

AALBORG UNIVERSITY

**Schrödinger operators on the half line:
Resolvent expansions and the Fermi Golden Rule at thresholds**

by

Arne Jensen and Gheorghe Nenciu

R-2005-34

November 2005

DEPARTMENT OF MATHEMATICAL SCIENCES
AALBORG UNIVERSITY

Fredrik Bajers Vej 7 G ■ DK-9220 Aalborg Øst ■ Denmark

Phone: +45 96 35 80 80 ■ Telefax: +45 98 15 81 29

URL: <http://www.math.aau.dk>



Schrödinger Operators on the Half Line: Resolvent Expansions and the Fermi Golden Rule at Thresholds

Arne Jensen* Gheorghe Nenciu^{†,‡}

Dedicated to K. B. Sinha on the occasion of his sixtieth birthday

Abstract

We consider Schrödinger operators $H = -d^2/dr^2 + V$ on $L^2([0, \infty))$ with the Dirichlet boundary condition. The potential V may be local or non-local, with polynomial decay at infinity. The point zero in the spectrum of H is classified, and asymptotic expansions of the resolvent around zero are obtained, with explicit expressions for the leading coefficients. These results are applied to the perturbation of an eigenvalue embedded at zero, and the corresponding modified form of the Fermi Golden Rule.

Keywords: Schrödinger operator, threshold eigenvalue, resonance, Fermi Golden Rule.

MSC: Primary: 81Q15. Secondary: 35J10, 47A10.

1 Introduction

This paper is a continuation of [5, 7], where expansions of the resolvents of Schrödinger type operators at thresholds, as well as the form of the Fermi

*Department of Mathematical Sciences, Aalborg University, Fredrik Bajers Vej 7G, DK-9220 Aalborg Ø, Denmark. E-mail: matarne@math.aau.dk

†Department of Theoretical Physics, University of Bucharest, P. O. Box MG11, 76900 Bucharest, Romania. E-mail: nenciu@barutu.fizica.unibuc.ro

‡Institute of Mathematics “Simion Stoilow” of the Romanian Academy, P. O. Box 1-764, RO-014700 Bucharest, Romania. E-mail: Gheorghe.Nenciu@imar.ro

Golden Rule (which actually goes back to Dirac), when perturbing a nondegenerate threshold eigenvalue, were obtained. While the methods and results in [5, 7] are to a large extent abstract, the examples discussed were restricted to Schrödinger operators in odd dimensions with local potentials. The aim of this paper is to show that the methods in [5, 7] allow to treat the non-local potentials in exactly the same manner as the local ones, although the properties of the corresponding operators can be quite different. For example, one can have zero as an eigenvalue in one dimension, or eigenfunctions for the zero eigenvalue with compact support (in this connection see e.g. [2]).

Let us briefly describe the results. Let H_0^D denote $-d^2/dr^2$ on $\mathcal{H} = L^2([0, \infty))$ with the Dirichlet boundary condition. Let V be a potential, which can be either local or non-local. We assume that V is a bounded self-adjoint operator on \mathcal{H} . Let $\mathcal{H}^s = L^{2,s}([0, \infty))$ denote the weighted space. Then we assume that V extends to a bounded operator from $\mathcal{H}^{-\beta/2}$ to $\mathcal{H}^{\beta/2}$ for a sufficiently large $\beta > 0$. Since we are concerned with threshold phenomena, the first step is to study the solutions of the equation $H\Psi = 0$. The result is that under the above conditions, for the solutions of $H\Psi = 0$ there are four possibilities.

- (i) No non-zero solutions. In this case zero is called a *regular* point for H .
- (ii) One non-zero solution in $L^\infty([0, \infty))$, but not in $L^2([0, \infty))$. In this case zero is called an *exceptional* point of the *first* kind for H .
- (iii) A finite number of linearly independent solutions, all belonging to $L^2([0, \infty))$. In this case zero is called an *exceptional* point of the *second* kind for H .
- (iv) A finite number of linearly independent solutions, which can be chosen such that all but one belong to $L^2([0, \infty))$. In this case zero is called an *exceptional* point of the *third* kind for H .

Let us note that if V is multiplication by a function, then only cases (i) and (ii) occur.

In all cases we obtain asymptotic expansions for the resolvent of H around the point zero. It is convenient to use the variable $\kappa = -i\sqrt{z}$ in these expansions. We have

$$(H + \kappa^2)^{-1} = \sum_{j=-2}^p \kappa^j G_j + \mathcal{O}(\kappa^{p+1})$$

as $\kappa \rightarrow 0$, in the topology of the bounded operators from \mathcal{H}^s to \mathcal{H}^{-s} for a sufficiently large s , depending on p and the classification of the point zero for H . We compute a few of the leading coefficients explicitly.

These results on asymptotic expansion for the resolvent, and the explicit expressions for the coefficients, are the main ingredients for the application of the results in [7], concerning the perturbation of an eigenvalue embedded at the threshold zero. The main result from [7] in the context of the Schrödinger operators on the half line considered above is as follows. Let $H = H_0^D + V$, where V satisfies Assumption 3.3 for a sufficiently large β . Let W be another potential satisfying the same assumption. We consider the family $H(\varepsilon) = H + \varepsilon W$ for $\varepsilon > 0$. Assume that 0 is a simple eigenvalue of H , with normalized eigenfunction Ψ_0 . Assume

$$b = \langle \Psi_0, W \Psi_0 \rangle > 0, \quad (1.1)$$

and that for some odd integer $\nu \geq -1$ we have

$$G_j = 0, \quad \text{for } j = -1, \dots, \nu - 2, \text{ and } g_\nu = \langle \Psi_0, W G_\nu W \Psi_0 \rangle \neq 0. \quad (1.2)$$

Then Theorem 3.7 in [7] gives the following result (the modified Fermi Golden Rule) on the survival probability for the state Ψ_0 under the evolution $\exp(-itH(\varepsilon))$, showing that for ε sufficiently small the eigenvalue zero of H becomes a resonance:

There exists $\varepsilon_0 > 0$, such that for $0 < \varepsilon < \varepsilon_0$ we have

$$\langle \Psi_0, e^{-itH(\varepsilon)} \Psi_0 \rangle = e^{-it\lambda(\varepsilon)} + \delta(\varepsilon, t), \quad t > 0. \quad (1.3)$$

Here $\lambda(\varepsilon) = x_0(\varepsilon) - i\Gamma(\varepsilon)$ with

$$\Gamma(\varepsilon) = -i^{\nu-1} g_\nu b^{\nu/2} \varepsilon^{2+(\nu/2)} (1 + \mathcal{O}(\varepsilon)), \quad (1.4)$$

$$x_0(\varepsilon) = b\varepsilon(1 + \mathcal{O}(\varepsilon)), \quad (1.5)$$

as $\varepsilon \rightarrow 0$. The error term satisfies

$$|\delta(\varepsilon, t)| \leq C\varepsilon^{p(\nu)}, \quad t > 0, \quad p(\nu) = \min\{2, (2 + \nu)/2\}. \quad (1.6)$$

As an application of the results on asymptotic expansion of the resolvent of H near zero we explicitly compute the coefficient g_ν in two cases.

