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R-2005-31

October 2005

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ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

AN INEQUALITY OF REARRANGEMENTS ON THE UNIT CIRCLE

CRISTINA DRAGHICI

Abstract. We prove that the integral of the product of two functions over a symmetric set in $\mathbb{S}^1 \times \mathbb{S}^1$, defined as $E = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$, where σ_1, σ_2 are diffeomorphisms of $\mathbb{S}^1$ with certain properties and d is the geodesic distance on $\mathbb{S}^1$, increases when we pass to their symmetric decreasing rearrangement. We also give a characterization of these diffeomorphisms σ_1, σ_2 for which the rearrangement inequality holds. As a consequence, we obtain the result for the integral of the function $\Psi(f(x), g(y))$ (Ψ a supermodular function) with a kernel given as $k[d(\sigma_1(x), \sigma_2(y))]$, with k decreasing.

1. Introduction

On a measure space (X, μ), the Hardy-Littlewood inequality asserts [4]:

$$\int_X f(x)g(x)\,d\mu(x) \leq \int_0^{\mu(X)} f^*(t)g^*(t)\,dt,$$

where f^* and g^* are the decreasing rearrangements of f and g, respectively. In what follows, $X = \mathbb{S}^1$, or $X = [-\pi, \pi]$, and the above inequality can be written as:

$$\int_{-\pi}^{\pi} f(x)g(x)\,dx \leq \int_{-\pi}^{\pi} f^\sharp(x)g^\sharp(x)\,dx, \tag{1.1}$$

with $f^\sharp$, $g^\sharp$ the symmetric decreasing rearrangements of f and g, given by $f^\sharp(x) = f^*(2|x|)$ and $g^\sharp(x) = g^*(2|x|)$.

These inequalities can be proved using *the layer-cake formula* [10]: Every measurable function $f : X \to \mathbb{R}_+$ can be written as an integral of the characteristic function of its level sets:

$$f(x) = \int_0^\infty \chi_{\{f>t\}}(x)\,dt. \tag{1.2}$$

A more general rearrangement inequality on $X = \mathbb{R}^n$ is the Riesz-Sobolev inequality:

$$\int_{\mathbb{R}^{2n}} f(x)g(y)h(x-y)\,dxdy \leq \int_{\mathbb{R}^{2n}} f^\sharp(x)g^\sharp(x)h^\sharp(x-y)\,dxdy, \tag{1.3}$$

where f, g, h are non-negative functions which vanish at infinity in a weak sense. The case $n = 1$ is due to Riesz in 1930 (see [12]), and the case $n > 1$ is due to Sobolev in 1938 (see [13]). The proof can be found in the book by Hardy, Littlewood, Pólya [9] which sets the beginning of the systematic study of rearrangement inequalities. A more general version of this inequality in $\mathbb{R}^n$, involving n functions can be found in [5].

1991 *Mathematics Subject Classification.* 26D15, 28A25, 52A10.

Key words and phrases. symmetrization, rearrangement, circle integral inequality, convolution.

The equivalent of (1.3) for three non-negative functions on the unit circle was proved by Baernstein [1]:

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(e^{i\phi})g(e^{i\theta})h(e^{i(\phi-\theta)})\,d\theta d\phi \le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f^\sharp(e^{i\phi})g^\sharp(e^{i\theta})h^\sharp(e^{i(\phi-\theta)})\,d\theta d\phi. \tag{1.4}$$

The proof of this inequality uses a variational principle applied to the convolution of characteristic functions of sets which does not seem to generalize in higher dimensions.

The Riesz-Sobolev inequality (1.3) is equivalent to the Brunn-Minkowski inequality from convex geometry [8, 11, 7] which states that if K and L are measurable sets in $\mathbb{R}^n$, then their Minkowski (pointwise) sum $K+L$ is related to the measure of the sets K and L by

$$V(K+L)^{1/n} \ge V(K)^{1/n} + V(L)^{1/n},$$

where V denotes the n-dimensional volume. An analog of this inequality for $\mathbb{S}^n$ is not known, and, since the proof of rearrangement inequalities in $\mathbb{R}^n$ require it, an analog of the Riesz-Sobolev inequality (1.3) is not known in $\mathbb{S}^n$, for $n>1$.

However, a partial result in $\mathbb{S}^n$ was proved by Baernstein and Taylor in [2]. They considered a version of the Riesz-Sobolev inequality where one of the functions is symmetric decreasing. They showed that, if $h=K$ is already symmetric decreasing then

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f(x)g(y)K(x\cdot y)\,d\sigma(x)d\sigma(y) \le \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f^\sharp(x)g^\sharp(y)K(x\cdot y)\,d\sigma(x)d\sigma(y),$$

where $d\sigma$ is the surface measure on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, $x\cdot y$ is the usual inner product and $K(t)$ is an increasing function on $[-1,1]$. Since $x\cdot y=\cos\alpha$, where α is the angle between the vectors x and y, we can write $K(x\cdot y)=k(d(x,y))$, with k decreasing. Here $d(x,y)$ is the great circle (geodesic) distance between x and y. Their proof is based on the polarization technique. They showed first that the inequality holds for the polarizations of f and g in any hyperplane and then they passed to the limit for the general case. They were led to this version of the Riesz-Sobolev inequality while trying to generalize a 2-dimensional result stating that u is subharmonic implies its star function is also subharmonic.

