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# A DIRECT PROOF OF SOBOLEV EMBEDDINGS FOR TRIEBEL–LIZORKIN SPACES, INCLUDING MIXED NORMS AND QUASI-HOMOGENEITY

JON JOHNSEN AND WINFRIED SICKEL

ABSTRACT. The article deals with a simplified proof of the Sobolev embedding theorem for Triebel–Lizorkin spaces (that contain the  $L_p$ -Sobolev spaces  $H_p^s$  as special cases). The method extends to a proof of the corresponding fact for general Triebel–Lizorkin spaces based on mixed  $L_p$ -norms. In this context a Nikol’skij–Plancherel–Polya inequality for sequences of functions satisfying a geometric rectangle condition is proved. The results extend also to anisotropic spaces of the quasi-homogeneous type.

## 1. INTRODUCTION

Since around 1977 the question of Sobolev embeddings of Triebel–Lizorkin spaces has been answered affirmatively, with a unified proof of the following

**Proposition 1.** *Let  $s, s_0 \in \mathbb{R}$  and  $p, p_0 \in ]0, \infty[$  be given such that*

$$s - \frac{n}{p} = s_0 - \frac{n}{p_0}, \quad p > p_0. \quad (1)$$

*There is then a continuous embedding  $F_{p_0, \infty}^{s_0}(\mathbb{R}^n) \hookrightarrow F_{p, q}^s(\mathbb{R}^n)$  for every  $q \in ]0, \infty]$ .*

Specifically this means that there exists a number  $C > 0$ , depending on the parameters, such that the following inequality is valid for every  $u \in F_{p_0, \infty}^{s_0}(\mathbb{R}^n)$ :

$$\left( \int_{\mathbb{R}^n} \left( \sum_{j=0}^{\infty} 2^{sjq} |u_j(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}^n} \left( \sup_{j \in \mathbb{N}_0} 2^{s_0 j} |u_j(x)| \right)^{p_0} dx \right)^{\frac{1}{p_0}}. \quad (2)$$

To explain the notation, note that a Littlewood–Paley decomposition  $1 = \sum_{j=0}^{\infty} \Phi_j(\xi)$  can be obtained by letting  $\Phi_j = \Psi_j - \Psi_{j-1}$  for  $j > 0$  when  $\Psi_j(\xi) = \psi(2^{-j}|\xi|)$  for some  $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$  fulfilling  $\psi(t) = 1$  for  $t < 1$  and  $\psi(t) = 0$  for  $t > 2$ .

By definition  $u \in \mathcal{S}'(\mathbb{R}^n)$  is in the Triebel–Lizorkin space  $F_{p, q}^s(\mathbb{R}^n)$  when the quantity on the left hand side of (2) is finite; hereby  $u_j =$

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$\mathcal{F}^{-1}(\Phi_j \hat{u})$ . As usual  $\|f\|_p$  denotes the norm of a function  $f$  in  $L_p(\mathbb{R}^n)$  with respect to the Lebesgue measure  $\lambda$  on  $\mathbb{R}^n$ .

Jawerth's original proof [Jaw77] of the Sobolev inequalities (2) was given for the homogeneous spaces  $\dot{F}_{p,q}^s$ , utilising the rewriting of  $L_p$ -norms via the distribution function ( $0 < p < \infty$ )

$$\int_{\mathbb{R}^n} |f(x)|^p dx = p \int_0^\infty t^{p-1} \lambda(\{x \mid |f(x)| > t\}) dt. \quad (3)$$

However, readers familiar with the subject may recall that for the inhomogeneous  $F_{p,q}^s$ , it is not possible for every  $t > 0$  to find  $j \in \mathbb{N}_0$  such that  $2^{jn/p} < t$  (unlike  $\dot{F}_{p,q}^s$ , where  $j \in \mathbb{Z}$ ), so Jawerth's proof needs an adaptation to this case. Triebel [Tri78, Tri83] introduced a splitting into the  $t$ -intervals  $]0, K[$  and  $]K, \infty[$ , where  $K$  essentially was the  $q^{\text{th}}$  root of  $\|u\|_{F_{p,q}^s}$ .

Since then this proof has widely been considered 'best possible'. Perhaps this is because the strategy of Jawerth and Triebel covers all  $p \in ]0, \infty[$  in an elegant way (at least for  $\dot{F}_{p,q}^s$ ), whilst previous attempts did not cover all cases.

In comparison, the Besov spaces  $B_{p,q}^s(\mathbb{R}^n)$  have corresponding embeddings with a well-known one-line proof based on the Nikol'skij–Plancherel–Polya inequality that we now recall.

For  $0 < p \leq r \leq \infty$  there exists a  $c > 0$  such that for every  $f \in \mathcal{S}'(\mathbb{R}^n) \cap L_p$  with  $\text{supp } \mathcal{F}f$  compact, say contained in the closed ball  $B(0, R) = \{\xi \in \mathbb{R}^n \mid |\xi| \leq R\}$ ,

$$\|f\|_r \leq cR^{\frac{n}{p}-\frac{n}{r}} \|f\|_p. \quad (4)$$

Applying this to each  $u_j$  ( $R = 2^{j+1}$ ) in  $\|u\|_{B_{p,q}^s} := (\sum 2^{sjq} \|u_j\|_p^q)^{\frac{1}{q}}$ , one finds at once that  $B_{p_0,q}^{s_0} \hookrightarrow B_{p,q}^s$  when (1) holds.

Generally  $F_{p,q}^s$ -spaces are rather more complicated to treat than  $B_{p,q}^s$ -spaces, so it would seem plausible that even the basic Sobolev inequalities in (2) are substantially more technical to achieve for the  $F_{p,q}^s$ .

But this is not so. Indeed, also for (2) there is a short proof based on the Nikol'skij–Plancherel–Polya inequality (4). The trick is to handle the infinite sum by means of the following result on the weighted sequence spaces  $\ell_q^s$ , that (for  $q < \infty$ ) is normed by

$$\|a_j\|_{\ell_q^s} = \left( \sum_{j=0}^{\infty} 2^{sjq} |a_j|^q \right)^{\frac{1}{q}}. \quad (5)$$

**Lemma 2.** *Let real numbers  $s_1 < s_0$  be given, and  $\theta \in ]0, 1[$ . For  $0 < q \leq \infty$  there is  $c > 0$  such that*

$$\|a_j\|_{\ell_q^{\theta s_0 + (1-\theta)s_1}} \leq c \|a_j\|_{\ell_\infty^{s_0}}^\theta \|a_j\|_{\ell_\infty^{s_1}}^{1-\theta} \quad (6)$$

*holds for all complex sequences  $(a_j)_{j \in \mathbb{N}_0}$  in  $\ell_\infty^{s_0}$ .*

This result was to our knowledge first crystallised in the works of Brezis and Mironescu [BM01]; cf. the elementary proof there.