The contents of the paper is as follows. In Section 2 we introduce some notation used in the rest of the paper. Section 3 forms the core of the paper and contains our results on the resolvent expansions for the free Schrödinger operator on the half line, and then for the Schrödinger operator with a general class of potentials, including non-local ones. In Section 4 we illustrate the general results by giving an explicit example with a rank 2 operator as the perturbation. Finally, Section 5 contains the results on the modified Fermi Golden Rule for the class of operators considered here.

Let us conclude with some remarks on the literature. Resolvent expansions of the type obtained here are typical for Schrödinger operators in odd dimensions, when the potential decays rapidly. Such results were obtained in [4, 3, 8]. More recently, a unified approach was developed in [5, 6]. It is this approach that we use here. Another approach to the threshold behavior is to use the Jost function. See for example [1, 11]. See also the cited papers for further references to results on resolvent expansions around thresholds.

2 Notation

Let H be a selfadjoint operator on a Hilbert space \mathcal{H} . Its resolvent is denoted by $R(z) = (H - z)^{-1}$. In the sequel we will often look at operators with essential spectrum equal to $[0, \infty)$, such that 0 is a threshold point. We will look at asymptotic expansions around this point for the resolvent. It is convenient to change the variable z by introducing $z = -\kappa^2$, with $\operatorname{Re} \kappa > 0$.

In the half line case there is a type of notation common in the physics literature that is very convenient. The resolvent will have an integral kernel $k(r, r')$, $r, r' \in [0, \infty)$. We introduce the two functions

$$r_{>} = \max\{r, r'\}, \quad r_{<} = \min\{r, r'\}. \quad (2.1)$$

We note a few properties for future reference

$$r_{>} + r_{<} = r + r', \quad r_{>} - r_{<} = |r - r'|, \quad r_{>} \cdot r_{<} = r \cdot r'. \quad (2.2)$$

The weighted L^2 -space on the half line is given by

$$\mathcal{H}^s = L^{2,s}([0, \infty)) = \left\{ f \in L^2_{\text{loc}}([0, \infty)) \mid \int_0^\infty |f(r)|^2 (1 + r^2)^s dr < \infty \right\}, \quad (2.3)$$

for $s \in \mathbf{R}$. We write $\mathcal{H} = \mathcal{H}^0 = L^2([0, \infty))$. We use the notation $\mathcal{B}(s_1, s_2)$ for the bounded operators from \mathcal{H}^{s_1} to \mathcal{H}^{s_2} .

The inner product $\langle \cdot, \cdot \rangle$ on \mathcal{H} is also used to denote the duality between \mathcal{H}^s and \mathcal{H}^{-s} . We use the bra and ket notation for operators from \mathcal{H}^s to \mathcal{H}^{-s} . For example, the operator $f \mapsto \int_0^\infty f(r) dr \cdot 1$ from \mathcal{H}^s to \mathcal{H}^{-s} for $s > 1/2$ is denoted by $|1\rangle\langle 1|$.

In the asymptotic expansions below there will be error terms in the norm topology of $\mathcal{B}(s_1, s_2)$ for specified values of the parameters s_1 and s_2 . Here $\kappa \in \{\zeta \mid 0 < |\zeta| < \delta, \operatorname{Re} \zeta > 0\}$ for a sufficiently small δ . We will use the standard notation $\mathcal{O}(\kappa^p)$ for these error terms.

3 Resolvent Expansions

In this section we first obtain the resolvent expansion of the free Schrödinger operator on the half line, and then for the Schrödinger operator with a general class of potentials, including non-local ones.

3.1 The Free Operator with the Dirichlet Boundary Condition

We denote by H_0^D the operator with the domain and action given by

$$\mathcal{D}(H_0^D) = \{f \in \mathcal{H} \mid f \in AC^2([0, \infty)), f(0) = 0\}, \quad H_0^D f = -\frac{d^2}{dr^2} f. \quad (3.1)$$

Here the space AC^2 denotes functions f that are continuously differentiable on $[0, \infty)$, with f' absolutely continuous (see [10]). It is well-known that this operator is selfadjoint.

The resolvent $R_0^D(z) = (H_0^D - z)^{-1}$ has the integral kernel (using $z = -\kappa^2$ as above)

$$K_0^D(\kappa; r, r') = -\frac{i}{\kappa} \sin(i\kappa r_{<}) e^{-\kappa r_{>}}, \quad (3.2)$$

which can be rewritten as

$$K_0^D(\kappa; r, r') = -\frac{1}{2\kappa} (e^{-\kappa(r_{>} + r_{<})} - e^{-\kappa(r_{>} - r_{<})}). \quad (3.3)$$

Using the Taylor expansion we can get the following result, as in [4, 3, 8].

Proposition 3.1. *The resolvent $R_0^D(-\kappa^2)$ has the following asymptotic expansion. Let $p \geq 0$ be an integer and let $s > p + \frac{3}{2}$. Then we have*

$$R_0^D(-\kappa^2) = \sum_{j=0}^p G_j^D \kappa^j + \mathcal{O}(\kappa^{p+1}) \quad (3.4)$$

in the norm topology of $\mathcal{B}(s, -s)$. The operators G_j^D are given explicitly in terms of their integral kernels by

$$G_j^D : \frac{(-1)^j}{2(j+1)!} ((r_{>} + r_{<})^{j+1} - (r_{>} - r_{<})^{j+1}). \quad (3.5)$$

Let $s_1, s_2 > \frac{1}{2}$ with $s_1 + s_2 > 2$. Then $G_0^D \in \mathcal{B}(s_1, -s_2)$. For $s > \frac{3}{2}$ we also have $G_0^D \in \mathcal{B}(\mathcal{H}^s, L^\infty([0, \infty)))$.

If $j \geq 1$ and $s > j + \frac{1}{2}$, then $G_j^D \in \mathcal{B}(s, -s)$.