In this paper we are interested in the case $n=1$ of this inequality with K replaced by the characteristic function of a symmetric set which does not depend on the distance between two points, but rather on the distance between their images under two diffeomorphisms σ_1, σ_2 of $\mathbb{S}^1$. We will also obtain a characterization of these diffeomorphisms for which the inequality holds. With the set E defined as

$$E=\{(x,y):\ d(\sigma_1(x),\sigma_2(y))\le\alpha\},$$

we will show that

$$\int_E f(x)g(y)\,dxdy \le \int_E f^\sharp(x)g^\sharp(y)\,dxdy, \tag{1.5}$$

for every $\alpha>0$. This result implies *the main result* of this paper, Theorem 3.6:

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \Psi(f(x),g(y))k[d(\sigma_1(x),\sigma_2(y))]\,dxdy$$
$$\le \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \Psi(f^\sharp(x),g^\sharp(y))k[d(\sigma_1(x),\sigma_2(y))]\,dxdy,$$

with k decreasing and Ψ the distribution function of a measure μ.

The paper is organized as follows: We will first prove (1.5) for f and g replaced by characteristic functions χ_A, χ_B, and σ_2 the identity. Then we will deduce the result (1.5) mentioned above, and we will show that we can replace the product $f(x)g(y)$ by a function $\Psi(f(x), g(y))$ and that we can replace χ_E by a decreasing function of the distance between $\sigma_1(x)$ and $\sigma_2(y)$, yielding Theorem 3.6.

2. Preliminaries

Recall that a function $f : I \to \mathbb{R}$, defined on an interval $I \subset \mathbb{R}$, is called convex if, for every $0 < \lambda < 1$ and every $a, b \in I$, the following inequality holds:

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b).$$

A convex function is differentiable almost everywhere on I and its derivative is increasing.

We denote by $\mathbb{S}^1$ the unit circle in $\mathbb{R}^2$, i.e., $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, and by $\mathbb{S}^1_+$ the upper half unit circle,

$$\mathbb{S}^1_+ = \{e^{i\theta} : 0 \leq \theta \leq \pi\}.$$

Definition 2.1. *A function $\sigma : \mathbb{S}^1_+ \to \mathbb{S}^1_+$ is called convex if the function $\sigma_1 : [0,\pi] \to [0,\pi]$, defined as :*

$$\sigma(e^{i\theta}) = e^{i\sigma_1(\theta)}, \qquad 0 \leq \theta \leq \pi,$$

is convex on $[0,\pi]$.

Let $f : \mathbb{S}^1 \to \mathbb{R}_+$ be a non-negative measurable function. We define its *distribution function*:

$$\lambda_f(t) = |\{f > t\}|, \qquad t \in [0,\infty),$$

where $\{f > t\} := \{z \in \mathbb{S}^1 : f(z) > t\}$ denote the level sets of f, and $|A|$ is the linear measure on $\mathbb{S}^1$ of A. Functions which have the same distribution function are called *equimeasurable.*

We define the *symmetric decreasing rearrangement* of f to be the function $f^\sharp : \mathbb{S}^1 \to \mathbb{R}_+$, given by:

$$f^\sharp(z) = \inf\{t : \lambda_f(t) \leq 2d(1,z)\},$$

where $d(1,z)$ is the geodesic distance on $\mathbb{S}^1$ between z and 1.

It is clear that $f^\sharp(z) = f^\sharp(\bar{z})$ and that $f^\sharp$ decreases as $d(1,z)$ increases. Also, f and $f^\sharp$ are equimeasurable.

If we write $z = e^{i\theta}$, $-\pi \leq \theta < \pi$, then $d(1,z) = d(1,e^{i\theta}) = |\theta|$, and we can think of f as a function of θ via the relation

$$\tilde{f}(\theta) = f(e^{i\theta}).$$

For $\tilde{f} : [-\pi,\pi] \to \mathbb{R}_+$, one defines its symmetric decreasing rearrangement as:

$$\tilde{f}^\sharp(\theta) = \inf\{t : \lambda_{\tilde{f}}(t) \leq 2|\theta|\},$$

where, as before, $\lambda_{\tilde{f}}(t) = |\{\tilde{f} > t\}|$, and thus, there is a one-to-one correspondence between $f^\sharp$ and $\tilde{f}^\sharp$, given by

$$\tilde{f}^\sharp(\theta) = f^\sharp(e^{i\theta}).$$

Whenever necessary, we will think of a function f defined on $\mathbb{S}^1$ as a function on $[-\pi,\pi]$. If $f = \chi_A$ is the characteristic function of a measurable set $A \subset \mathbb{S}^1$, then

$f^\sharp = \chi_{A^\sharp}$, where $A^\sharp$ is the open interval on the unit circle centered at 1, having the same linear measure as A.