In what follows we shall discuss certain generalizations of (4) and Proposition 1 to mixed  $L_p$ -norms and anisotropic smoothness. For the reader's convenience we shall first give the details of the proof in the isotropic (unmixed) case. Hence we prove Proposition 1; as it will become clear in Sections 2 and 3, the method carries over to the more general situation.

*Proof of Proposition 1 (based on (4), (6)).* For  $u \in F_{p_0, \infty}^{s_0}$  the claim is obtained by interpolating its  $F_{p, q}^s$ -norm between those of  $B_{\infty, \infty}^{s_1}$  with  $s_1 := s_0 - \frac{n}{p_0} = s - \frac{n}{p}$  and the given space  $F_{p_0, \infty}^{s_0}$ :

Since  $p > p_0$  there is some  $\theta \in ]0, 1[$  so that  $\theta \frac{n}{p_0} = \frac{n}{p}$ , hence

$$\theta s_0 + (1 - \theta) s_1 = \theta s_0 + (1 - \theta) (s_0 - \frac{n}{p_0}) = s - \frac{n}{p} + \theta \frac{n}{p_0} = s. \quad (7)$$

Clearly  $q < \infty$  suffices in (2), and for each  $x \in \mathbb{R}^n$  Lemma 2 gives,

$$\left( \sum_{j=0}^{\infty} 2^{s_j q} |u_j(x)|^q \right)^{\frac{1}{q}} \leq c \|u_j(x)\|_{\ell_{\infty}^{s_0}}^{\theta} (\sup_j 2^{s_1 j} \|u_j\|_{\infty})^{1-\theta}. \quad (8)$$

Here  $\|u_j\|_{\infty} \leq c' 2^{j \frac{n}{p_0}} \|u_j\|_{p_0}$  follows from (4) since  $u_j$  is in  $L_{p_0}$  with  $\text{supp } \mathcal{F}u_j \subset B(0, 2^{j+1})$ . The definition of  $s_1$  therefore yields

$$\sup_j 2^{s_1 j} \|u_j\|_{\infty} \leq c' \sup_j 2^{(s_1 + \frac{n}{p_0})j} \|u_j\|_{p_0} \leq c' \|u\|_{F_{p_0, \infty}^{s_0}}. \quad (9)$$

Hence the fact that  $\theta p = p_0$  gives, by taking  $L_p$ -norms in (8),

$$\|u\|_{F_{p, q}^s} \leq c \|u\|_{F_{p_0, \infty}^{s_0}}^{\theta} (c' \|u\|_{F_{p_0, \infty}^{s_0}})^{1-\theta} = c'' \|u\|_{F_{p_0, \infty}^{s_0}}. \quad (10)$$

That is, (2) is proved.  $\square$

*Remark 3.* For simplicity we have taken the sum-exponent  $q_0 = \infty$  on the right hand side of (2), but the other Sobolev inequalities can be recovered via the simple embedding  $F_{p_0, q_0}^{s_0} \hookrightarrow F_{p_0, \infty}^{s_0}$ .

The Nikol'skij–Plancherel–Polya inequality (4) will be generalised to mixed  $L_p$ -norms in Section 2 below. In the course of the proof given there, we utilise (4) in the above unmixed form; so to provide a complete overview we insert an argument for this (known) standard version.

*Proof of inequality (4).* It suffices to establish (4) when in addition to the stated assumptions  $u \in \mathcal{S}(\mathbb{R}^n)$ . To see this we take  $\hat{\psi} \in C^{\infty}(\mathbb{R}^n)$  such that  $\hat{\psi}(t) = 1$  for  $|t| \leq 1$ ,  $\hat{\psi}(t) = 0$  for  $|t| \geq 2$ , while  $\psi(0) = 1$ . Defining  $u_j \in \mathcal{S}(\mathbb{R}^n)$  by

$$u_j(x) := u(x) \psi\left(\frac{x}{j}\right), \quad (11)$$

then  $u_j$  has its spectrum in  $\text{supp } \hat{u} + B(0, \frac{2}{j})$ , for  $u(x)$  is  $\mathcal{O}((1 + |x|)^N)$  for some  $N \geq 0$  by the Paley–Wiener–Schwartz theorem ( $\hat{u} \in \mathcal{E}'$ ). There is only something to show for  $p < r$ , so  $p < \infty$  and hence

$$u_j \rightarrow u \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \cap L_p(\mathbb{R}^n). \quad (12)$$

So if (4) can be proved for all Schwartz functions with compact spectra, it will follow that  $(u_j)$  is a fundamental sequence in  $L_r$ , hence converging to some  $g \in L_r$  with  $g(x) = u(x)$  a.e.

Since  $\|\cdot\|_p$  is continuous  $L_p \rightarrow \mathbb{R}$ , the above would entail that

$$\|u\|_r \leq \lim_{j \rightarrow \infty} c(R + \frac{2}{j})^{\frac{n}{p} - \frac{n}{r}} \|u_j\|_p = cR^{\frac{n}{p} - \frac{n}{r}} \|u\|_p. \quad (13)$$

( $c > 0$  will be independent of  $u$  and the size of its spectrum.)

In the smooth case we proceed in the spirit of [Tri83, 1.3.2]. When  $u \in \mathcal{S}(\mathbb{R}^n)$  with  $\hat{u} \in C_{B(0,R)}^\infty$ , then  $\Psi(x) = R^n \psi(Rx)$  fulfils

$$u(x) = \mathcal{F}^{-1}(\hat{\Psi}\hat{u}) = \int_{\mathbb{R}^n} u(y)\Psi(x-y) dy. \quad (14)$$

For  $1 \leq p \leq \infty = r$  and  $p + p^* = pp^*$ , the above identity gives

$$|u(x)| \leq \|\Psi\|_{L_{p^*}} \|u\|_p \leq \|\psi\|_{L_{p^*}} R^{\frac{n}{p}} \|u\|_p, \quad (15)$$

hence (4). In case  $0 < p < 1$  and  $r = \infty$ ,

$$|u(x)| \leq \|\Psi\|_\infty (\sup |u(z)|)^{1-p} \int |u(y)|^p dy, \quad (16)$$

so the  $x$ -independence of the right hand side entails

$$|u(x)|^p \leq \|u\|_\infty^p \leq \|\Psi\|_\infty \int |u(y)|^p dy, \quad (17)$$

where  $\|\Psi\|_\infty^{\frac{1}{p}} = R^{\frac{n}{p}} \|\psi\|_\infty$ . This proves (4) for  $r = \infty$ .