Proof. The straightforward computations and estimates are omitted. \square

Remark 3.2. For future reference we note the expressions

$$G_0^D : r_<, \quad (3.6)$$

$$G_1^D : -r_<r_> = -r \cdot r', \quad (3.7)$$

$$G_2^D : \frac{1}{2}r_<r_>^2 + \frac{1}{6}r_<^3, \quad (3.8)$$

$$G_3^D : -\frac{1}{6}(r_>^3r_< + r_>r_<^3) = -\frac{1}{6}(r^3 \cdot r' + r \cdot (r')^3). \quad (3.9)$$

3.2 The Potential and the Factorization Method

We now add a potential V to H_0^D and find the asymptotic expansion of the resolvent of $H = H_0^D + V$ around zero. We will allow a rather general class of potentials, so we introduce the following assumption. We consider only bounded perturbations, however it is possible to extend the results to potentials with singularities.

Assumption 3.3. Let V be a bounded selfadjoint operator on \mathcal{H} , such that V extends to a bounded operator from $\mathcal{H}^{-\beta/2}$ to $\mathcal{H}^{\beta/2}$ for some $\beta > 2$. Assume that there exists a Hilbert space \mathcal{K} , a compact operator $v \in \mathcal{B}(\mathcal{H}^{-\beta/2}, \mathcal{K})$, and a selfadjoint operator $U \in \mathcal{B}(\mathcal{K})$ with $U^2 = I$, such that $V = v^*Uv$.

Remark 3.4. The factorization leads to a natural additive structure on the potentials. Assume that $V_j = v_j^*U_jv_j$, $j = 1, 2$, satisfy Assumption 3.3. Let $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Using matrix notation we define

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}. \quad (3.10)$$

Then it follows that $V = V_1 + V_2$ has the factorization $V = v^*Uv$ with the operators v and U defined in (3.10) and the space $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$.

Example 3.5. We give two examples, the first one a local perturbation, and the second one a non-local perturbation.

(i) Let V be multiplication by a realvalued function $V(r)$. Assume that

$$|V(r)| \leq C(1+r)^{-\beta}$$

for some $\beta > 2$. Take $\mathcal{K} = \mathcal{H}$ and let $v = v^*$ denote multiplication by $|V(r)|^{1/2}$. Let U denote multiplication by 1, if $V(r) \geq 0$, and by -1 , if $V(r) < 0$. Then all conditions in Assumption 3.3 are satisfied.

(ii) Let $\varphi \in \mathcal{H}^{\beta/2}$, and $\gamma \in \mathbf{R}$, $\gamma \neq 0$. Let $V = \gamma|\varphi\rangle\langle\varphi|$. It has the following factorization. Let $\mathcal{K} = \mathbf{C}$. Let $v: \mathcal{H}^{-\beta/2} \rightarrow \mathcal{K}$ be given by $v(f) = |\gamma|^{1/2} \langle\varphi, f\rangle$, and U multiplication by $\text{sign}(\gamma)$. Then $v^*(z) = z|\gamma|^{1/2} \varphi$, and we have $V = v^*Uv$. The generalization to an operator of rank N follows from Remark 3.4.

Write $H = H_0^D + V$ with V satisfying Assumption 3.3. We note the following result.

Lemma 3.6. *Let V satisfy Assumption 3.3. Then V is H_0^D -compact.*

Proof. We have

$$V(H_0^D + i)^{-1} = [V(1+r)^{\beta/2}][(1+r)^{-\beta/2}(H_0^D + i)^{-1}].$$

The first term $[\dots]$ is bounded by the assumption and the second term $[\dots]$ is compact by well-known arguments. \square

We now briefly recall the factorization method, as used in [5], but here extended to cover the non-local potentials. The starting point is the operator

$$M(\kappa) = U + v(H_0^D + \kappa^2)^{-1}v^*,$$

which is now a bounded operator on \mathcal{K} . The factored second resolvent equation is given by

$$R(-\kappa^2) = R_0^D(-\kappa^2) - R_0^D(-\kappa^2)v^*M(\kappa)^{-1}vR_0^D(-\kappa^2). \quad (3.11)$$

The first step in obtaining an asymptotic expansion for $R(-\kappa^2)$ is to study the invertibility of $M(\kappa)$ and the asymptotic expansion of the inverse. Inserting the asymptotic expansion (3.4) we get

$$M(\kappa) = \sum_{j=0}^p \kappa^j M_j + \mathcal{O}(\kappa^{p+1}), \quad (3.12)$$

provided $\beta > 2p + 3$. Here

$$M_0 = U + vG_0^Dv^* \quad \text{and} \quad M_j = vG_j^Dv^*, \quad j = 1, \dots, p. \quad (3.13)$$

3.3 Analysis of $\ker M_0$

We analyze the structure of $\ker M_0$ and the connection with the point zero in the spectrum of H .

Lemma 3.7. *Let Assumption 3.3 be satisfied with $\beta > 3$.*

(i) *Let $f \in \ker M_0$. Define $g = -G_0^D v^* f$. Then $Hg = 0$, with the derivatives in the sense of distributions. We have that $g \in L^\infty([0, \infty)) \cap C([0, \infty))$, with $g(0) = 0$. We have $g \in \mathcal{H}$, if and only if*

$$\langle vr, f \rangle_{\mathcal{X}} = 0. \quad (3.14)$$

(ii) *Assume $g \in \mathcal{H}^{-s} \cap C([0, \infty))$, $s \leq 3/2$, satisfies $g(0) = 0$ and $Hg = 0$, in the sense of distributions. Let $f = Uvg$. Then $f \in \ker M_0$.*

(iii) *Assume additionally that V is multiplication by a function. Let $f \in \ker M_0$, $f \neq 0$. Then $\langle vr, f \rangle \neq 0$, and $\dim \ker M_0 = 1$.*

Proof. Let $f \in \ker M_0$, and define $g = -G_0^D v^* f$. Then we have

$$g(r) = - \int_0^\infty r' (v^* f)(r') dr' - \int_r^\infty (r - r') (v^* f)(r') dr'.$$

Since $v^* f \in \mathcal{H}^s$ for some $s > 3/2$, the second term belongs to \mathcal{H} . The first term is a constant. Thus part (i) follows. For part (ii), assume $g \in \mathcal{H}^{-s} \cap C([0, \infty))$, $s \leq 3/2$, satisfies $g(0) = 0$ and $Hg = 0$, in the sense of distributions. Let $f = Uvg \in \mathcal{K}$. By assumption and definition we have

$$\frac{d^2}{dx^2} g = Vg = v^* f.$$

The mapping properties of v^* imply that $v^* f \in \mathcal{H}^s$ for some $s > 3/2$. Thus we can define

$$h(r) = - \int_r^\infty (r - r') (v^* f)(r') dr'.$$

Hence

$$\frac{d^2}{dx^2} h = v^* f.$$

We conclude that $\frac{d^2}{dr^2} (h - g) = 0$ in the sense of distributions, and thus for some $a, b \in \mathbf{C}$ we have $g(r) = h(r) + a + br$. Since $g \in \mathcal{H}^{-s}$, $s \leq 3/2$, and $h \in \mathcal{H}$, we conclude that $b = 0$. Since $g(0) = 0$ by assumption, we have

$$a = -h(0) = - \int_0^\infty r' v(r') f(r') dr.$$

Thus we have shown that

$$g(r) = - \int_0^\infty r' v(r') f(r') dr - \int_r^\infty (r - r') (v^* f)(r') dr' = -(G_0^D v^* f)(r),$$

such that

$$Uf = UUvg = vg = -vG_0^D v^* f,$$

or $M_0 f = 0$.