Next, we introduce the Hardy-Littlewood-Pólya preorder relation $\prec$ for non-negative functions defined on the interval $[-\pi, \pi]$. We say that (see [3, 4]):

$$f \prec F \qquad \text{iff} \int_{-t}^{t} f^\sharp(s)\, ds \le \int_{-t}^{t} F^\sharp(s)\, ds, \text{ for all } 0 \le t \le \pi.$$

This is equivalent to

$$\int_{-\pi}^{\pi} f^\sharp(s) h^\sharp(s)\, ds \le \int_{-\pi}^{\pi} F^\sharp(s) h^\sharp(s)\, ds,$$

for every positive symmetric decreasing function $h^\sharp$ defined on $[-\pi, \pi]$. To see this, write $h^\sharp(s) = \int_0^\infty \chi_{\{h^\sharp > t\}}(s)\, dt$ (this is the layer cake formula (1.2)), and, using Fubini's formula and the fact that $\{h^\sharp > t\} = (-l(t), l(t))$ is a symmetric interval,

$$\begin{aligned} \int_{-\pi}^{\pi} f^\sharp(s) h^\sharp(s)\, ds &= \int_0^\infty \left[\int_{-l(t)}^{l(t)} f^\sharp(s)\, ds \right] dt \\ &\le \int_0^\infty \left[\int_{-l(t)}^{l(t)} F^\sharp(s)\, ds \right] dt = \int_{-\pi}^{\pi} F^\sharp(s) h^\sharp(s)\, ds. \end{aligned}$$

Yet another equivalent characterization is:

$$f \prec F \Leftrightarrow \int_E f(s)\, ds \le \int_{E^\sharp} F(s)\, ds, \text{ for every } E \subset [-\pi, \pi].$$

The next result is well-known and it follows from the proof of the equality case in the Hardy-Littlewood inequality, presented by Lieb and Loss in [10, pp.82]. We will include a proof here for consistency.

Lemma 2.2. *Let $f : [-\pi, \pi] \to \mathbb{R}_+$ be a measurable function such that*

$$\int_{-t}^{t} f(x)\, dx \ge \int_{-t}^{t} f^\sharp(x)\, dx, \quad \textit{for every } 0 \le t \le \pi. \tag{2.1}$$

Then $f = f^\sharp$ a.e. on $[-\pi, \pi]$.

Proof. From (1.1) applied to $\chi_{(-t,t)}$ and f, it follows that we must have equality in (2.1), i.e.,

$$\int_{-t}^{t} f(x)\, dx = \int_{-t}^{t} f^\sharp(x)\, dx. \tag{2.2}$$

We will use the layer-cake formula to write $f(x) = \int_0^\infty \chi_{\{f>s\}}(x)\, ds$, and similarly for $f^\sharp(x)$.

Using (1.1), we obtain:

$$\int_{-t}^{t} \chi_{\{f>s\}}(x)\, dx \le \int_{-t}^{t} \chi_{\{f^\sharp>s\}}(x)\, dx, \qquad \text{for every } s \ge 0. \tag{2.3}$$

Fubini's theorem and (2.2) imply that:

$$\begin{aligned} \int_{-t}^{t} f(x)\, dx &= \int_0^\infty \left[\int_{-t}^{t} \chi_{\{f>s\}}(x)\, dx \right] ds \\ &= \int_0^\infty \left[\int_{-t}^{t} \chi_{\{f^\sharp>s\}}(x)\, dx \right] ds = \int_{-t}^{t} f^\sharp(x)\, dx. \end{aligned}$$

From this equality and (2.3) it follows that, for a fixed t, there exists a set of measure zero S_t, such that

$$\int_{-t}^{t} \chi_{\{f>s\}}(x)\,dx = \int_{-t}^{t} \chi_{\{f^\sharp>s\}}(x)\,dx, \qquad \text{for every } s \in (0,\infty) \setminus S_t.$$

Next, we choose T_N a countable dense set in $[0,\pi]$ and we denote by $S_{T_N} = \cup_{t\in T_N} S_t$. Then:

$$\int_{-t}^{t} \chi_{\{f>s\}}(x)\,dx = \int_{-t}^{t} \chi_{\{f^\sharp>s\}}(x)\,dx, \text{ for every } t \in T_N \text{ and } s \in (0,\infty) \setminus S_{T_N}. \tag{2.4}$$

Since for every fixed s, $t \to \int_{-t}^{t} \chi_{\{f>s\}}(x)\,dx$ is a continuous function of t, in fact (2.4) holds for every $0 \le t \le \pi$. Thus,

$$\int_{-t}^{t} \chi_{\{f>s\}}(x)\,dx = \int_{-t}^{t} \chi_{\{f^\sharp>s\}}(x)\,dx, \text{ for all } 0 \le t \le \pi \text{ and a.e. } s \in (0,\infty).$$

Now, let t be such that $\{f^\sharp > s\} = (-t,t)$. Then, it follows that $\{f > s\} = (-t,t) = \{f^\sharp > s\}$ a.e., and thus, $f = f^\sharp$ by the layer cake formula. □

The following result shows that $\int_{-t}^{t} f^\sharp(x)\,dx$ is attained as a supremum. A proof can be found in [4, Theorem 7.5, pp.82].