If  $r < \infty$  and  $p < 1$ , insertion of (17) into  $\|u\|_r$  gives

$$\begin{aligned} \left( \int |u(x)|^r dx \right)^{\frac{1}{r}} &\leq \left( \sup_z |u(z)|^{r-p} \right)^{\frac{1}{r}} \left( \int |u(x)|^p dx \right)^{\frac{1}{r}} \\ &\leq \|\Psi\|_\infty^{\frac{r-p}{rp}} \|u\|_p \leq (\|\psi\|_\infty R^n)^{\frac{1}{p} - \frac{1}{r}} \|u\|_p. \end{aligned} \quad (18)$$

For  $1 \leq p \leq r < \infty$  one can insert (15) instead. This completes the proof of (4), with  $c$  equal to a power of a norm of  $\psi$ .  $\square$

## 2. THE MIXED NORM CASE

One advantage of the above proof method is that it extends to spaces with mixed norms. In this set-up  $L_p(\mathbb{R}^n)$  is replaced by  $L_{\vec{p}}(\mathbb{R}^n)$  where  $\vec{p} = (p_1, \dots, p_n)$  for  $0 < p_j \leq \infty$ ,  $j = 1, \dots, n$ , and

$$\|f\|_{L_{\vec{p}}} = \left( \int_{\mathbb{R}} \left( \dots \left( \int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} \dots \right)^{\frac{p_n}{p_{n-1}}} dx_n \right)^{\frac{1}{p_n}}. \quad (19)$$

It is well known that such spaces enter the analysis of evolution problems for partial differential equations. To give one example, we mention

that of the famous weak solutions to the Navier–Stokes equation in an open domain  $\Omega \subset \mathbb{R}^n$ . In connection with the still open uniqueness question, it has been known since the 1960’s that weak solutions belonging to the additional class  $L_s(0, T; L_r(\Omega)^n)$  with  $\frac{2}{p} + \frac{n}{r} \leq 1$  are unique; cf. J.-L. Lions [Lio69, Thm. 6.9] or R. Temam [Tem84, Rem. III.3.6].

The corresponding Triebel–Lizorkin spaces  $F_{\vec{p}, q}^s(\mathbb{R}^n)$  consist of the  $u$  in  $\mathcal{S}'(\mathbb{R}^n)$  having finite quasi-norms ( $0 < p_j < \infty$ )

$$\|u\|_{F_{\vec{p}, q}^s} = \left\| \left( \sum_{j=0}^{\infty} 2^{sq_j} |u_j(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L_{\vec{p}}}. \quad (20)$$

The purpose is not to go into the general theory of such spaces here. Instead we want to show that the Sobolev embeddings follow directly from first principles, namely the definition (20) and a mixed norm version of the Nikol’skij–Plancherel–Polya inequality.

Before we turn to this, it is noted that also the mixed quasi-norm in  $L_{\vec{p}}$  is subadditive when raised to a power  $\lambda \in ]0, \min(1, p_1, \dots, p_n)[$ ,

$$\|f + g\|_{L_{\vec{p}}}^\lambda \leq \|f\|_{L_{\vec{p}}}^\lambda + \|g\|_{L_{\vec{p}}}^\lambda. \quad (21)$$

In fact  $\int |f+g|^p d\mu \leq \int |f|^p d\mu + \int |g|^p d\mu$  holds for arbitrary measures if  $0 < p \leq 1$ , so  $\|f+g\|_p \leq (\|f\|_p^\nu + \|g\|_p^\nu)^{1/\nu}$  for all  $\nu \in ]0, \min(1, p)[$ . Using this, it is easy to see that  $\|\cdot\|_q^\lambda$  is subadditive for  $\lambda \leq \min(1, p, q)$ . For  $\lambda = \min(1, p_1, \dots, p_n)$  a repeated use of this yields (21).

To prepare for the mixed norm version of the Nikol’skij–Plancherel–Polya inequality, we recall the Paley–Wiener–Schwartz Theorem in a precise version, cf. [Hör85, Ch. 7]: a distribution  $u \in \mathcal{S}'(\mathbb{R}^n)$  fulfils  $\text{supp } \hat{u} \Subset \mathbb{R}^n$  if and only if  $u$  extends to a function  $u(x + iy)$  on  $\mathbb{C}^n$  which is entire analytic and fulfils

$$|u(x + iy)| \leq C(1 + |x + iy|)^N e^{H(-y)}, \quad x, y \in \mathbb{R}^n \quad (22)$$

for some  $N \geq 0$  (the order of  $\hat{u}$ ); here  $H(y) = \sup\{y \cdot \xi \mid \xi \in \text{supp } \hat{u}\}$  is the supporting function of the compact set  $\text{supp } \hat{u}$ .

When  $u \in \mathcal{S}' \cap L_{\vec{p}}(\mathbb{R}^n)$  and  $\text{supp } \hat{u} \Subset \mathbb{R}^n$ , then  $u(x) = \mathcal{O}((1 + |x|)^N)$  for  $x \in \mathbb{R}^n$ , so  $u \in L_1^{\text{loc}}(\mathbb{R}^n)$  also if some  $p_k < 1$ ; and (22) also yields  $u(\varphi) = \int u(x)\varphi(x) dx$  for  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ . (In general elements of  $\mathcal{S}' \cap L_1^{\text{loc}}$  do not act on  $\mathcal{S}$  by integration; cf.  $e^x \cos(e^x) = \sin(e^x)'$  on  $\mathbb{R}$ .)

We also note that for a rectangle  $[-R_1, R_1] \times \dots \times [R_n, R_n]$ ,

$$H(y) = R_1|y_1| + \dots + R_n|y_n|. \quad (23)$$

Indeed,  $H(y)$  can be estimated by the triangle inequality, and equality is attained in one of the corners  $(\pm R_1, \dots, \pm R_n)$ . When  $\hat{u}$  is supported in this rectangle, and  $(x', x''), (y', y'') \in \mathbb{R}^{n'} \times \mathbb{R}^{n''}$ ,  $n = n' + n''$ , it follows when  $x'', y''$  are kept as parameters that  $U(x' + iy') :=$

$u((x', x'') + i(y', y''))$  is analytic on  $\mathbb{C}^{n'}$  and

$$|U(x' + iy')| \leq C'(1 + |(x', y')|)^N \exp\left(\sum_{n'} R_j |y_j|\right). \quad (24)$$

Therefore  $x' \mapsto U(x')$  is a tempered distribution on  $\mathbb{R}^{n'}$  with spectrum in  $\prod_{n'} [-R_j, R_j]$ , as one would expect.