Assume now that V is multiplication by a function \mathbf{V} , and that the factorization is chosen as above in Example 3.5. To prove part (iii), assume that $f \in \ker M_0$ and that $\langle vr, f \rangle = 0$. Let $g = -G_0^D v^* f$. Then $M_0 f = 0$ implies $f = Uvg$. Using $\langle vr, f \rangle = 0$, we find that g satisfies the homogeneous Volterra equation

$$g(r) = - \int_r^\infty (r - r') \mathbf{V}(r') g(r') dr'.$$

It follows by a standard iteration argument that $g = 0$, and then also $f = 0$. To prove the final statement, assume that we have $f_j \in \ker M_0$, and $f_j \neq 0$, $j = 1, 2$. Define $g_j = -G_0^D v^* f_j$. Then we can find an $\alpha \in \mathbf{C}$, such that $\langle vr, f_1 \rangle + \alpha \langle vr, f_2 \rangle = 0$. Thus we get

$$(g_1 + \alpha g_2)(r) = - \int_r^\infty (r - r') \mathbf{V}(r') (g_1 + \alpha g_2)(r') dr'.$$

It follows again by the iteration argument that $g_1 + \alpha g_2 = 0$, and then as above also $f_1 + \alpha f_2 = 0$. This concludes the proof of part (iii). \square

Remark 3.8. Let us note that for a local potential it suffices to assume $\beta > 2$ for the results in Lemma 3.7 to hold, since in this case we can use the mapping property of G_0^D given in Proposition 3.1.

We need the following result, which is analogous to [4, Lemma 2.6]. We include the proof here.

Lemma 3.9. *Assume that $f_j \in \mathcal{K}$, such that (3.14) holds for f_j , $j = 1, 2$. Then we have that*

$$\langle f_1, vG_2^D v^* f_2 \rangle = - \langle G_0^D v^* f_1, G_0^D v^* f_2 \rangle. \quad (3.15)$$

Proof. Let $g_j = -G_0^D v^* f_j$. Since (3.14) holds, we have that $g_j \in L^2([0, \infty))$. Furthermore, we have

$$\frac{d^2}{dr^2} g_j = v^* f_j \quad (3.16)$$

in the sense of distributions. We denote the Fourier transform on the line by $\widehat{\cdot}$. From (3.16) follows that we have

$$\xi^2 \widehat{g}_j(\xi) = -(v^* f_j)^\widehat{(\xi)}.$$

Since $v^* f_j \in \mathcal{H}^s$ for some $s > 3/2$, the Fourier transform $(v^* f_j)^\wedge$ is continuously differentiable, by the Sobolev embedding theorem. Since $\widehat{g}_j \in L^2(\mathbf{R})$, we must have

$$(v^* f_j)^\wedge(0) = 0, \quad \frac{d}{d\xi}(v^* f_j)^\wedge(0) = 0. \quad (3.17)$$

It follows from (3.7) that $G_1^D v^* f_j = 0$. Thus we have

$$\langle f_1, v G_2^D v^* f_2 \rangle = \lim_{\kappa \rightarrow 0} \frac{1}{\kappa^2} \langle v^* f_1, ((H_0^D + \kappa^2)^{-1} - G_0^D) v^* f_2 \rangle.$$

Now compute using the Fourier transform:

$$\begin{aligned} & \frac{1}{\kappa^2} \langle v^* f_1, ((H_0^D + \kappa^2)^{-1} - G_0^D) v^* f_2 \rangle \\ &= \frac{1}{\kappa^2} \int_{-\infty}^{\infty} \overline{(v^* f_1)^\wedge(\xi)} \left(\frac{1}{\xi^2 + \kappa^2} - \frac{1}{\xi^2} \right) (v^* f_2)^\wedge(\xi) d\xi. \\ &= \int_{-\infty}^{\infty} \overline{(v^* f_1)^\wedge(\xi)} \frac{-1}{(\xi^2 + \kappa^2)\xi^2} (v^* f_2)^\wedge(\xi) d\xi. \end{aligned}$$

It follows from (3.17) that

$$\frac{1}{\xi^2} (v^* f_j)^\wedge(\xi) \in L^2(\mathbf{R}).$$

Thus we can use dominated convergence and take the limit $\kappa \rightarrow 0$ under the integral sign above, to get the result. \square

3.4 Resolvent Expansions: Results

Let us now state the results obtained. We use the same terminology as in [4], since we have the same four possibilities for the point zero. We say that zero is a *regular point* for H , if $\dim \ker M_0 = 0$. We say that zero is an *exceptional point of the first kind*, if $\dim \ker M_0 = 1$, and there is an $f \in \ker M_0$ with $\langle vr, f \rangle \neq 0$. We say that zero is an *exceptional point of the second kind*, if $\dim \ker M_0 \geq 1$, and all $f \in \ker M_0$ satisfy $\langle vr, f \rangle = 0$. In this case zero is an eigenvalue for H of multiplicity $\dim \ker M_0$. Finally, we say that zero is an *exceptional point of the third kind*, if $\dim \ker M_0 \geq 2$, and there is an $f \in \ker M_0$ with $\langle vr, f \rangle \neq 0$.

We introduce the following notation. Let S denote the orthogonal projection onto $\ker M_0$. Then $M_0 + S$ is invertible in $\mathcal{B}(\mathcal{K})$. We write

$$J_0 = (M_0 + S)^{-1}. \quad (3.18)$$

Theorem 3.10. *Assume that zero is a regular point for H . Let $p \geq 1$ be an integer. Assume that $\beta > 2p + 3$ and $s > p + \frac{3}{2}$. Then we have the expansion*

$$R(-\kappa^2) = \sum_{j=0}^p \kappa^j G_j + \mathcal{O}(\kappa^{p+1}) \quad (3.19)$$

in the topology of $\mathcal{B}(s, -s)$. We have

$$G_0 = (I + G_0^D V)^{-1} G_0^D, \quad (3.20)$$

$$G_1 = (I + G_0^D V)^{-1} G_1^D (I + V G_0^D)^{-1}. \quad (3.21)$$

The kernels of the operators G_0^D and G_1^D are given in (3.6) and (3.7), respectively.

