{r}$. □

Remark. It follows from the proof of this theorem that if $\frac{k}{n} = \frac{1}{r}$ then the submatrix M has exactly one 1 in each row and then every column of A is equal to a column of M. In fact, Brualdi, Manber and Ross proved that any matrix with $\frac{k}{n} = \frac{1}{r}$ can be obtained from $I_{\frac{n}{k}} \otimes J_k$ by permuting rows and permuting columns.

In order to prove that a number belongs to B_r it is necessary to construct a matrix. In some but not all cases, Theorem 3 can be applied.

Example 7 *The following matrix shows that $\frac{4}{9} \in B_5$.*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The following theorem was obtained by running Algorithm 5 on a computer.

Theorem 8

$$B_1 = \{1\},$$

$$B_2 = \{\frac{1}{2}\},$$

$$B_3 = \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\},$$

$$B_4 = \{\frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}\},$$

$$B_5 = \{\frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}\},$$

$$B_6 = \{\frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{3}{11}, \frac{2}{7}, \frac{3}{10}, \frac{4}{13}, \frac{1}{3}, \frac{5}{14}, \frac{4}{11}, \frac{3}{8}, \frac{5}{13}, \frac{2}{5}, \frac{5}{12}, \frac{3}{7}, \frac{7}{16}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}, \frac{7}{15}, \frac{8}{17}, \frac{1}{2},$$
$$\frac{9}{17}, \frac{8}{15}, \frac{7}{13}, \frac{6}{11}, \frac{5}{9}, \frac{9}{16}, \frac{4}{7}, \frac{7}{12}, \frac{3}{5}, \frac{8}{13}, \frac{5}{8}, \frac{7}{11}, \frac{9}{14}, \frac{2}{3}, \frac{9}{13}, \frac{7}{10}, \frac{5}{7}, \frac{8}{11}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}\}.$$

$$B_7 = \{\frac{1}{7}, \frac{1}{6}, \frac{2}{11}, \frac{1}{5}, \frac{3}{14}, \frac{2}{9}, \frac{3}{13}, \frac{4}{17}, \frac{1}{4}, \frac{5}{19}, \frac{4}{15}, \frac{3}{11}, \frac{5}{18}, \frac{2}{7}, \frac{5}{17}, \frac{3}{10}, \frac{7}{23}, \frac{4}{13}, \frac{5}{16},$$
$$\frac{6}{19}, \frac{7}{22}, \frac{8}{25}, \frac{1}{3}, \frac{9}{26}, \frac{8}{23}, \frac{7}{20}, \frac{6}{17}, \frac{5}{14}, \frac{9}{25}, \frac{4}{11}, \frac{7}{19}, \frac{10}{27}, \frac{3}{8}, \frac{11}{29}, \frac{8}{21}, \frac{5}{13}, \frac{7}{18}, \frac{9}{23},$$
$$\frac{11}{28}, \frac{2}{5}, \frac{11}{27}, \frac{9}{22}, \frac{7}{17}, \frac{12}{29}, \frac{5}{12}, \frac{13}{31}, \frac{8}{19}, \frac{11}{26}, \frac{3}{7}, \frac{13}{30}, \frac{10}{23}, \frac{7}{16}, \frac{11}{25}, \frac{4}{9}, \frac{13}{29}, \frac{9}{20}, \frac{14}{31},$$
$$\frac{5}{11}, \frac{11}{24}, \frac{6}{13}, \frac{13}{28}, \frac{7}{15}, \frac{15}{32}, \frac{8}{17}, \frac{9}{19}, \frac{10}{21}, \frac{11}{23}, \frac{12}{25}, \frac{13}{27}, \frac{14}{29}, \frac{15}{31}, \frac{16}{33}, \frac{17}{35}, \frac{1}{2},$$
$$\frac{18}{35}, \frac{17}{33}, \frac{16}{31}, \frac{15}{29}, \frac{14}{27}, \frac{13}{25}, \frac{12}{23}, \frac{11}{21}, \frac{10}{19}, \frac{9}{17}, \frac{17}{32}, \frac{8}{15}, \frac{15}{28}, \frac{7}{13}, \frac{13}{24}, \frac{6}{11}, \frac{17}{31}, \frac{11}{20},$$
$$\frac{16}{29}, \frac{5}{9}, \frac{14}{25}, \frac{9}{16}, \frac{13}{23}, \frac{17}{30}, \frac{4}{7}, \frac{15}{26}, \frac{11}{19}, \frac{18}{31}, \frac{7}{12}, \frac{17}{29}, \frac{10}{17}, \frac{13}{22}, \frac{16}{27}, \frac{3}{5}, \frac{17}{28}, \frac{14}{23}, \frac{11}{18},$$
$$\frac{8}{13}, \frac{13}{21}, \frac{18}{29}, \frac{5}{8}, \frac{17}{27}, \frac{12}{19}, \frac{7}{11}, \frac{16}{25}, \frac{9}{14}, \frac{11}{17}, \frac{13}{20}, \frac{15}{23}, \frac{17}{26}, \frac{2}{3}, \frac{17}{25}, \frac{15}{22}, \frac{13}{19}, \frac{11}{16}, \frac{9}{13},$$
$$\frac{16}{23}, \frac{7}{10}, \frac{12}{17}, \frac{5}{7}, \frac{13}{18}, \frac{8}{11}, \frac{11}{15}, \frac{14}{19}, \frac{3}{4}, \frac{13}{17}, \frac{10}{13}, \frac{7}{9}, \frac{11}{14}, \frac{4}{5}, \frac{9}{11}, \frac{5}{6}, \frac{6}{7}\}$$