These facts are convenient for the proof of

**Theorem 4.** *Whenever  $0 < p_j \leq r_j \leq \infty$  for  $j = 1, \dots, n$ , there is a  $c > 0$  such that for every  $f \in \mathcal{S}'(\mathbb{R}^n) \cap L_{\vec{p}}$  with spectrum in a compact rectangle given by  $|\xi_k| \leq R_k$  for  $k = 1, \dots, n$ , i.e.*

$$\text{supp } \mathcal{F}f \subset [-R_1, R_1] \times \dots \times [-R_n, R_n], \quad (25)$$

it also holds that  $f \in L_{\vec{r}}(\mathbb{R}^n)$  and

$$\|f\|_{L_{\vec{r}}} \leq c \left( \prod_{k=1, \dots, n} R_k^{\frac{1}{p_k} - \frac{1}{r_k}} \right) \|f\|_{L_{\vec{p}}}. \quad (26)$$

This result is probably known, but in the present context we were unable to find a reference. In our proof the treatment of mixed norms with  $0 < p_j < 1$  is inspired by B. Stöckert's paper [Stö78, Satz 2.1].

The strategy of the proof is perhaps best described as a *succession of embeddings*, which means that (26) is realised as a composition of the following  $n$  embeddings, that each only affect a single coordinate direction:

$$L_{\vec{p}} \hookrightarrow L_{(r_1, p_2, \dots, p_n)} \hookrightarrow L_{(r_1, r_2, p_3, \dots, p_n)} \hookrightarrow \dots \hookrightarrow L_{\vec{r}}. \quad (27)$$

For  $r_k = p_k$  the  $k^{\text{th}}$  map is just the identity map from the  $k^{\text{th}}$  space to itself. This gives a convenient reduction to the case in which  $\vec{r}$  and  $\vec{p}$  differ in only one component, say  $r_m > p_m$  whilst  $r_j = p_j$  for  $j \neq m$ . Thereby some technicalities are circumvented. (Already (12) is troublesome to carry over to the mixed case, for if some  $p_k = \infty$  we cannot obtain convergence in the norm topology.)

*Proof.* 1°. We prove (26) by means of a succession of embeddings, as explained above. So it suffices to assume that  $r_m > p_m$  only holds for one value of  $m$ , and we can assume that this is  $m = n$ .

Indeed, seeing  $u$  as a function of  $x' = (x_1, \dots, x_m)$ , it was found in (24) ff. that its spectrum lies in  $[-R_1, R_1] \times \dots \times [-R_m, R_m]$ ; and by (22) it is in  $\mathcal{S}' \cap L_{(p_1, \dots, p_m)}$  on  $\mathbb{R}^m$ , so once the case  $m = n$  is covered, integration with respect to  $x''$  yields (26) in general when  $1 \leq m \leq n$ .

2°. If  $\psi \in \mathcal{S}(\mathbb{R})$  is such that  $\hat{\psi}(t) = 0$  for  $|t| \geq 2$  while  $\hat{\psi}(t) = 1$  for  $|t| \leq 1$ , we set  $\Psi(x_n) = R_n \psi(R_n x_n)$ . Clearly  $\hat{u}(\xi) = \hat{\Psi}(R_n^{-1} \xi_n) \hat{u}(\xi)$ , and we want to show that

$$u(x) = \int_{\mathbb{R}} u(x', y_n) \Psi(x_n - y_n) dy_n \quad \text{for } x \in \mathbb{R}^n. \quad (28)$$

The right hand side makes sense by (22), and for  $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\mathcal{F}_{x \rightarrow \xi} \left( \int \varphi(x', y_n) \Psi(x_n - y_n) dy_n \right) = \hat{\Psi}(\xi_n) \hat{\varphi}(\xi). \quad (29)$$

From this and the inversion formula  $\mathcal{F}^2 \varphi = (2\pi)^n \varphi(-\cdot)$  used twice,

$$\begin{aligned} \langle u, \varphi(-\cdot) \rangle &= \langle \hat{u}, \hat{\varphi} \rangle / (2\pi)^n = \langle \hat{u}, \hat{\Psi}(\xi_n) \hat{\varphi}(\xi) \rangle / (2\pi)^n \\ &= \langle u, \mathcal{F}_{\xi \rightarrow x}(\hat{\Psi}(\xi_n) \hat{\varphi}(\xi)) \rangle / (2\pi)^n \\ &= \iiint u(x) \Psi(y_n - x_n) \varphi(-x', -y_n) dx_n dy_n dx'. \end{aligned} \quad (30)$$

The last identity follows from Fubini's theorem and the polynomial growth. In particular  $(x', y_n) \mapsto \int u(x) \Psi(y_n - x_n) dx_n \varphi(-x', -y_n)$  is integrable, and since  $u$  itself acts by integration, this shows (28) in the set of locally integrable functions.

3°. In case  $p_k \geq 1$  for all  $k$  it follows from (28) and the generalised Minkowski inequality that

$$\|u(\cdot, x_n) |L_{p'}(\mathbb{R}^{n-1})\| \leq \int |\Psi(x_n - y_n)| \cdot \|u(\cdot, y_n) |L_{p'}\| dy_n. \quad (31)$$

The Hausdorff–Young inequality applies to the convolution in the last expression, so if  $q \geq 1$  is defined by  $\frac{1}{p_n} + \frac{1}{q} = 1 + \frac{1}{r_n}$ ,

$$\|u |L_{\bar{r}}\| \leq \|\Psi |L_q(\mathbb{R})\| \|u |L_{\bar{p}}\| \leq R^{1-\frac{1}{q}} \|\psi\|_q \|u |L_{\bar{p}}\|. \quad (32)$$

Because  $1 - \frac{1}{q} = \frac{1}{r_n} - \frac{1}{p_n}$ , this yields (26) for all classical exponents.