Theorem 3.11. *Let $p \geq 0$ be an integer, and let V satisfy Assumption 3.3 for some $\beta > 2p + 7$. Assume that zero is an exceptional point of the first kind for H . Assume that $s > p + \frac{7}{2}$. Then we have an asymptotic expansion*

$$R(-\kappa^2) = \sum_{j=-1}^p \kappa^j G_j + \mathcal{O}(\kappa^{p+1}) \quad (3.22)$$

in the topology of $\mathcal{B}(s, -s)$. We have

$$G_{-1} = |\Psi_c\rangle\langle\Psi_c|, \quad (3.23)$$

where

$$\Psi_c = \frac{\langle f, vr \rangle}{|\langle f, vr \rangle|^2} G_0^D v f,$$

for $f \in \ker M_0$, $\|f\| = 1$.

Theorem 3.12. *Let $p \geq 1$ be an integer, and let V satisfy Assumption 3.3 for some $\beta > 2p + 11$. Assume that zero is an exceptional point of the second kind for H . Assume that $s > p + \frac{11}{2}$. Then we have an asymptotic expansion*

$$R(-\kappa^2) = \sum_{j=-2}^p \kappa^j G_j + \mathcal{O}(\kappa^{p+1}) \quad (3.24)$$

in the topology of $\mathcal{B}(s, -s)$. We have

$$G_{-2} = P_0, \quad (3.25)$$

$$G_{-1} = 0, \quad (3.26)$$

$$\begin{aligned} G_0 = & G_0^D - G_0^D v^* J_0 v G_0^D - G_0^D v^* J_0 v G_2^D V P_0 - P_0 V G_2^D v^* J_0 v G_0^D \\ & + P_0 V G_4^D V P_0 + P_0 V G_2^D + G_2^D V P_0, \end{aligned} \quad (3.27)$$

$$\begin{aligned} G_1 = & G_1^D - G_1^D v^* J_0 v G_0^D - G_0^D v^* J_0 v G_1^D + G_3^D V P_0 + P_0 V G_3^D, \\ & + G_1^D v^* J_0 v G_2^D V P_0 + P_0 V G_2^D v^* J_0 v G_1^D. \end{aligned} \quad (3.28)$$

Here P_0 denotes the projection onto the zero eigenspace of H , and the operator J_0 is defined by (3.18).

Theorem 3.13. *Let $p \geq 0$ be an integer, and let V satisfy Assumption 3.3 for some $\beta > 2p + 11$. Assume that zero is an exceptional point of the third kind for H . Assume that $s > p + \frac{11}{2}$. Then we have an asymptotic expansion*

$$R(-\kappa^2) = \sum_{j=-2}^p \kappa^j G_j + \mathcal{O}(\kappa^{p+1}) \quad (3.29)$$

in the topology of $\mathcal{B}(s, -s)$. We have

$$G_{-2} = P_0, \quad (3.30)$$

$$G_{-1} = |\Psi_c\rangle\langle\Psi_c|. \quad (3.31)$$

Here P_0 is the orthogonal projection onto the zero eigenspace, and Ψ_c is the canonical zero resonance function defined in (3.50).

Remark 3.14. It is instructive to compare the results above with the results in the case dimension $d = 3$, see [4]. The operator we consider here is the angular momentum $\ell = 0$ component of $-\Delta + V$ on $L^2(\mathbf{R}^3)$, provided V commutes with rotations. In particular, we can only get zero as an eigenvalue for non-local V , and the expansion in the second exceptional case has coefficient $G_{-1} = 0$ (and in the third exceptional case this coefficient only contains the zero resonance term), consistent with the result in [4], where in the radial case this term lives in the $\ell = 1$ subspace, see [4, Remark 6.6].

3.5 Resolvent Expansions: Proofs

We now give some details on the proofs of the resolvent expansions.

Proof of Theorem 3.10

We give a brief outline of the proof. Since by assumption M_0 is invertible in \mathcal{K} , and since we assume $\beta > 2p + 3$, we can compute the inverse of $M(\kappa)$ up to an error term $\mathcal{O}(\kappa^{p+1})$ by using the Neumann series and the expansion (3.12). This expansion is then inserted into (3.11), leading to the existence of the expansion up to terms of order p , and to the two expressions

$$G_0 = G_0^D - G_0^D v^* M_0^{-1} v G_0^D$$

and

$$G_1 = (I - G_0^D v^* M_0^{-1} v) G_1^D (I - v^* M_0^{-1} v G_0^D).$$

Now we carry out the following computation

$$\begin{aligned}
I - G_0^D v^* M_0^{-1} v &= I - G_0^D v^* (U + v G_0^D v^*)^{-1} v \\
&= I - G_0^D v^* U (I + v G_0^D v^* U)^{-1} v \\
&= I - G_0^D v^* U v (I + G_0^D v^* U v)^{-1} \\
&= I - G_0^D V (I + G_0^D V)^{-1} \\
&= (I + G_0^D V)^{-1}.
\end{aligned}$$

Using this result, and its adjoint, we get the expressions in the theorem. It is easy to check that the above computations make sense between the weighted spaces.

Proof of Theorem 3.11

We assume that zero is an exceptional point of the first kind. Thus we have that $\dim \ker M_0 = 1$. Take $f \in \ker M_0$, $\|f\| = 1$. Let $S = |f\rangle\langle f|$ be the orthogonal projection onto $\ker M_0$. Assume $\beta > 2p + 7$. Let $q = p + 2$. Then by Proposition 3.1 we have an expansion

$$M(\kappa) = \sum_{j=0}^q \kappa^j M_j + \mathcal{O}(\kappa^{q+1}) = M_0 + \kappa \widetilde{M}_1(\kappa). \quad (3.32)$$

We now use [5, Corollary 2.2]. Thus $M(\kappa)$ is invertible, if and only if

$$m(\kappa) = \sum_{j=0}^{\infty} (-1)^j \kappa^j S \left[\widetilde{M}_1(\kappa) J_0 \right]^{j+1} S, \quad (3.33)$$

is invertible as an operator on $S\mathcal{K}$. We also recall the formula for the inverse from [5, Corollary 2.2].

$$M(\kappa)^{-1} = (M(\kappa) + S)^{-1} + \frac{1}{\kappa} (M(\kappa) + S)^{-1} S m(\kappa)^{-1} S (M(\kappa) + S)^{-1}. \quad (3.34)$$