Theorem 2.3. *(J. V. Ryff) For every measurable function f as in Lemma 2.2, there exists a measure preserving transformation T such that $f = f^\sharp \circ T$. This guarantees, for every t, the existence of a set $A \subset [-\pi,\pi]$ of measure $2t$ such that $\int_A f(x)\,dx = \int_{-t}^{t} f^\sharp(x)\,dx$.*

3. Main results: inequalities on the circle

Notation. As before, d is the geodesic distance, also called the arclength, on the unit circle $\mathbb{S}^1$. We have:

$$d(u,v) = d(u\bar{v},1), \qquad \text{for all } u,v \in \mathbb{S}^1, \tag{3.1}$$

where $\bar{v}$ denotes the complex conjugate of v.

We define, for $\alpha > 0$, the function:

$$\chi_\alpha(u,v) = \begin{cases} 1, & \text{if } d(u,v) \le \alpha, \\ 0, & \text{otherwise} \end{cases}$$

and we observe that $\chi_\alpha(u,v) = \chi_\alpha(u\bar{v},1)$, by (3.1).

We introduce a new function, which we call again $\chi_\alpha : \mathbb{S}^1 \to \mathbb{R}_+$, given by $\chi_\alpha(z) := \chi_\alpha(z,1)$, which is the characteristic function of the closed interval on $\mathbb{S}^1$ of linear length 2α, centered at 1.

We will make use, in what follows, of the relation:

$$\chi_\alpha(u\bar{v}) = \chi_\alpha(u,v), \qquad \text{for all } u,v \in \mathbb{S}^1. \tag{3.2}$$

Given two positive measurable functions $f, g : \mathbb{S}^1 \to \mathbb{R}_+$, their convolution, $f * g$, is defined to be the function:

$$\begin{aligned}(f * g)(z_0) &= \int_{\mathbb{S}^1} f(z_0\bar{z})g(z)\,dz \\ &= \int_{-\pi}^{\pi} f(e^{i(\theta_0-\theta)})g(e^{i\theta})\,d\theta,\end{aligned}$$

with $z_0 = e^{i\theta_0}$ and dz represents the arclength element on $\mathbb{S}^1$, usually denoted by $|dz|$.

Given three positive functions f, g, h defined on $\mathbb{S}^1$, we can write

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(e^{i(\theta-t)})g(e^{it})h(e^{i\theta})\,dt\,d\theta = (f * g * h^-)(1), \tag{3.3}$$

where $h^-(z) = h(\bar{z})$, i.e., $h^-(e^{i\theta}) = h(e^{-i\theta})$.

Theorem 3.1. *Let $\sigma : \mathbb{S}^1 \to \mathbb{S}^1$ be a C^1 diffeomorphism such that $\sigma(1) = 1$ and $\sigma(-1) = -1$. Additionally, we assume that $\sigma(\mathbb{S}^1_+) \subseteq \mathbb{S}^1_+$ and $\sigma(\mathbb{S}^1_-) \subseteq \mathbb{S}^1_-$. Let d be the geodesic distance on the unit circle, α be a positive real number, and we define the set $E = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma(x), y) \leq \alpha\}$. For $A, B \subset \mathbb{S}^1$ measurable sets, let*

$$I_\alpha(A, B) = \int_{\mathbb{S}^1\times\mathbb{S}^1} \chi_A(x)\chi_B(y)\chi_E(x, y)dx\,dy.$$

Then, for any A, B measurable subsets of $\mathbb{S}^1$, and $\alpha > 0$,

$$I_\alpha(A, B) \leq I_\alpha(A^\sharp, B^\sharp), \tag{3.4}$$

if and only if, σ is symmetric (i.e. $\overline{\sigma(z)} = \sigma(\bar{z})$, for every $z \in \mathbb{S}^1$) and convex on $\mathbb{S}^1_+$.

Proof. Sufficiency. We define $\sigma_1 : [-\pi, \pi) \to [-\pi, \pi)$ by $e^{\sigma_1(\theta)} := \sigma(e^{i\theta})$ and we assume that σ_1 is convex on $(0, \pi)$. Using change of variables, $(\sigma(x), y) = (u, v)$, the integral I_α becomes:

$$I_\alpha(A, B) = \int_{\mathbb{S}^1\times\mathbb{S}^1} \chi_{\sigma(A)}(u)\chi_B(v)\chi_\alpha(u, v)(\sigma^{-1})'(u)dudv.$$

With $\chi_\alpha(u, v) = \chi_\alpha(u\bar{v})$, as in (3.2), the above expression becomes:

$$I_\alpha(A, B) = \int_{\mathbb{S}^1\times\mathbb{S}^1} \chi_{\sigma(A)}(u)\chi_B(v)\chi_\alpha(u\bar{v})\psi(u)dudv, \tag{3.5}$$

where $\psi(e^{i\theta}) = \tau_1'(\theta)$ and τ_1 is defined by $\sigma^{-1}(e^{i\theta}) = e^{i\tau_1(\theta)}$, and is the inverse of σ_1.