4°. In general there is some  $s \in ]0, 1[$  such that  $\frac{p_k}{s} > 1$  for all  $k$ . With this (28) can be replaced by

$$|u(x)|^s \leq C(R_n) \int_{\mathbb{R}} |u(x', y_n)|^s |\Psi(x_n - y_n)|^s dy_n. \quad (33)$$

Indeed,  $y_n \mapsto u(x', y_n) \Psi(x_n - y_n)$  has spectrum in  $[-3R_n, 3R_n]$  (since that of  $u(x', \cdot)$  is contained in  $[-R_n, R_n]$ ). Hence (33) results from the next inequality, that follows from 2° and 3°,

$$|u(x)| \leq \|u(x', \cdot) \Psi(x_n - \cdot)\|_1 \leq c(3R_n)^{\frac{1}{s}-1} \|u(x', \cdot) \Psi(x_n - \cdot)\|_s. \quad (34)$$

Applying the generalised Minkowski inequality to (33) we first obtain

$$\| |u(\cdot, x_n)|^s |L_{p'/s}\| \leq C(R_n) \int |\Psi(x_n - y_n)|^s \| |u(\cdot, y_n)|^s |L_{p'/s}\| dy_n, \quad (35)$$

so by taking  $q \geq 1$  such that  $1 + \frac{1}{r_n/s} = \frac{1}{p_n/s} + \frac{1}{q/s}$ ,

$$\| |u|^s |L_{\bar{r}/s}\| \leq C(R_n) \| |\Psi|^s |L_{q/s}\| \| |u|^s |L_{\bar{p}/s}\|. \quad (36)$$

But this means that

$$\|u |L_{\bar{r}}\|^s \leq c' R_n^{1-s} R_n^{s-\frac{s}{q}} \|\psi\|_q^s \|u |L_{\bar{p}}\|^s, \quad (37)$$

and since  $1 - \frac{s}{q} = s(\frac{1}{p_n} - \frac{1}{r_n})$  the claim follows by taking roots.  $\square$



Using Lemma 2 and ideas from the proof of Proposition 1, the above theorem can now relatively easily be extended to a sequence version of the Nikol'skij–Plancherel–Polya inequality.

This version deals with sequences  $(f_j)$  in  $\mathcal{S}'(\mathbb{R}^n)$  that fulfill the following spectral condition, that we could describe as a *geometric rectangle condition*,

$$\text{supp } \mathcal{F}f_j \subset [-AR_1^j, AR_1^j] \times \cdots \times [-AR_n^j, AR_n^j]. \quad (38)$$

Here  $A > 0$  is a constant, while the fixed numbers  $R_1, \dots, R_n > 1$  define the rectangles.

The next inequality will give the Sobolev embeddings in Theorem 6 below at once, but the inequality is also interesting for other purposes; it was obtained in [ST87, Prop. 2.4.1] for  $n = 2$ .

**Theorem 5.** *When  $0 < p_k \leq r_k \leq \infty$  for  $k = 1, \dots, n$ , then there is for  $0 < q \leq \infty$  a number  $c > 0$  such that*

$$\left\| \left( \sum_{j=0}^{\infty} |f_j(\cdot)|^q \right)^{\frac{1}{q}} |L_{\vec{r}}| \right\| \leq c \left\| \sup_{j \in \mathbb{N}_0} \left( \prod_{k=1}^n R_k^{\frac{j}{p_k} - \frac{j}{r_k}} |f_j(\cdot)| \right) |L_{\vec{p}}| \right\| \quad (39)$$

for all sequences  $(f_j)$  in  $\mathcal{S}'(\mathbb{R}^n)$  fulfilling (38).

*Proof.* Using a succession of embeddings as in the proof of Theorem 4, it is enough to cover the case in which  $r_m \neq p_m$  only holds for  $m = n$ .

Furthermore it can be assumed that  $q < \min(r_1, \dots, r_n)$ . Then it is a consequence of Minkowski's inequality that the left hand side of (39) is less than

$$\left( \int_{\mathbb{R}} \left( \sum_{j=0}^{\infty} \|f_j(\cdot, x_n) |L_{p'}(\mathbb{R}^{n-1})\|^q \right)^{\frac{r_n}{q}} dx_n \right)^{\frac{1}{r_n}}. \quad (40)$$

To proceed we note that Lemma 2 also holds if the base of the exponential is shifted from 2 to  $R_n > 1$  (since  $R_n$  is a power of 2). We use this version below with  $\theta := \frac{p_n}{r_n} \in ]0, 1[$ , and setting  $s_0 = \frac{1}{p_n} - \frac{1}{r_n}$ ,  $s_1 = -\frac{1}{r_n}$ ,

$$\theta s_0 + (1 - \theta) s_1 = \frac{\theta}{p_n} - \frac{1}{r_n} = 0. \quad (41)$$

Applying this to the  $\ell_q^0$ -norm, the above integral is majorised by

$$c \left( \int_{\mathbb{R}} \left( \sup_j R_n^{s_0 j} \|f_j(\cdot, x_n) |L_{p'}|\| \right)^{\theta r_n} dx_n \right)^{\frac{1}{r_n}} \left( \sup_{j, x_n} R_n^{-\frac{j}{r_n}} \|f_j(\cdot, x_n) |L_{p'}|\| \right)^{1-\theta}. \quad (42)$$

Using Theorem 4 on the  $L_{(r', \infty)}$ -norm, one has for the last factor

$$\left( \dots \right)^{1-\theta} \leq c' \left( \sup_j R_n^{\frac{j}{p_n} - \frac{j}{r_n}} \|f_j |L_{\vec{p}}|\| \right)^{1-\theta} \leq c' \left\| \sup_j R_n^{\frac{j}{p_n} - \frac{j}{r_n}} |f_j |L_{\vec{p}}|\| \right\|^{1-\theta}. \quad (43)$$

The first factor  $\left( \dots \right)^{\theta/p_n}$  can be treated similarly, and it follows that (39) holds.  $\square$

Using Theorem 5 we arrive at the following Sobolev embedding.

**Theorem 6.** *Let  $s, t \in \mathbb{R}$ ,  $s > t$ , and  $p_j, r_j \in ]0, \infty[$  fulfil*

$$s - \frac{1}{p_1} - \dots - \frac{1}{p_n} = t - \frac{1}{r_1} - \dots - \frac{1}{r_n}, \quad \forall k: r_k \geq p_k. \quad (44)$$

*There is then a continuous embedding  $F_{\vec{p}, \infty}^s(\mathbb{R}^n) \hookrightarrow F_{\vec{r}, q}^t(\mathbb{R}^n)$  for every  $q \in ]0, \infty[$ .*

*Proof.* Since the ball  $B(0, A \cdot 2^j)$  is contained in the rectangle in (38) with  $R_k = 2$  for all  $k$ , it suffices to take a Littlewood–Paley decomposition of an arbitrary  $u$  in  $F_{\vec{p}, \infty}^s(\mathbb{R}^n)$  and then insert  $f_j := 2^{tj} u_j$  into (39).  $\square$

In the following section, the above embedding will be extended to a set-up with an additional anisotropy in  $s$ .