We see that the numbers in B_2, B_3, B_4 and B_5 are exactly the numbers that follow from Theorem 3 and Example 7.

The fact that B_r is finite does not exclude any particular rational number from B_r. But in the following theorem we give an upper bound on the numerator and the denominator of a number in B_r

Theorem 9 *Suppose that M_r is a number such that $|\det(R)| \le M_r$ for every $r \times r$ $\{0,1\}$ matrix. Suppose also that $\frac{a}{b} \in B_r$ and $\gcd(a,b) = 1$. Then $a \le M_r$ and $b \le 2M_r$.*

Proof By the proof of Theorem 2, there exists rational numbers $\alpha_1, \ldots, \alpha_r$, and a non-singular $r \times r$ $\{0,1\}$ matrix R such that $R(\alpha_1, \ldots, \alpha_r)^{\mathrm{t}} = \mathbf{1}_r$ and $\frac{b}{a} = \alpha_1 + \ldots + \alpha_r$. By Cramer's rule, $\det(R)$ is a common denominator of $\alpha_1, \ldots, \alpha_r$. Thus $a \le M_r$. By Theorem 3, $\frac{b-a}{b}$ is also in B_r and so $b - a \le M_r$. Thus $b \le 2M_r$. □

2 Applications to regular digraphs

If there exists a k-regular directed graph on n vertices with adjacency matrix of rank r then $\frac{k}{n} \in B_r$. Conversely, the following result shows that for any $q \in B_r$, $r \ge 2$, there exists a k-regular graph of order n, for some k and n, such that the adjacency matrix of the graph has rank r and $\frac{k}{n} = q$.

Proposition 10 *Suppose that there is an $n \times n$ $\{0,1\}$ matrix A so that $AJ = JA = kJ$ and rank $A = r \ge 2$. Then there exists a k-regular digraph on n vertices with adjacency matrix of rank r.*

Proof By Hall's theorem [6] on distinct representatives there exists a permutation of the rows of A so that the matrix A' obtained by this permutation has 0 on every diagonal entry. Thus A' is the adjacency matrix of a digraph. □

Corollary 11 *For $r \ge 2$, B_r is the set of numbers $\frac{k}{n}$ for which there exists a k-regular graph on n vertices with adjacency matrix of rank r.*

Digraphs for which equality holds in the inequality in Theorem 6 can be characterized.

Theorem 12 (Gimbert [5]) *Suppose that G is a k-regular digraph on n vertices with adjacency matrix of rank $\frac{n}{k}$. Then G is a line digraph.*

Proposition 13 *Suppose that G is a k-regular digraph on n vertices with adjacency matrix of rank r and $\frac{k}{n} = \frac{r-1}{r}$. Then G is an undirected complete r-partite graph.*

Proof Let A be the adjacency matrix of G. Equality holds in the first inequality of Theorem 6 for the matrix $J-A$. Thus it follows from the remark following the proof of Theorem 6 that if the (i,j)-entry and the (i,ℓ)-entry of A are both 0, then column j and column ℓ are equal. If the (i,j)-entry is 0 then, since the (i,i)-entry and the (j,j)-entry are 0, column i and column j are equal and the (j,i)-entry is 0. Thus A is symmetric and G is an undirected graph in which any two non-adjacent vertices have the same set of neighbours. This implies that G is a complete multipartite graph. □

3 Directed strongly regular graphs with eigenvalue 0

A directed strongly regular graph with parameters (n,k,μ,λ,t) is a k-regular directed graph on n vertices such that every vertex is incident with t undirected edges, and the number of paths of length 2 from a vertex x to a vertex y is λ if there is an edge directed from x to y and it is μ otherwise. This means the adjacency matrix A satisfies

$$A^2 = tI + \lambda A + \mu(J-I-A) \quad \text{and} \quad AJ = JA = kJ.$$

It will be assumed that $0<t<k$. These graphs were introduced by A. Duval [2], who also proved the following restrictions on the parameters.