Thus, we can write using convolution and (3.3):

$$I_\alpha(A, B) = [(\chi_{\sigma(A)} \cdot \psi) * \chi_\alpha * \chi_B^-](1),$$

where we used the fact that χ_α is a symmetric function.

It was proved in [1] (see also (1.4)) by Baernstein that, for any three positive measurable functions f, g, h on $\mathbb{S}^1$, the following inequality holds:

$$(f * g * h^-)(1) \leq (f^\sharp * g^\sharp * h^\sharp)(1). \tag{3.6}$$

One can replace h^- in the inequality above by h since they are equimeasurable functions. Thus, based on (3.6) and the fact that χ_α is symmetric decreasing, we conclude that:

$$I_\alpha(A, B) \leq [(\chi_{\sigma(A)} \cdot \psi)^\sharp * \chi_\alpha * \chi_{B^\sharp}](1). \tag{3.7}$$

Fact: If F is a positive symmetric decreasing function and if $f \prec F$ in the sense of Hardy-Littlewood-Pólya (i.e. $\sup\limits_{|G|=2\theta} \int_G f \leq \int_{-\theta}^{\theta} F$), then $f^\sharp$ in inequality (3.6) can be replaced by F. Indeed, $f \prec F$ is equivalent to $\int_{\mathbb{S}^1} f^\sharp(z)g^\sharp(z)\,dz \leq \int_{\mathbb{S}^1} F(z)g^\sharp(z)\,dz$, for all positive symmetric decreasing functions $g^\sharp$. Now, since $g^\sharp * h^\sharp$ is symmetric decreasing and since the convolution $(f^\sharp * g^\sharp * h^\sharp)(1)$ can be written as the integral of the product $f^\sharp(z)(g^\sharp * h^\sharp)(z)$, we conclude that:

$$(f^\sharp * g^\sharp * h^\sharp)(1) \leq (F * g^\sharp * h^\sharp)(1).$$

Therefore, using (3.7) and the Fact, we can prove (3.4) if we show that $\chi_{\sigma(A)}\psi \prec \chi_{\sigma(A^\sharp)}\psi$, i.e.

$$\int_E \chi_{\sigma(A)}\psi \leq \int_{E^\sharp} \chi_{\sigma(A^\sharp)}\psi. \tag{3.8}$$

Let $E' = \sigma^{-1}(E)$, and $E'' = \sigma^{-1}(E^\sharp)$. With these notations, inequality (3.8) becomes:

$$\int_{A\cap E'} dx \leq \int_{A^\sharp \cap E''} dx,$$

or equivalently, $|A \cap E'| \leq |A^\sharp \cap E''|$, which is true if $|E'| \leq |E''|$, since E'' is symmetric. Since ψ is symmetric decreasing, we have that $\int_E \psi(u)du \leq \int_{E^\sharp} \psi(u)du$, which is equivalent to $\int_{\sigma^{-1}(E)} dx \leq \int_{\sigma^{-1}(E^\sharp)} dx$, using change of variables. The latter inequality simply states that $|E'| \leq |E''|$, and the proof of the sufficiency is now complete.

Necessity. Dividing (3.5) by 2α, and letting α tend to zero, we obtain:

$$I_0(A,B) = \int_{\mathbb{S}^1} \chi_{\sigma(A)}(u)\chi_B(u)\psi(u)du,$$

and inequality (3.4) implies that:

$$I_0(A,B) \leq I_0(A^\sharp, B^\sharp). \tag{3.9}$$

With the notation $\tau = \sigma^{-1}$, ψ the Jacobian of τ, and $x = \tau(u)$, I_0 becomes:

$$I_0(A,B) = \int_{\mathbb{S}^1} \chi_A(x)\chi_{\tau(B)}(x)dx = |A \cap \tau(B)|. \tag{3.10}$$

First, we will show that the symmetry condition is necessary. Suppose τ is not symmetric. Then, there exists a point $x = e^{i\theta}$ in $\mathbb{S}^1_+$ such that $\overline{\tau(x)} \neq \tau(\bar{x})$. If we consider $A = \tau(\{e^{it} : |t| < \theta\})$ and $B = \{e^{it} : |t| < \theta\}$, then we have: $|A \cap \tau(B)| = |\tau(B)| > |A^\sharp \cap \tau(B^\sharp)|$, since $\tau(B^\sharp)$ is not symmetric and $|A| = |\tau(B)|$. But this contradicts (3.9) and therefore (3.4).