### 3. SPACES WITH MIXED NORMS AND QUASI-HOMOGENEOUS SMOOTHNESS.

As is well known, it is important, say for parabolic differential equations to consider spaces that are anisotropic in the quasi-homogeneous sense concerning the smoothness index  $s$ . This may be combined with the mixed Lebesgue norms in the way we now describe briefly.

Each coordinate  $x_j$  in  $\mathbb{R}^n$  is given a weight  $a_j \geq 1$ , and  $\vec{a} = (a_1, \dots, a_n)$ ; i.e.  $\vec{a} = (1, \dots, 1)$  is the case previously treated in this paper. With the anisotropic dilation  $t^{\vec{a}}x := (t^{a_1}x_1, \dots, t^{a_n}x_n)$  for  $t \geq 0$ , and  $t^{s\vec{a}}x := (t^s)^{\vec{a}}x$  for  $s \in \mathbb{R}$ , and in particular  $t^{-\vec{a}}x = (t^{-1})^{\vec{a}}x$ , the anisotropic distance function  $|x|_{\vec{a}}$  is introduced as the unique  $t > 0$  such that  $t^{-\vec{a}}x \in S^{n-1}$  (for  $x \neq 0$ ;  $|0|_{\vec{a}} = 0$ ); i.e.

$$\frac{x_1^2}{t^{2a_1}} + \dots + \frac{x_n^2}{t^{2a_n}} = 1. \quad (45)$$

For the reader's convenience we recall that  $|\cdot|_{\vec{a}}$  is  $C^\infty$  on  $\mathbb{R}^n \setminus \{0\}$  by the Implicit Function Theorem; the formula  $|t^{\vec{a}}x|_{\vec{a}} = t|x|_{\vec{a}}$  is seen directly, and this implies the triangle inequality:

$$|x + y|_{\vec{a}} \leq |x|_{\vec{a}} + |y|_{\vec{a}}. \quad (46)$$

Indeed, it should be shown for  $x \neq 0 \neq y$  that  $1 \geq \sum_{j=1}^n \frac{(x_j + y_j)^2}{(|x|_{\vec{a}} + |y|_{\vec{a}})^{2a_j}}$ , and since each fraction is invariant under  $(x, y) \mapsto (t^{\vec{a}}x, t^{\vec{a}}y)$ , it can be assumed that  $|x|_{\vec{a}} + |y|_{\vec{a}} = 1$ . Then  $0 < |x| \leq |x|_{\vec{a}} < 1$ , for

$$1 = \sum_{j=1}^n x_j^2 |x|_{\vec{a}}^{-2a_j} \geq \sum_{j=1}^n x_j^2 |x|_{\vec{a}}^{-2} = |x|^2 |x|_{\vec{a}}^{-2}. \quad (47)$$

With similar results for  $y$ , we get  $|x + y| \leq |x| + |y| \leq |x|_{\vec{a}} + |y|_{\vec{a}} \leq 1$ , whence the inequality after (46), as desired.

As a preparation, we need the following analogues of the inequalities between the  $\ell_\infty$ -,  $\ell_2$ - and  $\ell_1$ -norms on  $\mathbb{C}^n$ :

$$\max(|x_1|^{1/a_1}, \dots, |x_n|^{1/a_n}) \leq |x|_{\vec{a}} \leq |x_1|^{1/a_1} + \dots + |x_n|^{1/a_n}. \quad (48)$$

The inequality to the right follows from (46), since  $|(x_1, 0, \dots)|_{\vec{a}} = |x_1|^{1/a_1}$  etc. By taking  $t$  equal to the above maximum, the left hand side of (45) would be  $\geq 1$  (the maximum is attained), so that  $t \leq |\xi|_{\vec{a}}$ .

Along with  $|\cdot|_{\vec{a}}$ , the Littlewood–Paley decomposition is chosen as above but with the modification that  $\Psi_j(\xi) = \psi(2^{-j}|\xi|_{\vec{a}})$ . Then  $\Phi_j$  is supported in the anisotropic corona  $2^{j-1} \leq |\xi|_{\vec{a}} \leq 2^{j+1}$ . As usual  $u_j := \mathcal{F}^{-1}(\Phi_j \hat{u})$ , but here it is understood that it is the anisotropic distance function  $|\cdot|_{\vec{a}}$  that goes into the construction.

Using this, the anisotropic Triebel–Lizorkin space  $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ , with  $s \in \mathbb{R}$ ,  $\vec{p} \in ]0, \infty[^n$  and  $0 < q \leq \infty$ , consists of the  $u$  in  $\mathcal{S}'(\mathbb{R}^n)$  having finite quasi-norms

$$\|u\|_{F_{\vec{p},q}^{s,\vec{a}}} = \left\| \left( \sum_{j=0}^{\infty} 2^{sqj} |u_j(\cdot)|^q \right)^{\frac{1}{q}} |L_{\vec{p}}| \right\|. \quad (49)$$

The corresponding Besov space  $B_{\vec{p},\infty}^{s,\vec{a}}(\mathbb{R}^n)$ , now with  $0 < p_k \leq \infty$  for all  $k$ , is given by the quasi-norm

$$\|u\|_{B_{\vec{p},\infty}^{s,\vec{a}}} = \left( \sum_{j=0}^{\infty} 2^{sqj} \|u_j\|_{L_{\vec{p}}}^q \right)^{\frac{1}{q}}. \quad (50)$$

These are more precisely anisotropic spaces of the quasi-homogeneous and mixed norm type. In the case of usual Lebesgue norms, i.e.  $p_1 = \dots = p_n = p$ , such spaces have been used in general investigations, for example of microlocal properties of non-linear differential equations in [Yam88] and of pointwise multiplication in [Joh95]. For  $1 < p_k < \infty$ ,  $k = 1, \dots, n$  and positive  $s$ , anisotropic Triebel–Lizorkin spaces based on mixed  $L_p$ -norms have been investigated in the second edition of the famous book of Besov, Il'in and Nikol'skij [BIN96]. There they are introduced by means of differences. Cf. Remark 11 below.

Let us first note that  $\|\cdot\|_{F_{\vec{p},q}^{s,\vec{a}}}$  and  $\|\cdot\|_{B_{\vec{p},\infty}^{s,\vec{a}}}$  are quasi-norms. Indeed, (21) implies that for  $\lambda = \min(1, q, p_1, \dots, p_n)$  their powers  $\|\cdot\|^\lambda$  are subadditive; i.e. an analogue of (21) holds for them. So by taking  $\ell_{1/\lambda}$  and  $\ell_{(1/\lambda)^*}$ -norms,  $(\frac{1}{\lambda})^* = \frac{1}{1-\lambda}$ ,

$$\|f + g\| \leq (\|f\|^\lambda + \|g\|^\lambda)^{1/\lambda} \leq 2^{1-\lambda} (\|f\| + \|g\|). \quad (51)$$

When both  $q \geq 1$  and all  $p_k \geq 1$ , the spaces are therefore Banach spaces (in view of the completeness shown in Proposition 10 below).