Theorem 14 *Suppose that there exists a directed strongly regular graph with parameters (n,k,μ,λ,t).*

Then the parameters satisfy

$$k(k+(\mu-\lambda)) = t+(n-1)\mu \tag{1}$$

and

$$0 \le \lambda < t, \quad 0<\mu\le t, \quad -2(k-t-1) \le \mu-\lambda \le 2(k-t). \tag{2}$$

The eigenvalues of the adjacency matrix are

$$k > \rho = \frac{1}{2}(-(\mu-\lambda)+d) > \sigma = \frac{1}{2}(-(\mu-\lambda)-d),$$

for some positive integer d, where $d^2=(\mu-\lambda)^2+4(t-\mu)$. The multiplicities are

$$1,\ -\frac{k+\sigma(n-1)}{\rho-\sigma},\ \frac{k+\rho(n-1)}{\rho-\sigma}, \tag{3}$$

respectively.

We say that (n, k, μ, λ, t) is a feasible parameter set if the conditions (1) and (2) are satisfied and the multiplicities in (3) are positive integers.

We say that (n, k, μ, λ, t) is a realizable parameter set if there exists a directed strongly regular graph with these parameters. Thus realizable implies feasible. Our goal is to prove that many feasible parameter sets are not realizable.

We see from Theorem 14 that 0 is an eigenvalue if and only if $\rho = 0$. This is equivalent to $d = \mu - \lambda$, i.e., $t = \mu$. If 0 is an eigenvalue of the adjacency matrix then, since it can be diagonalized (see [8]), the rank of the adjacency matrix is the sum of multiplicities of non-zero eigenvalues, i.e.,

$$\text{rank } = 1 + \frac{k + \rho(n-1)}{\rho - \sigma} = 1 + \frac{k}{\mu - \lambda} = 1 + \frac{k}{d}.$$

Thus we define the rank of a feasible parameter set with $t = \mu$ to be $1 + \frac{k}{\mu - \lambda}$ (even if no directed strongly regular graph exists with these parameters).

If $\mu = t$ then equation (1) is equivalent to

$$k(k + d) = nt.$$

Contrary to Theorem 2, we now prove that for each $r \geq 3$ there are infinitely many values of $\frac{k}{n}$ for which there exists a feasible parameter set (n, k, μ, λ, t) with rank r.

Theorem 15 *Let r be a positive integer and let q be a rational number.*

Then there exists a feasible parameter set (n, k, μ, λ, t) with rank r and with $\frac{k}{n} = q$ if and only if $\frac{1}{r} \leq q \leq \frac{2r-3}{2r}$.

Proof

only if: Suppose that (n, k, μ, λ, t) is a feasible parameter set with rank r and $\frac{k}{n} = q$.

We have $k = d(r-1)$, $k + d = dr$ and $q = \frac{k}{n} = \frac{t}{dr}$. Thus from r, q and d we can compute

$$k = d(r-1), n = \frac{1}{q}k = \frac{1}{q}d(r-1), t = \mu = qdr, \lambda = \mu - d = (qr-1)d.$$

Since $\lambda \geq 0$,

$$qr - 1 \geq 0 \Rightarrow q \geq \frac{1}{r}.$$

From $d = \mu - \lambda \leq 2(k-t) = 2d(r-1-qr)$, we get

$$q \leq \frac{2r-3}{2r}.$$

if: Conversely, if $\frac{1}{r} \leq q \leq \frac{2r-3}{2r}$ and $q = \frac{a}{b}$ for integers a and b then it is easy to verify that

$$(n, k, \mu, \lambda, t) = ((r-1)b^2m, (r-1)abm, ra^2m, (ar-b)am, ra^2m)$$

is a feasible parameter set with rank r, for every positive integer m. □

We conclude that most of the feasible parameter sets with $t = \mu$ are not realizable. For $q \in B_r$, $q \leq \frac{2r-3}{2r}$, we do not know any general method to determine if there exists a realizable parameter set with $\frac{k}{n} = q$ and rank r. But it seems likely that a large fraction of the numbers in B_r can not be realized by a directed strongly regular graph, if r is large. In particular, we proved in [8] that $\frac{1}{4}$ and $\frac{1}{2}$ are the only numbers in B_4 that can be realized by a directed strongly regular graph.

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