Suppose now that τ_1 is symmetric, but not concave (or, equivalently, σ_1 is symmetric , but σ_1 is not convex on $(0,\pi)$). Then, there exist $e^{ib}, e^{ic} \in \mathbb{S}^1_+$ with $b, c \in (0,\pi)$ such that:

$$\frac{\tau_1(b)+\tau_1(c)}{2} > \tau_1(\frac{b+c}{2}). \tag{3.11}$$

Without loss of generality we can assume that $b > c$ and let us denote by $a = \frac{b+c}{2}$. Letting $B = \{e^{it} : -c < t < b\}$, it follows that $B^\sharp = \{e^{it} : -a < t < a\}$. We calculate $|\tau(B)| = \tau_1(b) - \tau_1(-c) = \tau_1(b) + \tau_1(c)$ and $|\tau(B^\sharp)| = 2\tau_1(a)$.

From (3.11) we obtain that $|\tau(B)| > |\tau(B^\sharp)|$ which shows that $I_0(\mathbb{S}^1, B) > I_0(\mathbb{S}^1, B^\sharp)$ and contradicts (3.4). Therefore, τ must also be concave. $\square$

Theorem 3.2. *Suppose we have two functions σ_1, σ_2 satisfying the conditions of σ in Theorem 3.1 and define $E = \{(x,y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$, for $\alpha \in \mathbb{R}_+$. Let*

$$I_\alpha(A,B) = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \chi_A(x)\chi_B(y)\chi_E(x,y)dxdy.$$

Then, for any A, B subsets of $\mathbb{S}^1$ and $\alpha > 0$,

$$I_\alpha(A,B) \leq I_\alpha(A^\sharp, B^\sharp), \tag{3.12}$$

if and only if σ_1 , σ_2 are symmetric and convex on $\mathbb{S}^1_+$.

Proof. Sufficiency. Very similar to Theorem 3.1. Using change of variables, $(\sigma_1(x), \sigma_2(y)) = (u,v)$, the integral becomes:

$$I_\alpha(A,B) = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \chi_{\sigma_1(A)}(u)\chi_{\sigma_2(B)}(v)\chi_\alpha(u\bar{v})\psi_1(u)\psi_2(v)dudv,$$

where ψ_1, ψ_2 are defined similarly to ψ in Theorem 3.1 (see (3.5)). Using convolution, this integral can be written as:

$$I_\alpha(A,B) = [(\chi_{\sigma_1(A)} \cdot \psi_1) * \chi_\alpha * (\chi_{\sigma_2(B)} \cdot \psi_2)^-](1).$$

We have already proven that $\chi_{\sigma_1(A)}\psi_1 \prec \chi_{\sigma_1(A^\sharp)}\psi_1$ and $\chi_{\sigma_2(B)}\psi_2 \prec \chi_{\sigma_2(B^\sharp)}\psi_2$, from which it follows that $I_\alpha(A,B) \leq I_\alpha(A^\sharp, B^\sharp)$.

Necessity. Using change of variable $v = \sigma_2(y)$, I_α becomes:

$$I_\alpha(A,B) = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \chi_A(x)\chi_{\{(x,v)\in\mathbb{S}^1\times\mathbb{S}^1 : d(\sigma_1(x),v)\leq\alpha\}}\chi_{\sigma_2(B)}(v)\psi_2(v)dxdv.$$

Dividing by α and letting $\alpha \to 0$, we obtain:

$$I_0(A,B) = \int_{\mathbb{S}^1} \chi_A(x)\chi_{\sigma_2(B)}(\sigma_1(x))\psi_2(\sigma_1(x))dx.$$

Inequality (3.12) of the theorem implies the following inequality:

$$I_0(A,B) \leq I_0(A^\sharp, B^\sharp), \tag{3.13}$$

for all subsets A and B of $\mathbb{S}^1$.

Now let $B = \mathbb{S}^1$ in the above identity. Then:

$$I_0(A,\mathbb{S}^1) = \int_{\mathbb{S}^1} \chi_A(x)\psi_2(\sigma_1(x))dx \leq \int_{\mathbb{S}^1} \chi_{A^\sharp}(x)\psi_2(\sigma_1(x))dx,$$

or equivalently,

$$\int_A \psi_2(\sigma_1(x))\, dx \leq \int_{A^\sharp} \psi_2(\sigma_1(x))\, dx,$$

for every measurable set $A \subset \mathbb{S}^1$. Since the inequality is true for every measurable set A, we conclude by Lemma 2.2 and Theorem 2.3 that $\psi_2 \circ \sigma_1$ is symmetric (i.e., $\psi_2(\sigma_1(z)) = \psi_2(\sigma_1(\bar{z}))$) and decreasing, which implies that ψ_2 is decreasing on $\mathbb{S}^1_+$. Likewise, $\psi_1 \circ \sigma_2$ is symmetric and decreasing on $\mathbb{S}^1_+$, implying that ψ_1 is decreasing on $\mathbb{S}^1_+$. Thus, σ_1^{-1} and σ_2^{-1} are concave on $\mathbb{S}^1_+$ and therefore, σ_1 and σ_2 are convex on $\mathbb{S}^1_+$.