As usual the simple embeddings  $F_{\vec{p},q}^{s,\vec{a}} \hookrightarrow F_{\vec{p},q}^{s-\varepsilon,\vec{a}}$  for  $\varepsilon > 0$ , respectively  $F_{\vec{p},q}^{s,\vec{a}} \hookrightarrow F_{\vec{p},r}^{s,\vec{a}}$  for  $q < r$  are easy to see. Similarly for  $B_{\vec{p},q}^{s,\vec{a}}$ . A direct argument and the generalised Minkowski inequality gives that

$$B_{\vec{p},\min(p_1,\dots,p_n)}^{s,\vec{a}} \hookrightarrow F_{\vec{p},q}^{s,\vec{a}} \hookrightarrow B_{\vec{p},\max(p_1,\dots,p_n)}^{s,\vec{a}}. \quad (52)$$

Next we show that the Sobolev embedding for these spaces is a direct consequence of the previous inequality for distributions fulfilling the geometric rectangle condition (38).

**Theorem 7.** *Let  $s, t \in \mathbb{R}$ ,  $s > t$ , and  $p_k, r_k \in ]0, \infty]$  fulfil*

$$s - \frac{a_1}{p_1} - \dots - \frac{a_n}{p_n} = t - \frac{a_1}{r_1} - \dots - \frac{a_n}{r_n}, \quad \forall k: r_k \geq p_k. \quad (53)$$

*For any  $q \in ]0, \infty]$  there are then continuous embeddings  $F_{\vec{p}, \infty}^{s, \vec{a}} \hookrightarrow F_{\vec{r}, q}^{t, \vec{a}}$  (all  $r_k < \infty$ ) and  $B_{\vec{p}, q}^{s, \vec{a}} \hookrightarrow B_{\vec{r}, q}^{t, \vec{a}}$  between these spaces over  $\mathbb{R}^n$ .*

*Proof.* Since  $\text{supp } \Phi_j$  is contained in the set where  $|\xi|_{\vec{a}} \leq 2^{j+1}$ , the left inequality in (48) gives for all  $k = 1, \dots, n$  that  $|\xi_k| \leq 2^{a_k + j a_k}$ . Hence  $\text{supp } \Phi_j$  is a subset of the rectangle in (38) for  $R_k = 2^{a_k}$  and  $A = 2^{\max(a_1, \dots, a_n)}$ .

Applying Theorem 5 to  $f_j := 2^{tj} u_j$  for an arbitrary  $u$  in  $F_{\vec{p}, \infty}^{s, \vec{a}}(\mathbb{R}^n)$  therefore yields that  $\|u\|_{F_{\vec{r}, q}^{t, \vec{a}}} \leq c \|u\|_{F_{\vec{p}, \infty}^{s, \vec{a}}}$  under the assumption in (53). The  $B$ -case follows from (26).  $\square$

*Remark 8.* In [BIN96, Thm. 29.10] embeddings of  $F$ -spaces into  $L_{\vec{r}}$  and into Besov spaces are studied. Our Theorem 7 supplements and improves (at least partly) the assertions stated there.

*Remark 9.* The proof of Theorem 7 gives a bit more than stated. Under the hypothesis of the theorem, and even in case some  $r_k = \infty$ , there is a constant  $c > 0$  such that

$$\left\| \left( \sum_{j=0}^{\infty} 2^{tj} |u_j(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L_{\vec{r}}} \leq c \|u\|_{F_{\vec{p}, \infty}^{s, \vec{a}}}. \quad (54)$$

If some  $r_k = \infty$  the left-hand side cannot be seen as a Triebel–Lizorkin norm, for already isotropic  $F_{\infty, q}^s$ -spaces require more than Littlewood–Paley decompositions for the definition. But the inequality is clearly a gratis consequence of the present methods.

Finally, as a supplement to the existing literature, we add the next result. It is another application of the mixed-norm version of the Nikol’skij–Plancherel–Polya inequality, cf. Theorem 4 above.

**Proposition 10.** *For  $s \in \mathbb{R}$ ,  $0 < p_k < \infty$  ( $1 \leq k \leq n$ ) and  $0 < q \leq \infty$  the Triebel–Lizorkin space  $F_{\vec{p}, q}^{s, \vec{a}}(\mathbb{R}^n)$  is a quasi-Banach space with continuous embeddings*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F_{\vec{p}, q}^{s, \vec{a}}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n). \quad (55)$$

*Analogous results hold for the  $B_{\vec{p}, q}^{s, \vec{a}}$  spaces with  $p_k \in ]0, \infty]$  for all  $k$ .*

*Proof.* By (52), it suffices to show (55) in the Besov case; and  $q = \infty$  is also enough. That  $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B_{\vec{p}, \infty}^{s, \vec{a}}(\mathbb{R}^n)$  is a direct consequence of the definition, with a proof similar to the isotropic one in [Tri83, 2.3.3].

The continuity of  $B_{\vec{p},\infty}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$  can also be carried over from [Tri83, 2.3.3]. In so doing, the anisotropies can be handled by Theorem 4 (like in the proof of Theorem 7), which for any  $u \in B_{\vec{p},\infty}^{s,\vec{a}}(\mathbb{R}^n)$  gives

$$\|u_j|L_\infty\| \leq c2^{j(\frac{a_1}{p_1}+\dots+\frac{a_n}{p_n})} \|u_j|L_{\vec{p}}\|. \quad (56)$$

Then one can set  $\tilde{\Phi}_j = \Phi_{j-1} + \Phi_j + \Phi_{j+1}$ , so that  $\tilde{\Phi}_j \equiv 1$  on  $\text{supp } \Phi_j$ ; this gives for  $u \in B_{\vec{p},\infty}^{s,\vec{a}}$  and  $\psi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned} |\langle u, \bar{\psi} \rangle| &\leq \sum_{j=0}^{\infty} |\langle u, \overline{\mathcal{F}^{-1}(\Phi_j \tilde{\Phi}_j \hat{\psi})} \rangle| \\ &\leq \sum_{j=0}^{\infty} c2^{j(\frac{a_1}{p_1}+\dots+\frac{a_n}{p_n})} \|u_j|L_{\vec{p}}\| \|\mathcal{F}^{-1}(\tilde{\Phi}_j \hat{\psi})|L_1\| \\ &\leq c' \|u|B_{\vec{p},\infty}^{s,\vec{a}}\| \sum_{j=0}^{\infty} 2^{(-s+\frac{a_1}{p_1}+\dots+\frac{a_n}{p_n})j} \|\mathcal{F}^{-1}(\tilde{\Phi}_j \hat{\psi})|L_1\|. \end{aligned} \quad (57)$$