It is easy to see that we have an expansion

$$m(\kappa) = \sum_{j=0}^{q-1} \kappa^j m_j + \mathcal{O}(\kappa^q),$$

where

$$m_0 = S M_1 S, \quad (3.35)$$

$$m_1 = S M_2 S - S M_1 J_0 M_1 S, \quad (3.36)$$

$$m_2 = S M_3 S - S M_1 J_0 M_2 S - S M_2 J_0 M_1 S + S M_1 J_0 M_1 J_0 M_1 S. \quad (3.37)$$

Using (3.7) we see that

$$m_0 = SM_1S = -|Svr\rangle\langle Svr| = -|\langle f, vr\rangle|^2 S. \quad (3.38)$$

Since $\langle f, vr\rangle \neq 0$, it follows that m_0 is invertible in $S\mathcal{K}$. The Neumann series then yields an expansion

$$m(\kappa)^{-1} = m_0^{-1} + \sum_{j=1}^{q-1} \kappa^j A_j + \mathcal{O}(\kappa^q).$$

The coefficients A_j are in principle computable, although the expressions rapidly get very complicated. This expansion is inserted into (3.34). We also use the Neumann series to expand

$$(M(\kappa) + S)^{-1} = J_0 + \sum_{j=1}^q \kappa^j \widetilde{M}_j + \mathcal{O}(\kappa^{q+1}).$$

This leads to an expansion

$$M(\kappa)^{-1} = \frac{1}{\kappa} S m_0^{-1} S + \sum_{j=0}^{q-2} \kappa^j B_j + \mathcal{O}(\kappa^{q-1}),$$

where we also used that $SJ_0 = J_0S = S$. We now use (3.11) together with the expansion above and the expansion of $R_0^D(-\kappa^2)$ from Proposition 3.1 to conclude that we have an expansion

$$R(-\kappa^2) = -\frac{1}{\kappa} G_0^D v^* S m_0^{-1} S v G_0^D + \sum_{j=0}^{q-2} \kappa^j G_j + \mathcal{O}(\kappa^{q-1}).$$

This concludes the proof of the theorem.

Proof of Theorem 3.13

Assume that zero is an exceptional point of the third kind for H . Thus $\dim \ker M_0 \geq 2$, and there exists an $f \in \ker M_0$ with $\langle vr, f\rangle \neq 0$. We repeat the computations in the proof of Theorem 3.11, although the assumptions are different. As above, S denotes the orthogonal projection onto $\ker M_0$. Given $p \geq 0$, assume $\beta > 2p + 11$, and let $q = p + 4$. Then $\beta > 2q + 3$, and for this q we have the expansion (3.32). We also have the expansion (3.33) and the expressions for the first three coefficients given in (3.35), (3.36), and (3.37), respectively. We have

$$m_0 = SM_1S = -|Svr\rangle\langle Svr|,$$

which by our assumption is a rank 1 operator. The orthogonal projection onto $\ker m_0$ is given by

$$S_1 = S + \frac{1}{\alpha} |Svr\rangle \langle Svr|, \quad \alpha = \|Svr\|_{\mathcal{K}}^2,$$

and by assumption $S_1 \neq 0$. Now we use the main idea in [5], the repeated application of Corollary 2.2. Applying it once more, we get

$$\begin{aligned} m(\kappa)^{-1} &= (m(\kappa) + S_1)^{-1} \\ &+ \frac{1}{\kappa} (m(\kappa) + S_1)^{-1} S_1 q(\kappa)^{-1} S_1 (m(\kappa) + S_1)^{-1}. \end{aligned} \quad (3.39)$$

with

$$\begin{aligned} q(\kappa) &= q_0 + \kappa q_1 + \dots + \mathcal{O}(\kappa^{q-1}) \\ &= S_1 m_1 S_1 + \kappa [S_1 m_2 S_1 - S_1 m_1 (m_0 + S_1)^{-1} m_1 S_1] \\ &+ \dots + \mathcal{O}(\kappa^{q-1}). \end{aligned} \quad (3.40)$$

Here the \dots are terms, whose coefficients can be computed explicitly. We must have that q_0 is invertible in $S_1 \mathcal{K}$. Otherwise, we can iterate the procedure, leading to a singularity in the expansion of $R(-\kappa^2)$ of type κ^{-j} with $j \geq 3$, contradicting the selfadjointness of H . Thus we have

$$q(\kappa)^{-1} = q_0^{-1} - \kappa q_0^{-1} q_1 q_0^{-1} + \dots + \mathcal{O}(\kappa^{q-1}). \quad (3.41)$$

It remains to perform the back-substitution, and to compute the coefficients. The back-substitution leads to

$$R(-\kappa^2) = \frac{1}{\kappa^2} G_{-2} + \frac{1}{\kappa} G_{-1} + \dots + \mathcal{O}(\kappa^{q-4}),$$

with expressions

$$G_{-2} = -G_0^D v S_1 q_0^{-1} S_1 v G_0^D, \quad (3.42)$$

$$\begin{aligned} G_{-1} &= G_0^D v S_1 q_0^{-1} S_1 m_2 S_1 q_0^{-1} S_1 v G_0^D \\ &- G_0^D v (S - S_1 q_0^{-1} S_1 m_1) (m_0 + S_1)^{-1} (S - m_1 S_1 q_0^{-1} S_1) v G_0^D. \end{aligned} \quad (3.43)$$

These expressions can be simplified. The computations are similar to the ones in [7], although there are some differences. Let P_0 denote the projection onto the eigenspace for eigenvalue zero for H .

Let us start by reformulating the result in Lemma 3.7. Let

$$T = -G_0^D v^* S_1 \quad \text{and} \quad \tilde{T} = U v P_0. \quad (3.44)$$

The operator T is a priori only bounded from \mathcal{K} to \mathcal{H}^{-s} for $s > 1/2$, but Lemma 3.7 shows that it is actually bounded from \mathcal{K} to \mathcal{H} , with $\text{Ran } T = P_0\mathcal{H}$. We also have that \tilde{T} is bounded from \mathcal{H} to \mathcal{K} , with $\text{Ran } \tilde{T} = S_1\mathcal{K}$. Now Lemma 3.9 implies that

$$T\tilde{T} = P_0 \quad \text{and} \quad \tilde{T}T = S_1 \quad (3.45)$$

The adjoint T^* is the closure of the operator $-S_1vG_0^D$. These observations lead to the result

$$S_1q_0^{-1}S_1 = -\tilde{T}\tilde{T}^*. \quad (3.46)$$

Now insert into (3.42) to get

$$G_{-2} = T\tilde{T}\tilde{T}^*T^* = P_0.$$

Then we note that

$$G_0^DvS_1q_0^{-1}S_1m_2S_1q_0^{-1}S_1vG_0^D = 0. \quad (3.47)$$

This result holds, since $S_1m_2S_1 = S_1G_3^Dv^*S_1 = 0$, as can be seen from the kernel (3.9) and the condition (3.14), which holds for all functions in the range of S_1 . As for the last term in (3.43), from (3.38) and (3.40) it follows that

$$(m_0 + S_1)^{-1} = S_1 - \frac{1}{\alpha^4}|Svr\rangle\langle Svr|, \quad (3.48)$$

$$(S - S_1q_0^{-1}S_1m_1)S_1 = 0. \quad (3.49)$$

Define

$$\Psi_c = \frac{1}{\|Svr\|^2}(G_0^Dv|Svr\rangle - P_0VG_2^Dv|Svr\rangle). \quad (3.50)$$