Next, we denote by $\tau = \sigma_1^{-1} \circ \sigma_2$. With this notation, I_0 becomes:

$$I_0(A,B) = \int_{\mathbb{S}^1} \chi_A(x)\chi_{\sigma_2(B)}(\sigma_1(x))[\psi_2 \circ \sigma_1](x)dx$$
$$= \int_{\mathbb{S}^1} \chi_A(x)\chi_{\tau(B)}(x)[\psi_2 \circ \sigma_1](x)dx = \int_{A\cap\tau(B)} [\psi_2 \circ \sigma_1](x)dx.$$

We will show that τ is symmetric, i.e., $\tau(\bar{x}) = \overline{\tau(x)}$, for every $x \in \mathbb{S}^1$. Suppose this is not the case. Then there exists $x = e^{i\theta}$, with $\theta \in (0,\pi)$, such that $\overline{\tau(x)} \neq \tau(\bar{x})$. Let $B = \{e^{it} : |t| < \theta\} = B^\sharp$ and $A = \tau(B) \neq A^\sharp$. Then, we have that $A^\sharp \cap \tau(B^\sharp) \subset A \cap \tau(B) = A$ and $|A \cap \tau(B)| > |A^\sharp \cap \tau(B^\sharp)|$. Since $\psi_2 \circ \sigma_1$ is positive, it follows that $I_0(A,B) > I_0(A^\sharp, B^\sharp)$, which contradicts (3.13). Thus, $\sigma_1^{-1} \circ \sigma_2$ is symmetric. We have shown before that $\psi_1 \circ \sigma_2$ is also symmetric.

Claim: $\sigma_1^{-1} \circ \sigma_2$ and $\psi_1 \circ \sigma_2$ symmetric imply σ_2 is symmetric.

Proof of claim: We define f_2 on the interval $[-\pi, \pi]$ as follows:

$$\sigma_2(e^{i\theta}) = e^{if_2(\theta)}.$$

Since $\psi_1 \circ \sigma_2$ is symmetric and $[\psi_1 \circ \sigma_2](e^{i\theta}) = \psi_1(e^{if_2(\theta)}) = \tau_1'(f_2(\theta))$, as in (3.5), it follows that $\tau_1' \circ f_2$ is even.

Since $[\sigma_1^{-1} \circ \sigma_2](e^{i\theta}) = e^{i\tau_1(f_2(\theta))}$ is symmetric, it follows that $\tau_1 \circ f_2$ is odd.

Now, $(\tau_1 \circ f_2)' = (\tau_1' \circ f_2) \cdot f_2'$ is even and $\tau_1' \circ f_2$ is also even (as we have previously shown) and nonzero, so that f_2' is even and thus f_2 is odd. Therefore σ_2 is symmetric and the proof of the claim is now complete.

Following exactly the same steps, we can show that σ_1 is symmetric. We have shown that σ_1, σ_2 are symmetric and convex on $\mathbb{S}^1_+$. □

Corollary 3.3. *With σ, α and $E = \{(x,y) \in \mathbb{S}^1 : d(\sigma(x), y) \le \alpha\}$, as in Theorem 3.1, we have the following result: For every $f, g : \mathbb{S}^1 \to \mathbb{R}_+$ positive measurable functions, and every $\alpha > 0$,*

$$\int_E f(x)g(y)\,dx\,dy \le \int_E f^\sharp(x)g^\sharp(y)\,dx\,dy, \tag{3.14}$$

if and only if, σ is symmetric, and convex on $\mathbb{S}^1_+$.

To sketch the proof, we write f and g as the integrals of their level sets, using the layer-cake representation formula (1.2):

$$f(x) = \int_0^\infty \chi_{\{f>t\}}(x)\,dt \qquad \text{and}$$
$$g(y) = \int_0^\infty \chi_{\{g>t\}}(y)\,dt,$$

and we notice that $\{f > t\}^\sharp = \{f^\sharp > t\}$ and $\{g > t\}^\sharp = \{g^\sharp > t\}$ so that inequality (3.14) reduces to the case where f and g are characteristic functions, and thus, Theorem 3.1 applies.

Corollary 3.4. *Let σ_1, σ_2 and $E = \{(x,y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), \sigma_2(y)) \le \alpha\}$ be as in Theorem 3.2. For every $f, g : \mathbb{S}^1 \to \mathbb{R}_+$ positive measurable functions, and every $\alpha > 0$,*

$$\int_E f(x)g(y)\,dx\,dy \le \int_E f^\sharp(x)g^\sharp(y)\,dx\,dy, \tag{3.15}$$

if and only if, σ_1 and σ_2 are symmetric, and convex on $\mathbb{S}^1_+$.