Formally the infinite series has the structure of a  $B_{1,1}^{\frac{a_1}{p_1}+\dots+\frac{a_n}{p_n}-s}$ -norm on  $\psi$ . However, the fact that the family  $(\tilde{\Phi}_j)$  appears instead of the decomposition  $(\Phi_j)$  is inconsequential, for the proof of the continuity of  $\mathcal{S} \hookrightarrow B_{p,q}^s$  also gives an estimate in this situation. So for some seminorm  $p_N$  on  $\mathcal{S}$  it holds that  $|\langle f, \bar{\psi} \rangle| \leq c'' \|u|B_{\vec{p},\infty}^{s,\vec{a}}\| p_N(\psi)$ , as desired.

Completeness is shown for the  $F$ -case ([Tri83, 2.3.3] is without details on this). Given a fundamental sequence  $(u_l)$  in  $F_{\vec{p},q}^{s,\vec{a}}$ , the just shown continuity gives that  $u_l - u_m$  belongs eventually to any given neighbourhood of 0 in  $\mathcal{S}'(\mathbb{R}^n)$ ; hence  $(u_l)$  converges in  $\mathcal{S}'$  to some  $u$ . So  $\check{\Phi}_j * u_l \rightarrow \check{\Phi}_j * u$  in  $\mathcal{S}'$  for  $l \rightarrow \infty$  (since the index  $l$  refers to the sequence, we shall write  $\check{\Phi}_j * u_l$  for the frequency modulations by means of the Littlewood–Paley decomposition). Hence  $\check{\Phi}_j * u_l(x) = \langle u_l, \check{\Phi}_j(x - \cdot) \rangle$  converges pointwisely to  $\check{\Phi}_j * u(x)$ .

It remains to prove that  $u \in F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$  and  $u_l \rightarrow u$  in the topology of this space. For  $q < \infty$  the sum over  $j \in \mathbb{N}_0$  can be seen as an integration w.r.t. the counting measure, and then  $n+1$  applications of Fatou's lemma give  $\|u|F_{\vec{p},q}^{s,\vec{a}}\| \leq \liminf \|u_l|F_{\vec{p},q}^{s,\vec{a}}\| < \infty$  (it is convenient that positive measurable functions always have integrals in  $[0, \infty]$  and that Tonelli's theorem implies the measurability during the successive integrations). Given  $\varepsilon > 0$  and  $N$  so that  $\|u_m - u_l|F_{\vec{p},q}^{s,\vec{a}}\| < \varepsilon$  for  $m, l > N$ , similar applications of Fatou's lemma gives

$$\begin{aligned} \|u - u_l|F_{\vec{p},q}^{s,\vec{a}}\| &\leq \|(\liminf_m \sum_j 2^{sjq} |\check{\Phi}_j * (u_m - u_l)|^q)^{1/q}|L_{\vec{p}}\| \\ &\leq \limsup_m \|u_m - u_l|F_{\vec{p},q}^{s,\vec{a}}\| < \varepsilon. \end{aligned} \quad (58)$$

This shows the convergence in  $F_{\vec{p},q}^{s,\vec{a}}$ . For  $q = \infty$  it is seen directly that  $\sup_j |\check{\Phi}_j * u(x)| \leq \liminf_l (\sup_j |\check{\Phi}_j * u_l(x)|)$ , and thence the conclusions follow as above. In the  $B$ -cases the ingredients are the same, only with the  $j$ -integration carried out last.  $\square$

*Remark 11.* To conclude we comment on the background.

Anisotropic Sobolev (or Bessel potential) spaces  $H_p^{s,\vec{a}}$  and Besov spaces  $B_{p,q}^{s,\vec{a}}$  (with  $1 < p < \infty$  respectively  $1 \leq p, q \leq \infty$ , partly with  $s > 0$ ) and in particular embedding relations between them have been investigated e.g. in the monographs of Nikol'skij [Nik75] and Besov, Il'in and Nikol'skij [BIN79], [BIN96]. Nikol'skij and his co-authors departed from a definition based on derivatives and differences. For a characterization of anisotropic Triebel–Lizorkin spaces  $F_{\vec{p},q}^{s,\vec{a}}$  by differences we refer to Yamazaki [Yam86b, Thm. 4.1] and Seeger [Se89] (for  $\vec{p} = (p, \dots, p)$ , but general  $\vec{a}$ ). Since Sobolev spaces  $H_p^{s,\vec{a}}$  represent particular cases of the Triebel–Lizorkin scale, there is some overlap between our work and these quoted books. In connection with anisotropic Triebel–Lizorkin spaces we would like to mention also Triebel [Tri77] and Stöckert, Triebel [StT79] for the Fourier-analytic characterization; and concerning the  $\varphi$ -transform and characterization by atoms, we refer to Dintelmann [Di96] and Farkas [Fa00].

As others before us, we have preferred to define the anisotropy in terms of the function  $|\cdot|_{\vec{a}}$ . This procedure is well known and goes back at least to the 1960's. A list of some basic properties of  $|\cdot|_{\vec{a}}$  can be found in [Yam86a], together with further historical remarks.

One advantage of using  $|\cdot|_{\vec{a}}$  is that it gives an efficient formalism where the powerful tools from Fourier analysis and distribution theory are easy to invoke. This is clearly illustrated by the rather manageable proofs of e.g. Theorems 5 and 7. In general the Fourier-analytic approach gives streamlined, if not simpler proofs of some basic properties of the spaces.

Finally we note that this paper has been partly motivated by some works of Weidemaier [Wei98, Wei02], whose results can be *roughly* summarised as follows: taking traces by setting  $x_1 = 0$  in the fully anisotropic  $F_{\vec{p},q}^{s,\vec{a}}$ -spaces, with  $s = 2$  and  $\vec{a} = (1, \dots, 1, 2)$ , leads to trace spaces in the *same* scale, and only to Besov spaces if  $p_1 = \dots = p_n$ .

Concerning the trace problem for the full scale  $F_{\vec{p},q}^{s,\vec{a}}$ , i.e. with general  $s$ ,  $\vec{a}$ ,  $\vec{p}$  and  $q$ , we have corroborated this conclusion in another joint work (in preparation). With the present article our intention was to extract some preparations that should be of independent interest.

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