Then a computation shows that we have

$$G_{-1} = |\Psi_c\rangle\langle\Psi_c|. \quad (3.51)$$

This concludes the proof of Theorem 3.13.

Proof of Theorem 3.12

We will not give the details of the proof of this theorem. It follows along the lines of the previous proofs. More precisely, if as above S is the orthogonal projection onto $\ker M_0$, then (3.34–3.37) hold true with $m_0 = 0$, and the argument leading to the invertibility of q_0 , see (3.40), gives the fact that M_1 is invertible. Then expanding in (3.34) and carrying the computation far enough, one finds the expressions in (3.25–3.28) for the first four coefficients explicitly, which are of interest in connection with the Fermi Golden Rule results below.

4 A Non-Local Potential Example

We will illustrate Theorem 3.13 by giving an explicit example, using a rank 2 perturbation. The example is constructed such that H has zero as an exceptional point of the third kind.

Let us define two functions in $L^2([0, \infty))$ as follows.

$$\phi_1(r) = \begin{cases} 0 & \text{for } 0 < r \leq 3, \\ 1 & \text{for } 3 < r < 4, \\ 0 & \text{for } 4 \leq r < \infty, \end{cases}$$

$$\phi_2(r) = \begin{cases} 0 & \text{for } 0 < r \leq 1, \\ 1 & \text{for } 1 < r < 2, \\ -\frac{3}{5} & \text{for } 2 \leq r \leq 3, \\ 0 & \text{for } 3 < r < \infty. \end{cases}$$

We have

$$\int_0^\infty r\phi_1(r)dr \neq 0 \quad \text{and} \quad \int_0^\infty r\phi_2(r)dr = 0. \quad (4.1)$$

As our potential we take

$$V = -\frac{3}{10}|\phi_1\rangle\langle\phi_1| - \frac{75}{28}|\phi_2\rangle\langle\phi_2|. \quad (4.2)$$

For the factorization we take $\mathcal{K} = \mathbf{C}^2$, and define $v \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ by

$$v(f) = \begin{bmatrix} \sqrt{\frac{3}{10}}\langle\phi_1, f\rangle \\ \sqrt{\frac{75}{28}}\langle\phi_2, f\rangle \end{bmatrix}. \quad (4.3)$$

We let $U = -I$, where I is the identity operator on \mathcal{K} . Then we have $V = v^*Uv$. Next we compute M_0 . Direct computation shows that

$$vG_0^Dv^* = I.$$

The constants in V were chosen to obtain this result. Thus $M_0 = 0$. Take

$$f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad f_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then

$$\langle vr, f_1 \rangle \neq 0 \quad \text{and} \quad \langle vr, f_2 \rangle = 0,$$

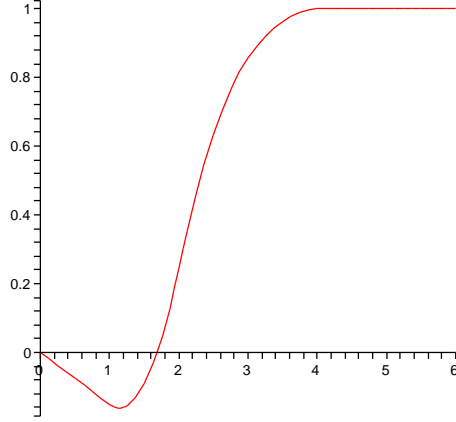


Figure 1: Canonical zero resonance function Ψ_c

due to (4.1). Thus zero is an exceptional point of the third kind for H with this potential. We can also find the resonance function and an eigenfunction explicitly. An eigenfunction is given by $-G_0^D v^* f_2$. Carrying out the computations, one finds after normalization

$$\Psi_0(r) = \sqrt{\frac{375}{98}} \begin{cases} -\frac{2}{5}r & \text{for } 0 < r \leq 1, \\ \frac{1}{2}r^2 - \frac{7}{5}r + \frac{1}{2} & \text{for } 1 < r < 2, \\ -\frac{3}{10}r^2 + \frac{9}{5}r - \frac{27}{10} & \text{for } 2 \leq r \leq 3, \\ 0 & \text{for } 3 < r < \infty. \end{cases} \quad (4.4)$$

Using this function and the expression (3.50) one gets

$$\Psi_c(r) = \begin{cases} -\frac{52}{343}r & \text{for } 0 < r \leq 1, \\ \frac{375}{686}r^2 - \frac{61}{49}r + \frac{375}{686} & \text{for } 1 < r < 2, \\ -\frac{225}{686}r^2 + \frac{773}{343}r - \frac{2025}{686} & \text{for } 2 \leq r \leq 3, \\ -\frac{1}{7}r^2 + \frac{8}{7}r - \frac{9}{7} & \text{for } 3 < r \leq 4, \\ 1 & \text{for } 4 < r < \infty. \end{cases} \quad (4.5)$$

The plots of the two functions are shown in Figure 1 and Figure 2, respectively.

The computations in this example have been made using the computer algebra system Maple.