The proof of Corollary 3.4 is indeed very similar to the proof of Corollary 3.3, in which one represents f and g as integrals of the characteristic functions of their level sets.

The next theorem is a generalization of the previous results, where one replaces the product by a function Ψ defined as follows:
$\Psi : \mathbb{R}^2_+ \to \mathbb{R}$ vanishes on the boundary of $\mathbb{R}^2_+$, i.e., $\Psi|_{\{x_1=0\}} = \Psi|_{\{x_2=0\}} = 0$, and

$$\Psi(x_1, x_2) + \Psi(y_1, y_2) \le \Psi(x_1 \wedge x_2, y_1 \wedge y_2) + \Psi(x_1 \vee x_2, y_1 \vee y_2).$$

If Ψ is twice continuously differentiable, then the above inequality is equivalent to $\partial_{12}\Psi \ge 0$.

Crowe, Zweibel and Rosenbloom [6] noticed that a continuous such Ψ is the distribution function of a Borel measure μ on $\mathbb{R}^2_+$, i.e.,

$$\Psi(s,t) = \mu([0,s) \times [0,t)), \tag{3.16}$$

and using Fubini's theorem:

$$\int \Psi(f(x), g(y))\, dx\, dy = \int_{\mathbb{R}^2_+} \left[\int \chi_{\{f>s\}}(x)\chi_{\{g>t\}}(y)\, dx\, dy \right] d\mu(s,t). \tag{3.17}$$

We are now ready to state our next result.

Theorem 3.5. *With σ_1, σ_2 and $E = \{(x,y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), \sigma_2(y)) \le \alpha\}$ as in Theorem 3.2, and Ψ the distribution function of a Borel measure μ on $\mathbb{R}^2_+$ as in (3.16), the following inequality holds for every $\alpha > 0$:*

$$\int_E \Psi(f(x), g(y))\, dx\, dy \le \int_E \Psi(f^\sharp(x), g^\sharp(y))\, dx\, dy,$$

if and only if, σ_1 and σ_2 are symmetric on $\mathbb{S}^1$, and convex on $\mathbb{S}^1_+$.

Again, we can reduce $\Psi(f(x), g(y))$ to a product of characteristic functions, using (3.17), and the result follows from Theorem 3.2.

The next theorem shows that we can replace the characteristic function of the set E by a decreasing function of the distance between $\sigma_1(x)$ and $\sigma_2(y)$, call it $k[d(\sigma_1(x), \sigma_2(y))]$.

Theorem 3.6. *Let σ_1, σ_2 be as in Theorem 3.2 and let $k : [0,\infty) \to [0,\infty)$ be a decreasing function, and Ψ the distribution function of a Borel measure μ on $\mathbb{R}^2_+$ as in (3.16). Then, the following inequality holds for every decreasing function k,*

$$\int_{\mathbb{S}^1} \int_{\mathbb{S}_1} \Psi(f(x), g(y)) k[d(\sigma_1(x), \sigma_2(y))]\, dxdy$$
$$\le \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f^\sharp(x), g^\sharp(y)) k[d(\sigma_1(x), \sigma_2(y))]\, dxdy,$$

if and only if, σ_1 and σ_2 are symmetric on $\mathbb{S}^1$, and convex on $\mathbb{S}^1_+$.

Proof. Using (1.2), we can write:

$$k(\tau) = \int_0^\infty \chi_{\{k>t\}}(\tau)\, dt = \int_0^\infty \chi_{[0,l(t)]}(\tau)\, dt,$$

and substituting $d(\sigma_1(x), \sigma_2(y))$ for τ in the above formula, we have

$$k[d(\sigma_1(x), \sigma_2(y))] = \int_0^\infty \chi_{[0,l(t)]}[d(\sigma_1(x), \sigma_2(y))]\, dt. \tag{3.18}$$

We define the set $E_{l(t)}$ as follows:

$$E_{l(t)} = \{(x,y) :\ d(\sigma_1(x),\sigma_2(y)) \le l(t)\}.$$

Then

$$\chi_{[0,l(t)]}[d(\sigma_1(x),\sigma_2(y)] = 1 \Leftrightarrow (x,y) \in E_{l(t)}.$$

Using this fact, (3.18), Fubini's theorem and Theorem 3.5 we obtain the conclusion of Theorem 3.6 by:

$$\begin{aligned}
&\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \Psi(f(x),g(y))k[d(\sigma_1(x),\sigma_2(y))]\,dxdy \\
&\qquad = \int_0^\infty \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \Psi(f(x),g(y))\chi_{E_{l(t)}}(x,y)\,dxdy\,dt \\
&\qquad\qquad \le \int_0^\infty \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \Psi(f^\sharp(x),g^\sharp(y))\chi_{E_{l(t)}}(x,y)\,dxdy\,dt \\
&\qquad\qquad\qquad = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \Psi(f^\sharp(x),g^\sharp(y))k[d(\sigma_1(x),\sigma_2(y))]\,dxdy.
\end{aligned}$$

□

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