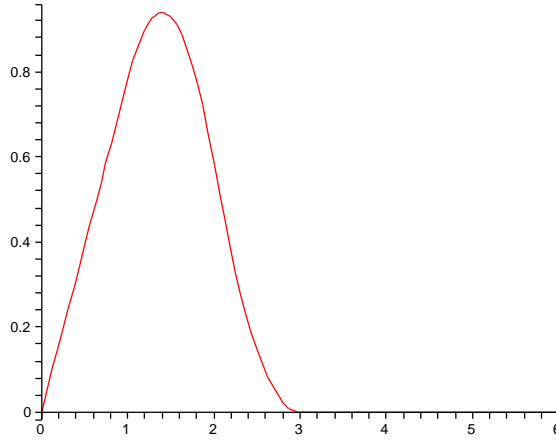


Figure 2: Normalized zero eigenfunction $-\Psi_0$

5 Application to the Fermi Golden Rule at Thresholds

We recall the main result from [7] in the context of the Schrödinger operators on the half line considered above. Let $H = H_0^D + V$, where V satisfies Assumption 3.3 for a sufficiently large β . Let W be another potential satisfying the same assumption. We consider the family $H(\varepsilon) = H + \varepsilon W$ for $\varepsilon > 0$. Assume that 0 is a simple eigenvalue of H , with normalized eigenfunction Ψ_0 . Assume

$$b = \langle \Psi_0, W\Psi_0 \rangle > 0. \quad (5.1)$$

The results in [7] show that under some additional assumptions the eigenvalue zero becomes a resonance for $H(\varepsilon)$ for ε sufficiently small. Here the concept of a resonance is the time-dependent one, as introduced in [9]. The additional assumption needed is that for some odd integer $\nu \geq -1$ we have

$$G_j = 0, \quad \text{for } j = -1, \dots, \nu - 2, \text{ and } g_\nu = \langle \Psi_0, WG_\nu W\Psi_0 \rangle \neq 0. \quad (5.2)$$

Here G_j denotes the coefficients in the asymptotic expansion for the resolvent of H around zero, as given in either Theorem 3.12 or Theorem 3.13. The main result in [7] gives the following result on the survival probability for the state Ψ_0 under the evolution $\exp(-itH(\varepsilon))$. There exists $\varepsilon_0 > 0$, such that for $0 < \varepsilon < \varepsilon_0$ we have

$$\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0 \rangle = e^{-it\lambda(\varepsilon)} + \delta(\varepsilon, t), \quad t > 0. \quad (5.3)$$

Here $\lambda(\varepsilon) = x_0(\varepsilon) - i\Gamma(\varepsilon)$ with

$$\Gamma(\varepsilon) = -i^{\nu-1} g_\nu b^{\nu/2} \varepsilon^{2+(\nu/2)} (1 + \mathcal{O}(\varepsilon)), \quad (5.4)$$

$$x_0(\varepsilon) = b\varepsilon(1 + \mathcal{O}(\varepsilon)), \quad (5.5)$$

as $\varepsilon \rightarrow 0$. The error term satisfies

$$|\delta(\varepsilon, t)| \leq C\varepsilon^{p(\nu)}, \quad t > 0, \quad p(\nu) = \min\{2, (2 + \nu)/2\}. \quad (5.6)$$

We state two corollaries to the results in this paper and in [7].

Corollary 5.1. *Let $H = H_0^D + V$ be a Schrödinger operator on the half-line, with V satisfying Assumption 3.3 for some $\beta > 17$. Assume that zero is an exceptional point of the second kind for H . The zero eigenfunction is denoted by Ψ_0 and is assumed to be simple. Let also W satisfy Assumption 3.3 for some $\beta > 17$. Assume that*

$$b = \langle \Psi_0, W\Psi_0 \rangle \neq 0, \quad (5.7)$$

$$g_1 = \langle \Psi_0, WG_1W\Psi_0 \rangle \neq 0. \quad (5.8)$$

Let $H(\varepsilon) = H + \varepsilon W$, $\varepsilon > 0$. The the results (5.3)–(5.6) hold with $\nu = 1$.

We note that an expression for g_1 can be obtained from (3.28).

Corollary 5.2. *Let $H = H_0^D + V$ be a Schrödinger operator on the half-line, with V satisfying Assumption 3.3 for some $\beta > 9$. Assume that zero is an exceptional point of the third kind for H . The zero eigenfunction is denoted by Ψ_0 and is assumed to be simple. The canonical resonance function is denoted by Ψ_c . Let also W satisfy Assumption 3.3 for some $\beta > 9$. Assume that*

$$b = \langle \Psi_0, W\Psi_0 \rangle \neq 0, \quad (5.9)$$

$$g_{-1} = \langle \Psi_0, WG_{-1}W\Psi_0 \rangle = |\langle \Psi_0, W\Psi_c \rangle|^2 \neq 0. \quad (5.10)$$

Let $H(\varepsilon) = H + \varepsilon W$, $\varepsilon > 0$. The the results (5.3)–(5.6) hold with $\nu = -1$.

This second Corollary is particularly interesting, since we can check the conditions (5.9) and (5.10) in the example given in Section 4. It is easy to see that one can get both $\langle \Psi_0, W\Psi_c \rangle \neq 0$ and $\langle \Psi_0, W\Psi_c \rangle = 0$, for both local and non-local perturbations W . Only in the first case can one apply directly the results from [7], due to the condition (5.2). The other case has not yet been investigated in detail.

One can also use the results on resolvent expansions to give examples using two channel models, as in [7]. We omit stating these results explicitly.

Acknowledgments A. Jensen was partially supported by a grant from the Danish Natural Sciences Research Council. G. Nenciu was partially supported by Aalborg University. G. Nenciu was supported in part by CNCSIS under Grant 905-13A/2005.

References

- [1] T. Aktosun, *Factorization and small-energy asymptotics for the radial Schrödinger equation*, J. Math. Phys. **41** (2000), 4262–4270.
- [2] W. O. Amrein, A.-M. Berthier, and V. Georgescu, *Lower bounds for zero energy eigenfunctions of Schrödinger operators*, Helv. Phys. Acta **57** (1984), 301–306.
- [3] A. Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in $L^2(\mathbf{R}^m)$, $m \geq 5$* , Duke Math. J. **47** (1980), 57–80.
- [4] A. Jensen and T. Kato, *Spectral properties of Schrödinger operators and time-decay of the wave functions*, Duke Math. J. **46** (1979), 583–611.
- [5] A. Jensen and G. Nenciu, *A unified approach to resolvent expansions at thresholds*, Rev. Math. Phys. **13** (2001), 717–754.
- [6] ———, *Erratum to the paper: “A unified approach to resolvent expansions at thresholds”*, Rev. Math. Phys. **16** (2004), 675–677.
- [7] ———, *The Fermi Golden Rule and Its Form at Thresholds in Odd Dimensions*, Comm. Math. Phys., to appear.
- [8] M. Murata, *Asymptotic expansions in time for solutions of Schrödinger-type equations*, J. Funct. Anal. **49** (1982), no. 1, 10–56.
- [9] A. Orth, *Quantum mechanical resonance and limiting absorption: the many body problem*, Comm. Math. Phys. **126** (1990), no. 3, 559–573.
- [10] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Revised and Enlarged Edition, Academic Press, New York 1980.
- [11] D. R. Yafaev, *The low energy scattering for slowly decreasing potentials*, Comm. Math. Phys. **85** (1982), 177–196.